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Minimal recurrence relations for connection coefficients between classical orthogonal polynomials: Discrete case[1]

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Abstract

We present a simple approach in order to compute recursively the connection coefficients between two families of classical (discrete) orthogonal polynomials (Charlier, Meixner, Kravchuk, Hahn), i.e., the coefficients $C_m(n)$ in the expression $P_n(x)=\sum_{m=0}^{n} C_m(n)Q_m(x)$, where $\{P_n(x)\}$ and $\{Q_m(x)\}$ belong to the aforementioned class of polynomials. This is done by adapting a general and systematic algorithm, recently developed by the authors, to the discrete classical situation. Moreover, extensions of this method allow to give new addition formulae and to estimate $C_m(n)$-asymptotics in limit relations between some families.

Keywords: Discrete orthogonal polynomials; Connection problems

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1. Introduction

The connection coefficients $C_m(n)$ between two families of orthogonal polynomials $\{P_n(x)\}$ and $\{Q_m(x)\}$ are defined by

$$P_n(x)=\sum_{m=0}^{n} C_m(n)Q_m(x). \tag{1}$$

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In recent papers [3, 10–12, 15] we have developed an algorithm building a recurrence relation in m only for $C_m(n)$ when $\{P_n(x)\}$ and $\{Q_m(x)\}$ belong to a large class of orthogonal polynomials called semi-classical orthogonal polynomials [4, 7], which contains the classical as a particular case. This algorithm has been already applied to classical continuous (Jacobi, Bessel, Laguerre and Hermite) [3, 10, 15] and to classical discrete (Hahn, Meixner, Kravchuk and Charlier) [11, 12, 15]. The algorithm acting on classical discrete can be summarized in the following way (see [10, 11] for details).

Let us assume that we know a linear difference operator $\mathscr{D}_{s,n}$ of order s with polynomial coefficients annihilating the polynomial $P_n(x)$ in Eq. (1), giving

$$\sum_{m=0}^{n} C_m(n)\mathscr{D}_{s,n}[Q_m(x)] = 0.$$

Using properties of classical discrete polynomials the expression $\mathscr{D}_{s,n}[Q_m(x)]$ can be expanded [10, 11] in a linear constant coefficient combination of the family $\{Q_m(x)\}$ which generates a recurrence relation in m for $C_m(n)$.

However, it is also possible to expand $\mathscr{D}_{s,n}[Q_m(x)]$ in other basis instead of $\{Q_m(x)\}$ (like $\{\Delta Q_m(x)\}$ or $\{\Delta\nabla Q_m(x)\}$), by using a so-called "Difference representation" (see Section 2) satisfied by the classical discrete orthogonal polynomials. It turns out that the length of the recurrence for $C_m(n)$ depends strongly on the choice inside these three bases and this property justify the name "MINIMAL" used in this article: The minimal recurrence relation is the shortest one in order given by the algorithm among these three bases.

The structure of the paper is as follows: in Section 2 we give the basic relations which will be useful in the rest of the manuscript. Section 3 deals with the connection problem involving classical discrete orthogonal polynomials. In this section we adapt the method described in [3] in order to find the *minimal* recurrence relations for connection coefficients between two families of classical discrete orthogonal polynomials. Moreover (see Section 3.3), particular examples are considered where the explicit recurrence relations for several connection coefficients are given. Finally, in Section 4 we present some related connection problems about addition formulae and limit relations which can be also treated with our algorithm.

2. Notations and basic properties

Given a polynomial $p(x)$, the forward and backward difference operators are defined as

$$\Delta p(x) = p(x+1) - p(x), \qquad \nabla p(x) = p(x) - p(x-1),$$

respectively. Both operators satisfy the following basic properties:

$$\begin{aligned} &\Delta\nabla = \nabla\Delta, \qquad \Delta = \nabla + \Delta\nabla, \\ &\Delta p(x) = \nabla p(x+1), \qquad \Delta[p(x)q(x)] = q(x)\Delta p(x) + p(x+1)\Delta q(x), \end{aligned} \tag{2}$$

where $p(x)$ and $q(x)$ are two arbitrary polynomials.

Let $\{P_n(x)\}$ be a family of monic classical discrete orthogonal polynomials. Then, they satisfy the orthogonality relation [2, 8, 13]

$$\sum_{x_i=a}^{b-1} P_l(x_i)P_m(x_i)\rho(x_i) = \delta_{lm}d_m^2,$$

where the weight $\rho(x)$ must be [2, 8] a solution of the Pearson-type difference equation:

$$\Delta[\sigma(x)\rho(x)] = \tau(x)\rho(x), \tag{3}$$

$\sigma(x)$ and $\tau(x)$ being polynomials of degree maximum two and one, respectively. The weight $\rho(x)$ (defined on $[a, b-1]$) is characterized by (3) and the following two conditions:

- $\rho(x_i) > 0$ for $a \leqslant x_i \leqslant b-1$ $(x_{i+1} = x_i + 1)$,
- $\sigma(x)\rho(x)x^k|_{x=a,b} = 0$ $(k \geqslant 0)$.

Besides the second-order difference equation [8]:

$$\mathscr{D}_{2,n}[P_n(x)] := \sigma(x)\Delta\nabla P_n(x) + \tau(x)\Delta P_n(x) + \lambda_n P_n(x) = 0, \tag{4}$$
$$(\lambda_n = -n\tau' - \tfrac{1}{2}n(n-1)\sigma''),$$

where $\sigma(x)$ and $\tau(x)$ are the same polynomials as in Eq. (3), the classical discrete orthogonal polynomials $\{P_n(x)\}$ with respect to $\rho(x)$ satisfy a number of properties which in turn provide characterizations of them (see e.g. [2, 8, 13]). We shall need here four of those properties.

