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An index theorem for Toeplitz operators on odd-dimensional manifolds with boundary

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Abstract

We establish an index theorem for Toeplitz operators on odd-dimensional spin manifolds with boundary. It may be thought of as an odd-dimensional analogue of the Atiyah–Patodi–Singer index theorem for Dirac operators on manifolds with boundary. In particular, there occurs naturally an invariant of η type associated to K^1 representatives on even-dimensional manifolds, which should be of independent interests. For example, it gives an intrinsic interpretation of the so called Wess–Zumino term in the WZW theory in physics.

Keywords: Toeplitz operators; Odd-dimensional manifolds; Index theorem; Eta type invariant

1. Introduction

On an even-dimensional smooth closed spin Riemannian manifold M, let $S(TM)$ be the corresponding bundle of spinors over M and E be a Hermitian vector bundle over M equipped with a Hermitian connection. The (twisted) Dirac operator $D^E : \Gamma(S(TM) \otimes E) \to \Gamma(S(TM) \otimes E)$ is elliptic and self-adjoint. Since $\dim M$ is even, the spinors split:

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$$S(TM) \otimes E = S^+(TM) \otimes E \oplus S^-(TM) \otimes E,$$

in terms of which the Dirac operator is off diagonal:

$$D^E = \begin{pmatrix} 0 & D_-^E \\ D_+^E & 0 \end{pmatrix}.$$

The Atiyah–Singer index theorem expresses the index of D_+^E in terms of the characteristic numbers:

$$\operatorname{ind} D_+^E = \langle \widehat{A}(TM)\operatorname{ch}(E), [M] \rangle,$$

where $\widehat{A}(TM)$ is the Hirzebruch $\widehat{A}$-class of TM, $\operatorname{ch}(E)$ is the Chern character of E (cf. [16, Chapter 1]).

Now let M be an *odd*-dimensional smooth closed spin Riemannian manifold. Any elliptic differential operator on M will have index zero. In this case, the appropriate index to consider is that of Toeplitz operators. It also fits perfectly with the interpretation of the index of Dirac operator on even-dimensional manifolds as a pairing between the even K-group and K-homology. Thus in the odd-dimensional case one considers the odd K-group and odd K-homology. An element of $K^{-1}(M)$ can be represented by a differentiable map from M into the general linear group

$$g : M \to GL(N, \mathbf{C}),$$

where N is a positive integer. As we mentioned the appropriate index pairing between the odd K-group and K-homology is given by that of the Toeplitz operator, defined as follows.

First of all, $L^2(S(TM) \otimes E)$, the natural L^2-completion of $\Gamma(S(TM) \otimes E)$, splits into an orthogonal direct sum as

$$L^2(S(TM) \otimes E) = \bigoplus_{\lambda \in \operatorname{Spec}(D^E)} E_\lambda,$$

where E_λ is the eigenspace associated to the eigenvalue λ of D^E. Set

$$L_+^2(S(TM) \otimes E) = \bigoplus_{\lambda \geqslant 0} E_\lambda,$$

and denote by $P_{\geqslant 0}^E$ the orthogonal projection from $L^2(S(TM) \otimes E)$ to $L_+^2(S(TM) \otimes E)$.

Now consider the trivial vector bundle $\mathbf{C}^N$ over M. We equip $\mathbf{C}^N$ with the canonical trivial metric and connection. Then $P_{\geqslant 0}^E$ extends naturally to an orthogonal projection from $L^2(S(TM) \otimes E \otimes \mathbf{C}^N)$ to $L_+^2(S(TM) \otimes E \otimes \mathbf{C}^N)$ by acting as identity on $\mathbf{C}^N$. We still denote this extension by $P_{\geqslant 0}^E$.

The map g can be interpreted as an automorphism of the trivial complex vector bundle $\mathbf{C}^N$. Moreover, g extends naturally to an action on $L^2(S(TM) \otimes E \otimes \mathbf{C}^N)$ by acting as identity on $L^2(S(TM) \otimes E)$. We still denote this extended action by g.

