JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 77, 288–296 (1980)

A Duality Theory for a Class of Non-Zero Sum Economic Games

T. R. JEFFERSON AND C. H. SCOTT

School of Mechanical and Industrial Engineering, The University of New South Wales Kensington, New South Wales 2033, Australia

Submitted by A. Mc Nabb

In this paper we consider n-person games in which each player has a convex strategy set over which his closed strictly quasi-concave payoff function is defined. The interaction of the players' strategies is via linear constraints in the form of a convex cone. An appropriate duality theory is developed and applied to an example with economic significance. The resulting analysis leads naturally to a means for solving such a game that merely involves the solution of a set of linear equations.

1. INTRODUCTION

It is now well known that for zero sum games with mixed strategies, the game played by one player is the linear programming dual of the game played by the other player [12]. Hence one may determine the equilibrium of the game by choosing the easier of the two linear programs and then use duality to find the solution for the other. This use of duality would be more valuable if we could attack games of a more complicated nature than zero sum games. We propose to do this in the sequel.

This research was motivated by the application of game theory to the analysis of economic equilibria. Although the study of economic equilibria was initiated by Walras [18] and Edgeworth [5] in the 19th century, it was the work of Shubik [17] that first analysed such problems within the framework of game theory. In this paper, we develop a duality theory for this class of games. This theory has value in that both additional insights are obtained and a new basis is established for computing economic equilibria. The approach we use to solve such games has perhaps been foreshadowed by the works on linear exchange models by Eaves [4] and Gale [6].

Specifically we consider n-person games with quasiconcave utility functions to be maximized by each player. The strategy set for each player is convex. Interaction among players is by the restriction of the vector of all players' strategies to a convex cone. We determine a dual game corresponding to this class of game.

0022-247X/80/090288–09$02.00/0

In a previous paper [8], we developed a duality theory for quasi-concave programs which provides the basis for the duality theory for games presented here. The important results are given below.

Let $[u_i, U_i]$ be the pair of a payoff function (utility function) for player i together with the strategy set U_i. We can associate a pair $[v_i, V_i]$ of dual quasi-concave payoff function v_i defined over a convex strategy set V_i.

DEFINITION. The utility transform of $[u_i, U_i]$ is a pair $[v_i, V_i]$ defined by

$$v_i(p^i) \triangleq \inf_{x^i \in U_i} \{-u_i(x^i) \mid \langle p^i, x^i \rangle \leqslant 0\},$$

$$V_i \triangleq \{p^i \mid \inf_{x^i \in U_i} \{-u_i(x^i) \mid \langle p^i, x^i \rangle \leqslant 0\} > -\infty\}.$$

Here $\langle \cdot, \cdot \rangle$ denotes the usual inner product in finite-dimensional Euclidean space.

By construction, for $x^i \in U_i$, $p^i \in V_i$, and $\langle p^i, x^i \rangle \leqslant 0$ we have the utility inequality holding:

$$u_i(x^i) + v_i(p^i) \leqslant 0.$$

The dual function v_i has economic significance and is related to the indirect utility function first introduced by Roy [16]. The constraint $\langle p^i, x^i \rangle \leqslant 0$ can be thought of as a budget constraint formed by the inner product of primal and dual variables, commodity and price. For recent economic results in this area, the reader is referred to Lau [9] or Diewert [2].

Of interest is the condition for the utility inequality to hold at equality. This requires the use of a generalization of gradients as quasi-concave functions are not necessarily differentiable.

DEFINITION. The local supergradient of a quasi-concave function $u_i(x^i)$, $x^i \in U_i$ is a set defined by

$$\partial^{\mathrm{loc}} u_i(x^i) \triangleq \{p^i \mid D_{\Delta x^i} u_i(x^i) \leqslant \langle \Delta x^i, p^i \rangle,$$

$\forall$ feasible directions Δx^i with respect to $U_i\}$.

Here $D_{\Delta x^i} u_i(x^i)$ is the directional derivative of u_i at x^i in the direction Δx^i. For differentiable functions, the local supergradient contains the single element: the gradient. The local supergradient is a generalisation of the well-known concept of supergradient (subgradient) used in convex analysis. Penot [13] has developed a similar concept. The requirement of quasi-concavity guarantees existence of the directional derivative if we allow values to be taken over the extended real line.