First, as any orthogonal polynomial sequence [1], the monic family $\{P_n(x)\}$ verifies a three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + B_nP_n(x) + C_nP_{n-1}(x) \quad (n \geqslant 0), \tag{5}$$
$$P_{-1}(x) = 0; \quad P_0(x) = 1 \quad (C_n \neq 0).$$

Second, one has the so-called forward or Δ-Structure relation (see, e.g., [2, 8, 13])

$$[\sigma(x) + \tau(x)]\Delta P_n(x) = \alpha_nP_{n+1}(x) + \beta_nP_n(x) + \gamma_nP_{n-1}(x) \quad (n \geqslant 0), \tag{6}$$
$$P_{-1}(x) = 0; \quad P_0(x) = 1.$$

Third, the (monic) classical discrete orthogonal polynomials satisfy the corresponding backward or ∇-Structure relation (see, e.g., [8, 13])

$$\sigma(x)\nabla P_n(x) = \tilde{\alpha}_nP_{n+1}(x) + \tilde{\beta}_n(x)P_n(x) + \tilde{\gamma}_nP_{n-1}(x) \quad (n \geqslant 0), \tag{7}$$
$$P_{-1}(x) = 0; \quad P_0(x) = 1.$$

And fourth, there exists the not so well-known Δ-Difference representation [8, 13]

$$P_n(x) = \frac{1}{n+1}\Delta P_{n+1}(x) + F_n\Delta P_n(x) + G_n\Delta P_{n-1}(x) \quad (n \geqslant 0), \tag{8}$$
$$P_{-1}(x) = 0; \quad P_0(x) = 1.$$

Remark 1. It is interesting to notice here that all the x-independent coefficients appearing of the above four structural properties (5)–(8) can be expressed [2, 8, 13] in terms of the polynomials

$\sigma(x)$ and $\tau(x)$ which characterize the Pearson weight equation (3). For the sake of completeness, an appendix has been included at the end of the manuscript giving these expressions. Moreover, the appendix also includes two tables (see Tables 2 and 3) where the specific values of these coefficients are collected for each monic classical discrete orthogonal family.

Concerning notations for the connection problem (1), the family $\{P_n(x)\}$ will satisfy Eqs. (4)–(8), while for the family $\{Q_m(x)\}$ the upper bar notation will be used; i.e., this family will satisfy Eqs. ($\overline{4}$)–($\overline{8}$), with overlined coefficients $(\overline{\sigma}, \overline{\tau}, \overline{\lambda}_m)$, $(\overline{B}_m, \overline{C}_m)$, $(\overline{\alpha}_m, \overline{\beta}_m, \overline{\gamma}_m)$ and $(\overline{F}_m, \overline{G}_m)$, respectively.

Remark 2. As we have already mentioned, throughout the paper the normalization for both $\{P_n(x)\}$ and $\{Q_m(x)\}$ families in (1) will be to consider monic polynomials. Notice that this can be done without loss of generality in what concerns the connection problems. This is so because if other normalizations are considered, say $\widetilde{P}_n(x) = N_n P_n(x)$ and $\widetilde{Q}_m(x) = M_m Q_m(x)$, it is easy to check that the new connection coefficients $\widetilde{C}_m(n)$ between the families $\{\widetilde{P}_n(x)\}$ and $\{\widetilde{Q}_m(x)\}$ are given by $\widetilde{C}_m(n) = N_n^{-1} M_m C_m(n)$.

3. Minimal recurrence relations for connection coefficients between classical discrete orthogonal polynomials

As in [11], the first step to obtain a recurrence relation for the connection coefficients consists in applying the difference operator $\mathscr{D}_{2,n}$, defined in (4), to both sides of the connection problem (1). This gives

$$\mathscr{D}_{2,n}[P_n] = \sum_{m=0}^{n} C_m(n)\{\sigma(x)\Delta\nabla Q_m(x) + \tau(x)\Delta Q_m(x) + \lambda_n Q_m(x)\} = 0. \tag{9}$$

Then, the searched recurrence relation comes out (with a shift of indices) after expanding, in a linear constant coefficients combination of linearly independent polynomials, the expression:

$$S_{m,n}(x) := \sigma(x)\Delta\nabla Q_m(x) + \tau(x)\Delta Q_m(x) + \lambda_n Q_m(x). \tag{10}$$

As pointed out in the introduction, the algorithm developed in [10, 11] chooses the $\{Q_m(x)\}$ family as expanding basis for (10), giving a recurrence of maximum order eight (cf. [11, Table 2], where there is a misprint: the word “order” should be replaced by “number of terms”).

As shown below (see Section 3.1), the minimal recurrence relation (i.e. the shortest one in order) for the connection coefficients in (1) is attached by using the $\{\Delta\nabla Q_m(x)\}$ basis and it is of maximum order four, which strongly decrease when Hahn polynomials are not present in the connection problem (see Table 1 below). Moreover, when the families $\{P_n(x)\}$ and $\{Q_m(x)\}$ in the connection problem (1) satisfy two difference equations (4) and ($\overline{4}$) respectively, with $\sigma(x) = \overline{\sigma}(x)$, then the order of the recurrence (see Section 3.2) is reduced to two using $\{\Delta Q_m(x)\}$ instead of the aforementioned $\{\Delta\nabla Q_m(x)\}$. All of these considerations are summarized in Table 1. Of course, further reductions in the order could appear for some specific values of the parameters.