With the above data given, the Toeplitz operator T_g^E can be defined as

$$T_g^E = P_{\geqslant 0}^E g P_{\geqslant 0}^E : L_+^2(S(TM) \otimes E \otimes \mathbf{C}^N) \to L_+^2(S(TM) \otimes E \otimes \mathbf{C}^N).$$

The first important fact is that T_g^E is a Fredholm operator. Moreover, it is equivalent to an elliptic pseudodifferential operator of order zero. Thus one can compute its index by using the Atiyah–Singer index theorem [1], as was indicated in the paper of Baum, Douglas [4]:

$$\operatorname{ind} T_g^E = -\big\langle \widehat{A}(TM)\operatorname{ch}(E)\operatorname{ch}(g), [M]\big\rangle, \tag{1.1}$$

where $\operatorname{ch}(g)$ is the odd Chern character associated to g (cf. [16, Chapter 1]).

There is also an analytic proof of (1.1) by using heat kernels. For this one first note that by a simple deformation, one may well assume that g is unitary. Then a result of Booss and Wojciechowski (cf. [6]) shows that the computation of $\operatorname{ind} T_g^E$ is equivalent to the computation of the spectral flow of the linear family of self-adjoint elliptic operators, acting of $\Gamma(S(TM) \otimes E \otimes \mathbf{C}^N)$, which connects D^E and gD^Eg^{-1}. The resulting spectral flow can then be computed by variations of η-invariants cf. [2,5], where the heat kernels are naturally involved. These ideas have been extended in [8] to give a heat kernel proof of a family extension of (1.1).

The purpose of this paper is to establish a generalization of (1.1) to the case where M is a spin manifold with boundary ∂M, by extending the above heat kernel proof strategy. We wish to point out that when $g|_{\partial M}$ is the identity, such a generalization can be reduced easily to a result of Douglas and Wojciechowski [9]. Thus the main concern for us in this paper will be the case where $g|_{\partial M}$ is not the identity.

A full statement of our main result will be given in Section 2 (Theorem 2.3). Here we only point out that our formula may be viewed as an odd-dimensional analogue of the Atiyah–Patodi–Singer index theorem [2] for Dirac operators on even-dimensional manifolds with boundary. In particular, a very interesting invariant of η-type for even-dimensional manifolds and K^1 representatives appears in our formula, which plays a role similar to that played by the η-invariant term in the Atiyah–Patodi–Singer index theorem. There is also an interesting new integer term here, a triple Maslov index introduced in [12].

This paper is organized as follows. In Section 2, we introduce the notations and state the main result of this paper. In Section 3, we introduce a perturbation to overcome a technical difficulty and prove an index formula for the perturbed Toeplitz operator. In Section 4, we compare the index of Toeplitz operator and that of the perturbed one and prove our main result. In Section 5 we discuss some generalizations of the main result proved in Sections 3, 4, including the basic properties of the η-type invariant mentioned above. We also include Appendix A in which we outline a new proof of (1.1) by a simple use of the Atiyah–Patodi–Singer index theorem.

2. An index theorem for Toeplitz operators on manifolds with boundary

In this section, we state the main result of this paper, which extends (1.1) to manifolds with boundary.

This section is organized as follows. In Section 2.1, we present our basic geometric data and define the Toeplitz operators on manifolds with boundary. In Section 2.2, we define an η-type invariant for K^1 representatives on even-dimensional manifolds, which will appear in the statement of the main result. In Section 2.3, we state the main result of this paper, the proof of which will be presented in the next two sections.

2.1. Toeplitz operators on manifolds with boundary

Let M be an odd-dimensional oriented spin manifold with boundary ∂M. We assume that M carries a fixed spin structure. Then ∂M carries the canonically induced orientation and spin structure. Let g^{TM} be a Riemannian metric on TM such that it is of product structure near the boundary ∂M. That is, there is a tubular neighborhood, which, without loss of generality, can be taken to be $[0,1)\times\partial M\subset M$ with $\partial M=\{0\}\times\partial M$ such that

$$g^{TM}\big|_{[0,1)\times\partial M}=dx^2\oplus g^{T\partial M}, \tag{2.1}$$

where $x\in[0,1)$ is the geodesic distance to ∂M and $g^{T\partial M}$ is the restriction of g^{TM} on ∂M. Let ∇^{TM} be the Levi-Civita connection of g^{TM}. Let $S(TM)$ be the Hermitian bundle of spinors associated to (M,g^{TM}). Then ∇^{TM} extends naturally to a Hermitian connection $\nabla^{S(TM)}$ on $S(TM)$.