THEOREM 1. *The directional derivative of a quasi-concave function $u_i(x^i)$ at $x^i \in U_i$ exists.*

Proof. Consider $u_i(x^i)$ and a feasible direction Δx^i. Suppose

$$\limsup_{\lambda \downarrow 0} \frac{u_i(x^i + \lambda \Delta x^i) - u_i(x^i)}{\lambda} \neq \liminf_{\lambda \downarrow 0} \frac{u_i(x^i + \lambda \Delta x^i) - u_i(x^i)}{\lambda}.$$

Then there exist a finite number of points z_k on the line segment from $x^i + \Delta x^i$ to x^i such that

$$u_i(z_k) > u_i(z_{k+1}) < u_i(z_{k+2}) \qquad \text{for} \quad k = 1, 3, 5, \text{etc.}$$

This contradicts the quasi-concavity of $u_i(x^i)$.

However, the local supergradient set may be null. For example, the local supergradient for the function

$$\begin{aligned} f(x) &= x/2, \qquad & x \leqslant 0, \\ &= x, & x \geqslant 0, \end{aligned}$$

is

$$\begin{aligned} \partial^{\text{loc}} f(x) &= \{\tfrac{1}{2}\}, \qquad & x < 0, \\ &= \phi, & x = 0, \\ &= \{1\}, & x > 0. \end{aligned}$$

The natural definition for a generalisation of the gradient at $x = 0$ is the interval $[\frac{1}{2}, 1]$. The generalised gradient of Clarke [3] yields this result. However it is defined for Lipschitz functions and does not allow for the possibility of jumps. To capture the properties of closed quasi-concave functions we extend the concept of local supergradient to the gradient set.

DEFINITION. The gradient set of a function $u_i(x^i)$ at a point x^i is the set defined by

$$\begin{aligned} \partial u_i(x^i) &\triangleq \partial^{\text{loc}} u_i(x^i), \qquad & \text{if} \quad \partial^{\text{loc}} u_i(x^i) \neq \varnothing, \\ &\triangleq \text{co} \bigcup_z \lim_{\lambda \downarrow 0} \partial^{\text{loc}} u_i(x^i + \lambda z), & \text{otherwise,} \end{aligned}$$

where co denotes the convex hull of the set.

The gradient set captures the properties of the generalised gradient while being able to handle jumps by means of the local supergradient.

THEOREM 2. *Let* $[u_i, U_i]$ *have the following properties*:

(i) u_i is closed strictly concave.

(ii) $[v_i, V_i]$ is the utility transform of $[u_i, U_i]$.

(iii) $z^i \in U_i$ is the optimal point of $\sup_{x^i \in U_i}(u_i(x^i))$ if it exists.

For $\bar{x}^i \in U_i$, $\bar{p}^i \in V_i$, $\langle \bar{p}^i, \bar{x}^i \rangle \leqslant 0$ and $\langle z^i, \bar{p}^i \rangle > 0$, we have that

$$u_i(\bar{x}^i) + v_i(\bar{p}^i) = 0$$

iff either

$$\lambda_i \bar{p}^i \in \partial u_i(\bar{x}^i), \qquad \lambda_i > 0, \tag{I}$$

or

$$\begin{aligned} &\text{(a)} \quad v_i \bar{x}^i \in \partial v_i(\bar{p}^i), \qquad v_i > 0, \\ &\text{(b)} \quad u_i(\bar{x}^i) = \sup_{y^i \in U_i} \{u_i(y^i) \mid \langle \bar{p}^i, y^i > \leqslant 0\}. \end{aligned} \tag{II}$$

Proof. See [8].

From Theorem 2, we may determine a primal solution in terms of the variable x^i from the dual variable p^i and vice versa. In the next section we develop this relationship as part of a duality theory for games. The theory is applied to an economic example in the final section. We conclude this introduction with a theorem that gives the properties of the dual function v_i.

THEOREM 3. $v_i(p^i)$ *is quasi-concave and positively homogeneous of degree zero.* V_i *is a convex cone.* $v_i(p^i)$ *is closed if* $u_i(x^i)$ *is closed.*

Proof. See [8].