Let us now describe the algorithms when $\{\Delta\nabla Q_m\}$ and $\{\Delta Q_m\}$ are chosen as expanding basis for expression (10).

Table 1
Order of the "minimal" recurrence relation satisfied by the connection coefficients between two families of classical discrete orthogonal polynomials and the corresponding expanding basis. The column represents the $\{P_n(x)\}$ monic family in Eq. (1) and the row the $\{Q_m(x)\}$ monic family also with the notation of Eq. (1)

$\{Q_m\}$ family $\longrightarrow$ $\{P_n\}$ family $\downarrow$	Charlier $C_n^{(a)}(x)$	Meixner $M_n^{(\delta,\nu)}(x)$	Kravchuk $K_n^{(a)}(x;N)$	Hahn $H_n^{(a,b)}(x;N)$
Charlier $C_n^{(\mu)}(x)$	Order 1 $\{\Delta Q_m\}$	Order 2 $\{\Delta\nabla Q_m\}$	Order 2 $\{\Delta\nabla Q_m\}$	Order 4 $\{\Delta\nabla Q_m\}$
Meixner $M_n^{(\gamma,\mu)}(x)$	Order 2 $\{\Delta\nabla Q_m\}$	Order 2 $\{\Delta Q_m\}$	Order 2 $\{\Delta\nabla Q_m\}$	Order 4 $\{\Delta\nabla Q_m\}$
Kravchuk $K_n^{(\mu)}(x;M)$	Order 2 $\{\Delta\nabla Q_m\}$	Order 2 $\{\Delta\nabla Q_m\}$	Order 2 $\{\Delta Q_m\}$	Order 4 $\{\Delta\nabla Q_m\}$
Hahn $H_n^{(c,d)}(x;L)$	Order 4 $\{\Delta\nabla Q_m\}$	Order 4 $\{\Delta\nabla Q_m\}$	Order 4 $\{\Delta\nabla Q_m\}$	Order 4 $\{\Delta Q_m\}$

3.1. Using the $\{\Delta\nabla Q_m\}$ basis

To consider the basis $\{\Delta\nabla Q_m(x)\}$ one can proceed as follows. First, applying ∇ to ($\overline{8}$) and using (2) one obtains

$$\Delta Q_m(x) = \frac{1}{m+1}\Delta\nabla Q_{m+1}(x) + (\overline{F}_m + 1)\Delta\nabla Q_m(x) + \overline{G}_m\Delta\nabla Q_{m-1}(x).$$

Then, this expression together with ($\overline{8}$) allows to write

$$Q_m(x) = \sum_{j=m-2}^{m+2} a_{m,j}\Delta\nabla Q_j(x), \tag{11}$$

with

$$a_{m,m+2} = \frac{1}{(m+1)(m+2)}, \qquad a_{m,m+1} = \frac{\overline{F}_m + \overline{F}_{m+1} + 1}{m+1},$$
$$a_{m,m} = \overline{F}_m(\overline{F}_m + 1) + \frac{\overline{G}_m}{m} + \frac{\overline{G}_{m+1}}{m+1},$$
$$a_{m,m-1} = \overline{G}_m(\overline{F}_m + \overline{F}_{m-1} + 1), \qquad a_{m,m-2} = \overline{G}_m\overline{G}_{m-1}.$$

Second, from ($\overline{5}$) and ($\overline{8}$) one has

$$\tau(x)\Delta Q_m(x) = \sum_{j=m-2}^{m+2} a_{m,j}^{(1)}\Delta\nabla Q_j(x), \tag{12}$$

with

$$a^{(1)}_{m,m+2} = \frac{m\tau'}{(m+1)(m+2)},$$
$$a^{(1)}_{m,m+1} = \frac{\tau'[m(\overline{F}_{m+1}+1) + \overline{B}_m - \overline{F}_m - 1] + \tau(0)}{m+1},$$
$$a^{(1)}_{m,m} = \tau'\left[\frac{m\overline{G}_{m+1}}{m+1} + \frac{\overline{C}_m - \overline{G}_m}{m}\right] + (\overline{F}_m + 1)[\tau'(\overline{B}_m - \overline{F}_m - 1) + \tau(0)],$$
$$a^{(1)}_{m,m-1} = \tau'[\overline{G}_m(\overline{B}_m - \overline{F}_m - \overline{F}_{m-1}) + \overline{C}_m(\overline{F}_{m-1}+1)] + \tau(0)\overline{G}_m,$$
$$a^{(1)}_{m,m-2} = \tau'(\overline{C}_m - \overline{G}_m)\overline{G}_{m-1}.$$

And third, ($\overline{6}$) and ($\overline{8}$) give

$$\sigma(x)\Delta\nabla Q_m(x) = \sum_{j=m-2}^{m+2} a^{(2)}_{m,j}\Delta\nabla Q_j(x), \tag{13}$$

with

$$a^{(2)}_{m,m+2} = \frac{m(m-1)\sigma''}{2(m+1)(m+2)},$$
$$a^{(2)}_{m,m+1} = \frac{m-1}{m+1}\left\{\frac{\sigma''}{2}[(\overline{B}_{m+1} + \overline{B}_m - 2(1 + \overline{F}_m + \overline{F}_{m+1})] + \sigma'(0)\right\},$$
$$a^{(2)}_{m,m} = \frac{\sigma''}{2}\left[\frac{m-1}{m+1}(\overline{C}_{m+1} - 2\overline{G}_{m+1}) + \frac{m-2}{m}(\overline{C}_m - 2\overline{G}_m) + (\overline{B}_m - 2\overline{F}_m - 1)^2\right]$$
$$+\sigma'(0)(\overline{B}_m - 2\overline{F}_m - 1) + \sigma(0),$$
$$a^{(2)}_{m,m-1} = (\overline{C}_m - 2\overline{G}_m)\left[\frac{\sigma''}{2}[\overline{B}_m + \overline{B}_{m-1} - 2(\overline{F}_m + \overline{F}_{m-1} + 1)] + \sigma'(0)\right],$$
$$a^{(2)}_{m,m-2} = \frac{\sigma''}{2}(\overline{C}_m - 2\overline{G}_m)(\overline{C}_{m-1} - 2\overline{G}_{m-1}).$$