Let E be a Hermitian vector bundle over M. Let ∇^E be a Hermitian connection on E. We assume that the Hermitian metric g^E on E and connection ∇^E are of product structure over $[0,1)\times\partial M$. That is, if we denote $\pi:[0,1)\times\partial M\to\partial M$ the natural projection, then

$$g^E\big|_{[0,1)\times\partial M}=\pi^*\big(g^E\big|_{\partial M}\big),\qquad \nabla^E\big|_{[0,1)\times\partial M}=\pi^*\big(\nabla^E\big|_{\partial M}\big). \tag{2.2}$$

For any $X\in TM$, we extend the Clifford action $c(X)$ of X on $S(TM)$ to an action on $S(TM)\otimes E$ by acting as identity on E, and still denote this extended action by $c(X)$. Let $\nabla^{S(TM)\otimes E}$ be the tensor product connection on $S(TM)\otimes E$ obtained from $\nabla^{S(TM)}$ and ∇^E.

The canonical (twisted) Dirac operator D^E is defined by

$$D^E=\sum_{i=1}^{\dim M}c(e_i)\nabla^{S(TM)\otimes E}_{e_i}:\Gamma\big(S(TM)\otimes E\big)\to\Gamma\big(S(TM)\otimes E\big), \tag{2.3}$$

where $e_1,\dots,e_{\dim M}$ is an orthonormal basis of TM. By (2.1) and (2.2), over $[0,1)\times\partial M$, one has

$$D^E=c\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}+\pi^*D^E_{\partial M}\right), \tag{2.4}$$

where $D^E_{\partial M}:\Gamma((S(TM)\otimes E)|_{\partial M})\to\Gamma((S(TM)\otimes E)|_{\partial M})$ is the induced Dirac operator on ∂M.

We now introduce the APS type boundary conditions for D^E. The induced Dirac operator on the boundary, $D^E_{\partial M}$, is elliptic and self-adjoint. Let $L^2_+((S(TM)\otimes E)|_{\partial M})$ be the space of the direct sum of eigenspaces of positive eigenvalues of $D^E_{\partial M}$. Let $P_{\partial M}$ denote the orthogonal projection operator from $L^2((S(TM)\otimes E)|_{\partial M})$ to $L^2_+((S(TM)\otimes E)|_{\partial M})$ (for simplicity we suppress the dependence on E).

As is well known, the APS projection $P_{\partial M}$ is an elliptic global boundary condition for D^E. However, to get self-adjoint boundary conditions, we need to modify it by a Lagrangian subspace of $\ker D^E_{\partial M}$, namely, a subspace L of $\ker D^E_{\partial M}$ such that $c(\frac{\partial}{\partial x})L=L^\perp\cap(\ker D^E_{\partial M})$. Since ∂M bounds M, by the cobordism invariance of the index, such Lagrangian subspaces always exist.

The modified APS projection is obtained by adding the projection onto the Lagrangian subspace. Let $P_{\partial M}(L)$ denote the orthogonal projection operator from $L^2((S(TM)\otimes E)|_{\partial M})$ to $L^2_+((S(TM)\otimes E)|_{\partial M})\oplus L$:

$$P_{\partial M}(L) = P_{\partial M} + P_L, \tag{2.5}$$

where P_L denotes the orthogonal projection from $L^2((S(TM)\otimes E)|_{\partial M})$ to L.

The pair $(D^E, P_{\partial M}(L))$ forms a self-adjoint elliptic boundary problem, and $P_{\partial M}(L)$ is called an Atiyah–Patodi–Singer boundary condition associated to L. We will also denote the corresponding elliptic self-adjoint operator by $D^E_{P_{\partial M}(L)}$.