2. DUALITY

In this section we consider n-person games in which each player i seeks to maximize his utility $u_i(x^i)$ over his convex strategy set U_i. $u_i(x^i)$ are closed strictly quasi-concave functions and the players interact linearly in their strategies. Thus a primal game is of the following form.

Game G. The payoff vector for the game is $u(x) = (u_1(x^1), u_2(x^2),..., u_n(x^n))$ with constraints

$$x^i \in U_i, \qquad i = 1,..., n,$$

and

$$x = (x^1, x^2,..., x^n) \in \chi,$$

where χ is a convex cone.

The corresponding dual game to G is given as

Game H. The payoff vector is $v(p) = (v_1(p^1), v_2(p^2), \ldots, v_n(p^n))$ with constraints

$$p^i \in V_i, \qquad i = 1, \ldots, n$$

and

$$p = (p^1, p^2, \ldots, p^n) \in \chi^*,$$

where $[v_i, V_i]$ is the utility transform of $[u_i, U_i]$ and χ^* is the dual cone of χ, i.e.,

$$\chi^* \triangleq \{p \mid \langle p, x\rangle \leqslant 0,\ x \in \chi\}.$$

The equilibrium conditions are

$$\text{(a)} \quad u_i(x^i) + v_i(p^i) = 0, \qquad i = 1, \ldots, n,$$
$$\langle x^i, p^i\rangle = 0, \qquad i = 1, \ldots, n,$$

and

$$\text{(b)} \quad \lambda_i p^i \in \partial u^i(x^i), \qquad \lambda_i > 0, \qquad i = 1, \ldots, n,$$
$$\nu_i x^i \in \partial v_i(p^i), \qquad \nu_i > 0, \qquad i = 1, \ldots, n,$$
$$u_i(x^i) = \sup_{y^i \in U_i} \{u_i(y^i) \mid \langle p^i, y^i\rangle \leqslant 0\}, \qquad i = 1, \ldots, n.$$

The inequality $\langle x^i, p^i\rangle \leqslant 0$ appearing in the utility transform is called the linking inequality; λ_i is the primal linking variable and ν_i is the dual linking variable. By construction, we have that $v_i(p^i) \leqslant -u_i(0)$ where 0 is the do-nothing strategy.

THEOREM 4. *If games* G *and* H *are well defined, the equilibrium conditions* (*a*) *and* (*b*) *are both necessary and sufficient for an equilibrium for games* G *and* H.

Proof. Let $\bar{x}$ be an equilibrium solution for game G. For any player i, there exists p^i such that $\lambda_i p^i \in \partial u_i(\bar{x}_i)$, $\lambda_i > 0$ and $\langle x^i - \bar{x}^i, p^i\rangle \leqslant 0$ for feasible $X \in \chi$. In particular, let $x = \alpha\bar{x} \in \chi$ and thus, $\langle \bar{x}^i, p^i\rangle = 0$. By Theorem 1,

$$u_i(\bar{x}^i) + v_i(p^i) = 0,$$
$$\nu_i x^i \in \partial v_i(p^i), \qquad \nu_i > 0, \qquad i = 1, \ldots, m$$
$$u_i(x^i) = \sup_{y^i \in U_i} \{u_i(y^i) \mid \langle p^i, y^i\rangle \leqslant 0\}, \qquad i = 1, \ldots, m.$$

Let $\bar{p}$ be an equilibrium solution to game H. Let x satisfy condition (II) of Theorem 2. Since $\bar{p}$ is an equilibrium solution $\langle \bar{p}^i, x^i \rangle = 0$ by a similar argument as previously.

By Theorem 2,

$$u_i(x^i) + v_i(\bar{p}^i) = 0$$

and we have the equilibrium conditions.

We now assume that the equilibrium conditions (a) and (b) are satisfied for $\bar{x}$, $\bar{p}$. Suppose $\bar{x}$ is not an equilibrium for game G. In this case there exists a player i such that $\lambda_i \bar{p}^i \in \partial u_i(\bar{x}^i)$ and an equilibrium point x for game G such that $\langle x^i - \bar{x}^i, \bar{p}^i \rangle > 0$ for some player. By assumption $\langle \bar{x}^i, \bar{p}^i \rangle = 0$ and $\langle x^i, \bar{p}^i \rangle = 0$ from the first part of the proof. Hence we have a contradiction. Thus $\bar{x}$ must be an equilibrium solution to game G.