Now, inserting (11)–(13) in Eq. (9) one obtains

$$\sum_{m=0}^{n} C_m(n)\left\{\sum_{j=m-2}^{m+2} \Omega_{m,j}(n)\Delta\nabla Q_j(x)\right\} = 0,$$
$$\Omega_{m,j}(n) := a^{(2)}_{m,j} + a^{(1)}_{m,j} + \lambda_n a_{m,j}.$$

Finally, after an appropriate shift of indices, this latter expression provides the backward "minimal" recurrence relation of order four which can be written as

$$\sum_{s=0}^{4} \Omega_{m+s,m+2}(n)C_{m+s}(n) = 0, \quad 0 \leqslant m \leqslant n-1, \tag{14}$$

being the initial conditions given by $C_{n+s}(n) = 0$ $(s = 1,2,3)$ and $C_n(n) = 1$, since monic polynomials have been considered.

3.2. Using the $\{\Delta Q_m(x)\}$ basis

When the difference equations (4) and ($\bar{4}$) satisfied by the polynomials $P_n(x)$ and $Q_m(x)$ in (1), respectively, are such that $\sigma(x) = \bar{\sigma}(x)$, then the minimal recurrence relation for the connection coefficients appears when $\{\Delta Q_m(x)\}$ is the expanding basis for expression (10). For this reason, this basis should be used when dealing with the Charlier–Charlier, Meixner–Meixner, Kravchuk–Kra-wtchouck and Hahn–Hahn connnection problems (see Table 1).

The algorithm in this case is as follows. First, the difference equation ($\bar{4}$) gives

$$\sigma(x)\Delta\nabla Q_m(x) = \bar{\sigma}(x)\Delta\nabla Q_m(x) = -\bar{\tau}(x)\Delta Q_m(x) - \bar{\lambda}_m Q_m(x).$$

So, Eq. (10) can be rewritten as

$$\sum_{m=0}^{n} C_m(n)[(\tau(x) - \bar{\tau}(x))\Delta Q_m(x) + (\lambda_n - \bar{\lambda}_m)Q_m(x)] = 0. \tag{15}$$

To expand this expression in the $\{\Delta Q_m(x)\}$ basis one has: first, from the difference representation ($\bar{8}$),

$$(\lambda_n - \bar{\lambda}_m)Q_m(x) = \sum_{j=m-1}^{m+1} b_{m,j}\Delta Q_m(x), \tag{16}$$

with

$$b_{m,m+1} = \frac{\lambda_n - \bar{\lambda}_m}{m+1}, \qquad b_{m,m} = (\lambda_n - \bar{\lambda}_m)\bar{F}_m, \qquad b_{m,m-1} = (\lambda_n - \bar{\lambda}_m)\bar{G}_m.$$

And second, use of ($\bar{5}$) and ($\bar{8}$) gives

$$[\tau(x) - \bar{\tau}(x)]\Delta Q_m(x) = \sum_{j=m-1}^{m+1} b^{(1)}_{m,j}\Delta Q_m(x), \tag{17}$$

with

$$b^{(1)}_{m,m+1} = \frac{m}{m+1}(\tau' - \bar{\tau}'),$$
$$b^{(1)}_{m,m} = (\tau' - \bar{\tau}')(\bar{B}_m - \bar{F}_m - 1) + (\tau(0) - \bar{\tau}(0)),$$
$$b^{(1)}_{m,m-1} = (\tau' - \bar{\tau}')(\bar{C}_m - \bar{G}_m).$$

Then, inserting (16) and (17) into (15) we obtain

$$\sum_{m=0}^{n} C_m(n)\left\{\sum_{j=m-1}^{m+1} \Lambda_{m,j}(n)\Delta Q_m(x)\right\} = 0, \quad \Lambda_{m,j}(n) = b_{m,j} + b^{(1)}_{m,j}.$$

Finally, after an appropriate shift of indices, this latter expression provides the backward "minimal" recurrence relation of order two which can be written as

$$\sum_{s=-1}^{1} \Lambda_{m+s,m}(n)C_{m+s}(n) = 0, \quad 1 \leqslant m \leqslant n, \tag{18}$$

Table 2
Data for monic Charlier, Meixner and Kravchuk polynomials

	Charlier $C_n^{(\mu)}(x)$ $(\mu > 0)$	Meixner $M_n^{(\gamma,\mu)}(x)$ $(\gamma > 0,\ \mu \in (0,1))$	Kravchuk $K_n^{(a)}(x;N)$, $n \leqslant N$ $(a \in (0,1),\ N \in \mathbb{Z}^+)$
$\sigma(x)$	x	x	x
$\tau(x)$	$\mu - x$	$\gamma\mu - x(1-\mu)$	$\dfrac{Na - x}{1-a}$
λ_n	n	$n(1-\mu)$	$\dfrac{n}{1-a}$
B_n	$n+\mu$	$\dfrac{\gamma\mu + n(1+\mu)}{1-\mu}$	$n + a(N-2n)$
C_n	μn	$\dfrac{\mu n(\gamma+n-1)}{(1-\mu)^2}$	$an(1-a)(N-n+1)$
α_n	0	0	0
β_n	0	μn	$\dfrac{an}{a-1}$
γ_n	μn	$\dfrac{\mu n(\gamma+n-1)}{1-\mu}$	$an(1-n+N)$
F_n	0	$\dfrac{\mu}{1-\mu}$	$-a$
G_n	0	0	0

being the initial conditions given by $C_{n+1}(n)=0$ and $C_n(n)=1$, since monic polynomials have been considered.