Let $L^{2,+}_{P_{\partial M}(L)}(S(TM)\otimes E))$ be the space of the direct sum of eigenspaces of *non-negative* eigenvalues of $D^E_{P_{\partial M}(L)}$. This can be viewed as an analog of the Hardy space. We denote by $P_{P_{\partial M}(L)}$ the orthogonal projection from $L^2(S(TM)\otimes E)$ to $L^{2,+}_{P_{\partial M}(L)}(S(TM)\otimes E)$.

Let $N>0$ be a positive integer, let $\mathbf{C}^N$ be the trivial complex vector bundle over M of rank N, which carries the trivial Hermitian metric and the trivial Hermitian connection. Then all the above construction can be developed in the same way if one replaces E by $E\otimes\mathbf{C}^N$. And all the operators considered here extend to act on $\mathbf{C}^N$ by identity. If there is no confusion we will also denote them by their original notation.

Now let $g: M\to GL(N,\mathbf{C})$ be a smooth automorphism of $\mathbf{C}^N$. With simple deformation, we can assume that g is unitary. That is, $g: M\to U(N)$. Furthermore, we make the assumption that g is of product structure over $[0,1)\times\partial M$, that is,

$$g|_{[0,1)\times\partial M} = \pi^*(g|_{\partial M}). \tag{2.6}$$

Clearly, g extends to an action on $S(TM)\otimes E\otimes\mathbf{C}^N$ by acting as identity on $S(TM)\otimes E$. We still denote this extended action by g.

Since g is unitary, one verifies easily that the operator $gP_{\partial M}(L)g^{-1}$ is again an orthogonal projection on $L^2((S(TM)\otimes E\otimes\mathbf{C}^N)|_{\partial M})$, and that $gP_{\partial M}(L)g^{-1}-P_{\partial M}(L)$ is a pseudodifferential operator of order less than zero. Moreover, the pair $(D^E, gP_{\partial M}(L)g^{-1})$ forms a self-adjoint elliptic boundary problem. We denote its associated elliptic self-adjoint operator by $D^E_{gP_{\partial M}(L)g^{-1}}$. Thus $D^E_{gP_{\partial M}(L)g^{-1}}$ has the boundary condition which is the conjugation by g of the previous APS type condition.

The necessity of using the conjugated boundary condition here is from the fact that, if $s\in L^2(S(TM)\otimes E\otimes\mathbf{C}^N)$ verifies $P_{\partial M}(L)(s|_{\partial M})=0$, then gs verifies $gP_{\partial M}(L)g^{-1}((gs)|_{\partial M})=0$.

Thus, consider also the analog of Hardy space for the conjugated boundary value problem, $L^{2,+}_{gP_{\partial M}(L)g^{-1}}(S(TM)\otimes E\otimes\mathbf{C}^N)$ which is the space of the direct sum of eigenspaces of *non-negative* eigenvalues of $D^E_{gP_{\partial M}(L)g^{-1}}$. Let $P_{gP_{\partial M}(L)g^{-1}}$ denote the orthogonal projection from $L^2(S(TM)\otimes E\otimes\mathbf{C}^N)$ to $L^{2,+}_{gP_{\partial M}(L)g^{-1}}(S(TM)\otimes E\otimes\mathbf{C}^N)$.

Definition 2.1. The Toeplitz operator $T^E_g(L)$ is defined by

$$\begin{gathered} T^E_g(L) = P_{gP_{\partial M}(L)g^{-1}}\circ g\circ P_{P_{\partial M}(L)}: \\ L^{2,+}_{P_{\partial M}(L)}\big(S(TM)\otimes E\otimes\mathbf{C}^N\big)\to L^{2,+}_{gP_{\partial M}(L)g^{-1}}\big(S(TM)\otimes E\otimes\mathbf{C}^N\big). \end{gathered}\tag{2.7}$$

One verifies that $T^E_g(L)$ is a Fredholm operator. The main purpose of this paper is to establish an index formula for it in terms of geometric data.

2.2. Perturbation

The analysis of the conjugated elliptic boundary value problem $D^E_{gP_{\partial M}(L)g^{-1}}$ turns out to be surprisingly subtle and difficult. To circumvent this difficulty, we now construct a perturbation of the original problem.