The fact that $\bar{p}$ is an equilibrium solution to the game H is proved similarly.

COROLLARY. *If the v_i's are differentiable, a dual equilibrium may be found by solving*:

$$x \in \chi, \qquad p \in \chi^*,$$

where

$$p = (p^1, p^2, \dots, p^n)^\tau, \qquad x = (x^1, x^2, \dots, x^n)^\tau,$$

and

$$x_j^i = 1/v_i(p^i)\ \partial v_i(p^i)/\partial p_j^i.$$

3. EXAMPLE

Consider the following primal game.

Game I. The payoff vector is $u(x) = (u_1(x^1), u_2(x^2), \dots, u_n(x^n))$, where $u_i(x^i) = \sum_j \gamma_j^i \log(x_j^i + e_j^i)$ and $\gamma_j^i \geqslant 0$, e_j^i are given parameters, subject to

$$x_j^i \geqslant -e_j^i \qquad \forall i,j$$

and

$$x = (x^1, x^2, \dots, x^n) \in \chi.$$

Let $v_i(p^i)$ be the utility transform of $u_i(x^i)$. Hence

$$v_i(p^i) = \inf_{x_j^i \geqslant -e_j^i} \{-u_i(x^i) \mid \langle p^i, x^i \rangle \leqslant 0\}.$$

The infimum is attained when

$$\lambda_i p_j^i = \gamma_j^i/(x_j^i + e_j^i), \qquad \langle p^i, x^i \rangle = 0$$

or

$$x_j^i = -e_j^i + \gamma_j^i/(\lambda_i p_j^i).$$

From the orthogonality of p^i and x^i, we obtain

$$\lambda_i(p^i) = \sum_j \gamma_j^i \Big/ \Big(\sum_j p_j^i e_j^i\Big)$$

and hence

$$x_j^i = -e_j^i + \Big(\gamma_j^i \Big/ \sum_k \gamma_k^i\Big)\Big(\sum_k p_k^i e_k^i / p_j^i\Big).$$

The utility transform is then

$$v_i(p^i) = \sum_j \gamma_j^i \log \left(\Big(\sum_k \gamma_k^i/\gamma_j^i\Big)\Big(p_j^i \Big/ \sum_k p_k^i e_k^i\Big)\right)$$

and

$$p^i \geqslant 0.$$

The dual game to I is

Game J. The payoff vector is $v(p) = (v_1(p^1), v_2(p^2),..., v_n(p^n))$, where $v_i(p^i)$ is defined as above, subject to

$$p^i \geqslant 0, \qquad \forall i$$

and

$$p = (p^1, p^2,..., p^n) \in \chi^*.$$

It is straightforward to show that

$$\frac{\partial v_i(p^i)}{\partial p_i^i} \Big/ v_i(p^i) = \left(\frac{\gamma_k^i}{\sum_j \gamma_j^i}\right)\left(\frac{\sum_j p_j^i e_j^i}{p_k^i}\right) - e_k^i.$$

By the corollary to Theorem 5, a dual equilibrium may be found by solving

$$x \in \chi, \qquad p \in \chi^*$$

$$x_k^i = \left(\gamma_k^i \Big/ \sum_j \gamma_j^i\right)\left(\sum_j p_j^i e_j^i / p_k^i\right) - e_k^i, \qquad \forall i, k.$$

In the particular case that $\chi = \{x \mid \sum_i x^i = 0\}$, we have that $\chi^* = \{p \mid p^i = p^k, \forall i, k\}$ and the duality equations reduce to

$$\sum_i \left(\gamma_k^i \sum_j \bar{p}_j e_j^i \Big/ \sum_j \gamma_j^i - \bar{p}_k e_k^i\right) = 0, \qquad \forall k$$

which is a set of linear equations in the $\bar{p}_j = p_j^i$, $\forall i$. We may set any one variable to 1 and solve the remaining $m-1$ linear equations. Therefore we can solve an exchange problem involving $m \times n$ (the number of commodities times the number of consumers) by solving $m-1$ linear equations: a significant reduction.

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