3.3. Examples

As illustration of the recurrences which these algorithms provide, we consider now several examples giving for each of them, the concrete connection problem, the minimal recurrence relation for the corresponding connection coefficients (together with the initial conditions) and, sometimes, the solution of this recurrence. The expanding basis for expression (10) used in each example is the one listed in Table 1. Notations for discrete classical orthogonal polynomials are described in Tables 2 and 3 (see the appendix).

For a different approach (also recursive) to compute these connection coefficients see [6].

1. *Charlier–Charlier*

- Connection problem: $C_n^{(a)}(x) = \sum_{m=0}^{n} C_m(n) C_m^{(b)}(x)$.
- Minimal recurrence relation (of order one)

$$(a-b)mC_m(n) + (n-m+1)C_{m-1}(n) = 0 \quad (1 \leqslant m \leqslant n).$$

- Initial condition: $C_n(n) = 1$.
- Solution: $C_m(n) = \binom{n}{m}(b-a)^{n-m}$ $(0 \leqslant m \leqslant n)$.

This expression coincide with the one obtained in [11, Section 5.1], where the family $\{Q_m(x)\}$ was used as expanding basis for expression (10). On the other hand, notice that these $C_m(n)$-coefficients are always positive when $n-m$ is an even number. However, if $n-m$ is odd, the additional condition $b>a$ must hold.

Table 3
Data for monic Hahn polynomials

	Hahn $H_n^{(a,b)}(x;N)$, $n < N$ $a,b > -1$, $N \in \mathbb{Z}^+$
$\sigma(x)$	$x(N+a-x)$
$\tau(x)$	$(b+1)(N-1)-(a+b+2)x$
λ_n	$n(a+b+n+1)$
B_n	$\dfrac{a-b+2N-2}{4}+\dfrac{(b^2-a^2)(a+b+2N)}{4(a+b+2n)(a+b+2n+2)}$
C_n	$\dfrac{n(N-n)(a+n)(b+n)(a+b+n)(a+b+N+n)}{(a+b+2n-1)(a+b+2n)^2(a+b+2n+1)}$
α_n	$-n$
β_n	$\dfrac{n(1+a+b+n)(N(a-b)-2n(a+b+n+1)-(a+b+ab+b^2))}{(a+b+2n)(2+a+b+2n)}$
γ_n	$\dfrac{n(a+n)(b+n)(N-n)(a+b+n+N)(a+b+n)(a+b+n+1)}{(-1+a+b+2n)(a+b+2n)^2(1+a+b+2n)}$
F_n	$\dfrac{N(a-b)-a-b-ab-b^2-2n(1+a+b+n)}{(a+b+2n)(2+a+b+2n)}$
G_n	$\dfrac{n(a+n)(b+n)(-N+n)(a+b+N+n)}{(-1+a+b+2n)(a+b+2n)^2(1+a+b+2n)}$

2. *Meixner–Charlier*
- Connection problem: $M_n^{(\gamma,\mu)}(x) = \sum_{m=0}^{n} C_m(n) C_m^{(a)}(x)$.
- Minimal recurrence relation (of order two)

$$(1-\mu)(1-m+n)C_{m-1}(n) + m(\mu(a+\gamma+m-1)-a)C_m(n) \\ +am\mu(m+1)C_{m+1}(n) = 0 \quad (1 \leqslant m \leqslant n). \tag{19}$$

- Initial conditions: $C_{n+1}(n) = 0$ and $C_n(n) = 1$.

3. *Kravchuk–Charlier*
- Connection problem: $K_n^{(a)}(x;N) = \sum_{m=0}^{n} C_m(n) C_m^{(\mu)}(x)$.
- Minimal recurrence relation (of order two)

$$(m-n-1)C_{m-1}(n) + m(\mu + a(m-N-1))C_m(n) \\ +am\mu(m+1)C_{m+1}(n) = 0 \quad (1 \leqslant m \leqslant n). \tag{20}$$

- Initial conditions: $C_{n+1}(n) = 0$ and $C_n(n) = 1$.

4. *Charlier–Meixner*
- Connection problem: $C_n^{(a)}(x) = \sum_{m=0}^{n} C_m(n) M_m^{(\gamma,\mu)}(x)$.

- Minimal recurrence relation (of order two)

$$m(m+1)(\gamma+m)\mu^2 C_{m+1}(n)$$
$$+m(1-\mu)(\mu(2m-n+\gamma+a-1)-a)C_m(n)$$
$$+(1-\mu)^2(m-n-1)C_{m-1}(n)=0 \quad (1\leqslant m\leqslant n).$$

- Initial conditions: $C_{n+1}(n)=0$ and $C_n(n)=1$.