Let $\psi = \psi(x)$ be a cut off function which is identically 1 in the ϵ-tubular neighborhood of ∂M ($\epsilon > 0$ sufficiently small) and vanishes outside the 2ϵ-tubular neighborhood of ∂M. Consider the Dirac type operator

$$D^\psi = (1-\psi)D^E + \psi g D^E g^{-1}. \tag{2.8}$$

The effect of this perturbation is that, near the boundary, the operator D^ψ is actually given by the conjugation of D^E, and therefore, the elliptic boundary problem $(D^\psi, gP_{\partial M}(L)g^{-1})$ is now the conjugation of the APS boundary problem $(D^E, P_{\partial M}(L))$.

All previous consideration applies to $(D^\psi, gP_{\partial M}(L)g^{-1})$ and its associated self-adjoint elliptic operator $D^\psi_{gP_{\partial M}(L)g^{-1}}$. In particular, we have the perturbed Toeplitz operator

$$\begin{aligned} T^E_{g,\psi}(L) = P^\psi_{gP_{\partial M}(L)g^{-1}} \circ g \circ P_{P_{\partial M}(L)}:\\ L^{2,+}_{P_{\partial M}(L)}\big(S(TM)\otimes E\otimes \mathbf{C}^N\big) \to L^{2,+,\psi}_{gP_{\partial M}(L)g^{-1}}\big(S(TM)\otimes E\otimes \mathbf{C}^N\big), \end{aligned} \tag{2.9}$$

where $P^\psi_{gP_{\partial M}(L)g^{-1}}$ is the APS projection associated to $D^\psi_{gP_{\partial M}(L)g^{-1}}$, whose range is denoted by $L^{2,+,\psi}_{gP_{\partial M}(L)g^{-1}}(S(TM)\otimes E\otimes \mathbf{C}^N)$.

We will also need to consider the conjugation of D^ψ:

$$D^{\psi,g} = g^{-1}D^\psi g = D^E + (1-\psi)g^{-1}\big[D^E, g\big]. \tag{2.10}$$

2.3. An invariant of η-type for even-dimensional manifolds

Given an even-dimensional closed spin manifold X, we consider the cylinder $[0,1]\times X$ with the product metric. Let $g: X \to U(N)$ be a map from X into the unitary group which extends trivially to the cylinder. Similarly, $E \to X$ is an Hermitian vector bundle which is also extended trivially to the cylinder. We make the assumption that $\operatorname{ind} D^E_+ = 0$ on X.

Consider the analog of $D^{\psi,g}$ as defined in (2.10), but now on the cylinder $[0,1]\times X$ and denote it by $D^{\psi,g}_{[0,1]}$. We equip it with the boundary condition $P_X(L)$ on one of the boundary component $\{0\}\times X$ and the boundary condition $\mathrm{Id} - g^{-1}P_X(L)g$ on the other boundary component $\{1\}\times X$ (note that the Lagrangian subspace L exists by our assumption of vanishing index). Then $(D^{\psi,g}_{[0,1]}, P_X(L), \mathrm{Id} - g^{-1}P_X(L)g)$ forms a self-adjoint elliptic boundary problem. For simplicity, we will still denote the corresponding elliptic self-adjoint operator by $D^{\psi,g}_{[0,1]}$.

Let $\eta(D^{\psi,g}_{[0,1]}, s)$ be the η-function of $D^{\psi,g}_{[0,1]}$ which, when $\mathrm{Re}(s) \gg 0$, is defined by

$$\eta\big(D^{\psi,g}_{[0,1]}, s\big) = \sum_{\lambda\neq 0} \frac{\operatorname{sgn}(\lambda)}{|\lambda|^s}, \tag{2.11}$$

where λ runs through the nonzero eigenvalues of $D^{\psi,g}_{[0,1]}$.