5. *Meixner–Meixner*

- Connection problem: $M_n^{(\gamma,\mu)}(x)=\sum_{m=0}^{n} C_m(n) M_m^{(\delta,\nu)}(x)$.
- Minimal recurrence relation (of order two)

$$(1-\mu)(n-m+1)(\nu-1)^2 C_{m-1}(n)$$
$$+m(\nu-1)((2m-n+\delta-1)\nu$$
$$+\mu(1-\gamma-m+(\gamma-\delta-m+n)\nu))C_m(n)$$
$$+m(m+1)(\delta+m)(\mu-\nu)\nu C_{m+1}(n)=0 \quad (1\leqslant m\leqslant n). \tag{21}$$

- Initial conditions: $C_{n+1}(n)=0$ and $C_n(n)=1$.

If $\mu=\nu$, Eq. (21) reduces to a first-order recurrence relation, whose solution is

$$C_m(n)=\binom{n}{m}\left(\frac{\mu}{1-\mu}\right)^{n-m}(\gamma-\delta)_{n-m} \quad (0\leqslant m\leqslant n).$$

In case $\gamma=\delta$ the result is

$$C_m(n)=\binom{n}{m}\frac{\Gamma(n+\delta)}{\Gamma(m+\delta)}\left(\frac{\nu-\mu}{(1-\mu)(1-\nu)}\right)^{n-m} \quad (0\leqslant m\leqslant n).$$

For $\gamma=\delta=1$ (discrete Laguerre polynomials), this expression is also given in [11, Section 5.2] where there is a missprint: the first factor $(n-m)$ should be $(n-m)!$.

6. *Kravchuk–Meixner*

- Connection problem: $K_n^{(a)}(x;N)=\sum_{m=0}^{n} C_m(n) M_m^{(\gamma,\mu)}(x)$.
- Minimal recurrence relation (of order two)

$$(\mu-1)^2(m-n-1)C_{m-1}(n)$$
$$+m(\mu-1)(\mu(1-\gamma-2m+n)+a(N-m+1)(1-\mu))C_m(n)$$
$$+m(m+1)(\gamma+m)\mu(a+\mu(1-a))C_{m+1}(n)=0 \quad (1\leqslant m\leqslant n). \tag{22}$$

- Initial conditions: $C_{n+1}(n)=0$ and $C_n(n)=1$.

Remark 3. The well-known relation between Kravchuk and Meixner polynomials can be written as

$$K_n^{(a)}(x;N)=M_n^{(-N,a/(a-1))}(x). \tag{23}$$

This choice of Meixner parameters reduces the recurrence relation (22) to a first-order one, now independent on N:

$$am(n-m)C_m(n)+(m-n-1)C_{m-1}(n)=0,$$

which gives $C_m(n)=0$ for all $0\leqslant m\leqslant n-1$ in accordance with Eq. (23).

7. *Charlier–Kravchuk*
- Connection problem: $C_n^{(\mu)}(x)=\sum_{m=0}^{n} C_m(n)K_m^{(a)}(x;N)$.
- Minimal recurrence relation (of order two)

$$(n-m+1)C_{m-1}(n)+m(\mu+a(2m-n-N-1))C_m(n)$$
$$+a^2m(m+1)(N-m)C_{m+1}(n)=0 \quad (1\leqslant m\leqslant n).$$

- Initial conditions: $C_{n+1}(n)=0$ and $C_n(n)=1$.
- Solution:

$$C_m(n)=\binom{n}{m}(N-n+1)_{n-m}\sum_{j=0}^{n-m}(-1)^j\binom{n-m}{j}\frac{\mu^j a^{n-m-j}}{(N-n+1)_j}.$$

8. *Meixner–Kravchuk*
- Connection problem: $M_n^{(\gamma,\mu)}(x)=\sum_{m=0}^{n} C_m(n)K_m^{(a)}(x;N)$.
- Minimal recurrence relation (of order two)

$$(1-\mu)(n-m+1)C_{m-1}(n)$$
$$+m(\mu(m+\gamma-1)+a(\mu-1)(1-2m+n+N))C_m(n)$$
$$+am(m+1)(a+\mu(1-a))(N-m)C_{m+1}(n)=0 \quad (1\leqslant m\leqslant n). \tag{24}$$

- Initial conditions: $C_{n+1}(n)=0$ and $C_n(n)=1$.

Remark 4. Now, the relation between Kravchuk and Meixner polynomials can be written in the form

$$M_n^{(\gamma,\mu)}(x)=K_n^{(\mu/(\mu-1))}(x;-\gamma). \tag{25}$$

This choice of Kravchuk parameters reduces Eq. (24) to a first-order recurrence relation independent of γ:

$$m\mu(n-m)C_m(n)+(\mu-1)(m-n-1)C_{m-1}(n)=0$$

and gives again $C_m(n)=0$ for $0\leqslant m\leqslant n-1$, in accordance with (25).

9. *Kravchuk–Kravchuk*
- Connection problem: $K_n^{(a)}(x;N)=\sum_{m=0}^{n}C_m(n)K_m^{(b)}(x;M)$.
- Minimal recurrence relation (of order two)

$$\begin{aligned}&(m-n-1)C_{m-1}(n)\\&\quad+m(b(1-2m+M+n)+a(m-N-1))C_m(n)\\&\quad+b(b-a)m(m+1)(m-M)C_{m+1}(n)=0\quad(1\leqslant m\leqslant n).\end{aligned}\tag{26}$$

- Initial conditions: $C_{n+1}(n)=0$ and $C_n(n)=1$.

In case $N=M$ the solution of (26) is

$$C_m(n)=\binom{n}{m}(b-a)^{n-m}(N-n+1)_{n-m},\quad 0\leqslant m\leqslant n.$$

This formula agrees with the one obtained in [11], where the family $\{Q_m(x)\}$ was used as expanding basis for (10) giving rise to a third-order recurrence relation for the connection coefficients (see [11, Section 5.3]).

In case $a=b$ the solution of (26) gives the (new) result

$$C_m(n)=\binom{n}{m}a^{n-m}(M-N)_{n-m}\quad(0\leqslant m\leqslant n).$$

Remark 5. Connection problems involving Hahn (or Hahn–Eberlein) discrete polynomials could be also considered in a similar way as in the above examples. In fact, the corresponding minimal recurrences for the connection coefficients can be obtained by inserting the data of Hahn polynomials (see Appendix, Table 3) in Eqs. (14) or (18). However, these recurrences are (in general) lengthy. In spite of this, they can be handled and solved in some simple cases (see e.g. [11, 15]) by using Mathematica [14] computer algebra system. More results in this direction are now under investigation.

4. Two related connection problems

As pointed out in the introduction, this section deals with two kinds of problems close related with connection problems, namely, the computation of addition formulas and the obtention of asymptotics for connection coefficients in limit relations between classical discrete orthogonal polynomials.

4.1. Addition formulas

Let us consider an addition formula, i.e. an expression of type

$$P_n(x+y)=\sum_{m=0}^{n}C_m(n;y)P_m(x),\tag{27}$$

where y is a x-independent parameter and $\{P_n(x)\}$ is a classical discrete orthogonal polynomial family. Hence, its members are solutions of (4) and then, the polynomial $P_n(x+y)$ satisfy the corresponding difference equation, i.e.

$$\mathscr{D}_{2,n,y}[P_n(x+y)] := \sigma(x+y)\Delta\nabla P_n(x+y) + \tau(x+y)\Delta P_n(x+y) + \lambda_n P_n(x+y) = 0.$$

On applying this $\mathscr{D}_{2,n,y}$-operator to (27) one obtains

$$\sum_{m=0}^{n} C_m(n;y)[\sigma(x+y)\Delta\nabla P_m(x) + \tau(x+y)\Delta P_m(x) + \lambda_n P_m(x)] = 0,$$

which, taking into account that $\sigma(x+y) = \sigma(x) + y\sigma'(x) + \frac{y^2}{2}\sigma''$ and $\tau(x+y) = \tau(x) + y\tau'$, can be written as follows:

$$\sum_{m=0}^{n} C_m(n;y)\left[y\sigma'' x\Delta\nabla P_m(x) + \left(y\sigma'(0) + \frac{y^2}{2}\sigma''\right)\Delta\nabla P_m(x) + y\tau'\Delta P_m(x) + (\lambda_n - \lambda_m)P_m(x)\right] = 0.$$

So, we are now able to expand this latter expression in a linear constant coefficients combination of one of the three families $\{P_m(x)\}$, $\{\Delta P_m(x)\}$ or $\{\Delta\nabla P_m(x)\}$ giving three different recurrences for the connection coefficients. Since in Eq. (27) one always has $\sigma(x+y) \neq \sigma(x)$, the minimal one is reached by using the $\{\Delta\nabla P_m(x)\}$ as the expanding basis.

As illustration let us consider some examples.

1. *Charlier polynomials*

- Addition formula: $C_n^{(\mu)}(x+y) = \sum_{m=0}^{n} C_m(n;y) C_m^{(\mu)}(x)$.
- Minimal recurrence relation (of first order)

$$(m-1)(n-m-y+1)C_{m-1}(n;y) + (n-m+2)C_{m-2}(n;y) = 0, \quad (1 \leqslant m \leqslant n).$$

- Initial conditions: $C_n(n;y) = 1$.
- Solution: $C_m(n;y) = \binom{n}{m}(-1)^{n-m}(-y)_{n-m}$ $(0 \leqslant m \leqslant n)$.

The same expression was given in [11] from a second-order recurrence obtained by using $\{P_m(x)\}$ as expanding basis.

2. *Meixner polynomials*

- Addition formula: $M_n^{(\gamma,\mu)}(x+y) = \sum_{m=0}^{n} C_m(n;y) M_m^{(\gamma,\mu)}(x)$.
- Minimal recurrence relation (of second order)

$$\mu(m-1)m(m-n)C_m(n;y) + (\mu-1)(m-1)((1+\mu)(n-m+y+1)-2y)C_{m-1}(n;y) + (\mu-1)^2(m-n-2)C_{m-2}(n;y) = 0 \quad (1 \leqslant m \leqslant n).$$

• Initial conditions: $C_{n+1}(n;y)=0$ and $C_n(n;y)=1$.
Notice that the recurrence and the initial conditions being independent of γ the connection coefficients $C_m(n;y)$ too.

3. *Kravchuk polynomials*

• Addition formula: $K_n^{(\mu)}(x+y;N)=\sum_{m=0}^{n} C_m(n;y)K_m^{(\mu)}(x;N)$.

• Minimal recurrence relation (of second order)

$$\begin{aligned}&\mu(\mu-1)m(m-1)(m-n)C_m(n;y)\\&\quad+(m-1)((m-n-1)(1-2\mu)+y)C_{m-1}(n;y)\\&\quad+(m-n-2)C_{m-2}(n;y)=0\quad(1\leqslant m\leqslant n).\end{aligned}$$

• Initial conditions: $C_{n+1}(n;y)=0$ and $C_n(n;y)=1$.
Similarly to what happens in Meixner case, the connection coefficients do not depend on the parameter N.