By [7,9,14], one knows that the η-function $\eta(D^{\psi,g}_{[0,1]}, s)$ admits a meromorphic extension to $\mathbf{C}$ with $s=0$ a regular point (and only simple poles). One then defines, as in [2], the η-invariant of $D^{\psi,g}_{[0,1]}$, denoted by $\eta(D^{\psi,g}_{[0,1]})$, to be the value at $s=0$ of $\eta(D^{\psi,g}_{[0,1]}, s)$, and the reduced η-invariant by

$$\bar{\eta}\big(D^{\psi,g}_{[0,1]}\big) = \frac{\dim\ker D^{\psi,g}_{[0,1]} + \eta(D^{\psi,g}_{[0,1]})}{2}. \tag{2.12}$$

In our application, we will apply this construction to the cylinder $[0,1]\times\partial M$, i.e., $X=\partial M$ is a boundary. We point out in passing that the invariant $\bar{\eta}(D^{\psi,g}_{[0,a]})$, similarly constructed on a cylinder $[0,a]\times X$, does not depend on the radial size of the cylinder $a>0$ by a rescaling argument (cf. [14, Proposition 2.16]).

Definition 2.2. We define an invariant of η type for the complex vector bundle E on the even-dimensional manifold X (with vanishing index) and the K^1 representative g by

$$\bar{\eta}(X,g) = \bar{\eta}\big(D^{\psi,g}_{[0,1]}\big) - \mathrm{sf}\big\{D^{\psi,g}_{[0,1]}(s),\ 0\leqslant s\leqslant 1\big\}, \tag{2.13}$$

where $D^{\psi,g}_{[0,1]}(s)$ is a path connecting $g^{-1}D^E g$ with $D^{\psi,g}_{[0,1]}$ defined by

$$D^{\psi,g}(s) = D^E + (1-s\psi)g^{-1}\big[D^E, g\big] \tag{2.14}$$

on $[0,1]\times X$, with the boundary condition $P_X(L)$ on $\{0\}\times X$ and the boundary condition $\mathrm{Id} - g^{-1}P_X(L)g$ at $\{1\}\times X$.

We will show in Section 5 that $\bar{\eta}(X,g)$ does not depend on the cut-off function ψ.

2.4. *An index theorem for $T^E_g(L)$*

Recall that $g: M\to U(N)$. Let d be the trivial Hermitian connection on $\mathbf{C}^N|_M$. Thus $g^{-1}dg$ defines a $\Gamma(\mathrm{End}(\mathbf{C}^N))$-valued 1-form on M. Let $\mathrm{ch}(g,d)$ denote the odd Chern character form of g defined by (cf. [16, Chapter 1])

$$\mathrm{ch}(g,d) = \sum_{n=0}^{\frac{\dim M-1}{2}} \frac{n!}{(2n+1)!}\,\mathrm{Tr}\big[\big(g^{-1}\,dg\big)^{2n+1}\big]. \tag{2.15}$$

Recall also that ∇^{TM} is the Levi-Civita connection associated to the Riemannian metric g^{TM}, and ∇^E is the Hermitian connection on E. Let $R^{TM}=(\nabla^{TM})^2$ (respectively $R^E=(\nabla^E)^2$) be the curvature of ∇^{TM} (respectively ∇^E).

Let $\mathcal{P}_M$ denote the Calderón projection associated to $D^{E\otimes\mathbf{C}^N}$ on M (cf. [6]). Then $\mathcal{P}_M$ is an orthogonal projection on $L^2((S(TM)\otimes E\otimes\mathbf{C}^N)|_{\partial M})$, and that $\mathcal{P}_M - P_{\partial M}(L)$ is a pseudodifferential operator of order less than zero.

Let $\tau_\mu(gP_{\partial M}(L)g^{-1}, P_{\partial M}(L), \mathcal{P}_M)\in\mathbf{Z}$ be the Maslov triple index in the sense of Kirk and Lesch [12, Definition 6.8].

We can now state the main result of this paper as follows.