4.2. $C_m(n)$-asymptotics in limit relations

In some situations a polynomial family $P_n(x,p_i)$ tends to another family $Q_m(x,q_j)$ when the parameters p_i and q_j tends to particular limiting values (p_i and q_j being related). From the recurrence relation for coefficients in the connection problem:

$$P_n(x,p_i)=\sum_{m=0}^{n} C_m(n,p_i,q_j)\,Q_m(x,q_j),$$

it is possible to estimate the asymptotic behaviour of $C_m(n,p_i,q_j)$ in the limiting parameter.

The algorithmic part is the same as in Section 3. The asymptotic behaviour is afterward computed from the recurrence relation given by the algorithm. This is illustrated by means of the following two examples.

4.2.1. Kravchuk to Charlier

The limit property between these two discrete polynomial families is [5]

$$\lim_{N\to\infty} K_n^{(a/N)}(x;N)=C_n^{(a)}(x).\tag{28}$$

This suggests to consider the connection problem

$$K_n^{(a/N)}(x;N)=\sum_{m=0}^{n} C_m(n)\,C_n^{(a)}(x).$$

Then, Eq. (20) with the appropriate parameters gives

$$\begin{aligned}&a^2m(m+1)C_{m+1}(n)+a(m-1)mC_m(n)\\&\quad+(m-n-1)NC_{m-1}(n)=0,\end{aligned}\tag{29}$$

with initial conditions $C_{n+1}(n)=0$ and $C_n(n)=1$.

Of course, when N goes to infinity $C_{n-1}(n)=an(n-1)/N$ goes to zero and from Eq. (29) all $C_m(n)$ $(0\leqslant m\leqslant n-1)$ go also to zero. The interest of this development is to give explicitly the behaviour in $(1/N)^k$ in limit property (28). The expression of the first few connection coefficients is

$$C_{n-2}(n)=\frac{a^2n(n-1)((n-1)(n-2)+N)}{2\,N^2},$$

$$C_{n-3}(n)=\frac{a^3n(n-1)(n-2)}{6\,N^3}((n-1)(n-2)(n-3)+N(3n-5)),$$

and the behaviour in the general case is

$$C_m(n)\sim\binom{n}{m}a^{n-m}\frac{1}{N^{[(n-m+1)/2]}}B_{K-C}(n,m),$$

where $B_{K-C}(n,m)$ can be computed recursively, and the symbol $\sim$ is used in the sense of [9, p.4].

4.2.2. Meixner to Charlier

The limit property [5] is now $\lim_{\gamma\to\infty}M_n^{(\gamma,a/(a+\gamma))}(x)=C_n^{(a)}(x)$, and for the connection problem

$$M_n^{(\gamma,a/(a+\gamma))}(x)=\sum_{m=0}^{n}C_m(n)\;C_n^{(a)}(x),$$

the recurrence relation (19) reduces to

$$a^2m(1+m)C_{m+1}(n)+am(m-1)C_m(n)+\gamma(1-m+n)C_{m-1}(n)=0,$$

with $C_{n+1}(n)=0$ and $C_n(n)=1$. The general behaviour is now

$$C_m(n)\sim\binom{n}{m}a^{n-m}\frac{1}{\gamma^{[(n-m+1)/2]}}B_{M-C}(n,m),$$

where $B_{M-C}(n,m)$ can be computed recursively.

Remark 6. As already mentioned in Remark 5 for connection problems of the form (1), addition formulas and asymptotics for $C_m(n)$ in limit relations including Hahn (or Hahn–Eberlein) discrete polynomials involve lengthy computations. Nevertheless, they could be also handled with our algorithm and the help of Mathematica [14]. Results in this direction are in progress.

Appendix. Data for monic classical discrete orthogonal polynomials

With the notation

$$\eta_n=-\frac{\tau(0)+n\sigma'(0)-n^2\sigma''/2}{\tau'+n\sigma''},$$

the expressions of the coefficients appearing in Eqs. (5)–(8) in terms of the polynomials σ and τ characterizing the Pearson weight equation (see Remark 1) are as follows:

- Coefficients of the three-term recurrence relation (5).

$$B_n = \frac{-\tau(0)(\tau' - \sigma'') + n(\tau' - 2\sigma'(0) - 2\tau')(\tau' + \sigma''(n-1)/2)}{(\sigma''(n-1) + \tau')(\sigma'' n + \tau')},$$

$$C_n = -\frac{n(\tau' + \frac{\sigma''}{2}(n-2))}{(\tau' + \sigma''(2n-3)/2)(\tau' + \sigma''(2n-1)/2)}[\sigma(\eta_{n-1}) + \tau(\eta_{n-1})].$$

- Coefficients of the Δ-structure relation (6).

$$\alpha_n = \frac{n\sigma''}{2}, \qquad \beta_n = -\frac{1}{2}\left[\tau(B_n) - n\left(\tau' + \frac{\sigma''}{2}(n-1)\right)\right],$$

$$\gamma_n = -\left(\tau' + \frac{\sigma''}{2}(n-1)\right) C_n.$$

- Coefficients of the ∇-structure relation (7).

$$\widetilde{\alpha}_n = \alpha_n, \quad \widetilde{\beta}_n = \beta_n + \lambda_n, \quad \widetilde{\gamma}_n = \gamma_n,$$

being λ_n the difference equation coefficient given in (4).

- Coefficients of the Δ-difference representation (8).

$$F_n = \frac{\tau(B_n)}{2n(\tau' + \sigma''(n-1)/2)} - \frac{1}{2}, \qquad G_n = -\frac{C_n\,\sigma''}{2\tau' + \sigma''(n-2)}.$$

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