Theorem 2.3. *The following identity holds,*

$$\operatorname{ind} T_g^E(L) = -\left(\frac{1}{2\pi\sqrt{-1}}\right)^{(\dim M+1)/2} \int_M \widehat{A}\left(R^{TM}\right) \operatorname{Tr}\left[\exp\left(-R^E\right)\right] \operatorname{ch}(g, d)$$
$$- \bar{\eta}(\partial M, g) + \tau_\mu\left(g P_{\partial M}(L) g^{-1}, P_{\partial M}(L), \mathcal{P}_M\right). \tag{2.16}$$

Remark 2.4. We will show in Theorem 5.5 that the same formula holds without the product type assumption (2.6). Also, the spin assumption can be relaxed and the same result holds for general Dirac type operators, in particular, spin^c Dirac operators.

Remark 2.5. Our formula (2.16) is closely related to the so called WZW theory in physics [15]. When $\partial M = S^2$ or a compact Riemann surface and E is trivial, the local term in (2.16) is precisely the Wess–Zumino term, which allows an integer ambiguity, in the WZW theory. Thus, our eta-invariant $\bar{\eta}(\partial M, g)$ gives an intrinsic interpretation of the Wess–Zumino term without passing to the bounding 3-manifold. In fact, for $\partial M = S^2$, it can be further reduced to a local term on S^2 by using Bott's periodicity, see Remark 5.9.

The following immediate consequence is of independent interests and will be studied further in Section 5.

Corollary 2.6. *The number*

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^{(\dim M+1)/2} \int_M \widehat{A}\left(R^{TM}\right) \operatorname{Tr}\left[\exp\left(-R^E\right)\right] \operatorname{ch}(g, d) + \bar{\eta}(\partial M, g)$$

is an integer.

The next two sections will be devoted to a proof of Theorem 2.3.

3. η-Invariants, spectral flow and the index of the perturbed Toeplitz operator

In this section, we prove an index formula for the perturbed Toeplitz operator $T_{g,\psi}^E(L)$. The strategy follows from that of the heat kernel proof of (1.1) sketched in Section 1. However, as we are dealing with the case of manifolds with boundary, we must make necessary modifications at each step of the procedure.

This section is organized as follows. In Section 3.1, we reduce the computation of $\operatorname{ind} T_{g,\psi}^E(L)$ to the computation of a spectral flow of a natural family of self-adjoint elliptic operators on manifolds with boundary. In Section 3.2, we reduce the computation of the above mentioned spectral flow to a computation of certain η-invariants as well as their variations. We then apply a result of Kirk, Lesch [12, Theorem 7.7] to reduce the proof of Theorem 2.3 to a computation of certain local index term arising from the variations of η-invariants. In Section 3.3, we prove the index formula by computing the local index term.

3.1. Spectral flow and the index of perturbed Toeplitz operators

Recall that D^{ψ} defined in (2.8) is the perturbed Dirac operator on M acting on $\Gamma(S(TM) \otimes E \otimes \mathbf{C}^N)$, and $g : M \to U(N)$ is a smooth map.

For any $u \in [0, 1]$, in view of (2.10), set

$$D^{\psi,g}(u) = (1-u)D^E + ug^{-1}D^{\psi}g = D^E + u(1-\psi)g^{-1}\big[D^E, g\big]. \tag{3.1}$$

Then for each $u \in [0, 1]$, the boundary condition $P_{\partial M}(L)$ is still a self-adjoint elliptic boundary condition for $D^{\psi,g}(u)$. We denote the corresponding self-adjoint elliptic operator by $D^{\psi,g}_{P_{\partial M}(L)}(u)$, which depends smoothly on $u \in [0, 1]$.

Let $\mathrm{sf}(D^{\psi,g}_{P_{\partial M}(L)}(u),\ 0 \leqslant u \leqslant 1)$ be the spectral flow of the this one parameter family of elliptic self-adjoint operators in the sense of Atiyah et al. [3].

The following result generalizes a theorem of Booss, Wojciechowski (cf. [6, Theorem 17.17]) for closed manifolds.

Theorem 3.1. *We have,*

$$\operatorname{ind} T^E_{g,\psi}(L) = -\,\mathrm{sf}\big(D^{\psi,g}_{P_{\partial M}(L)}(u),\ 0 \leqslant u \leqslant 1\big). \tag{3.2}$$

Proof. We use the method in the proof of [8, Theorem 4.4], which extends the Booss–Wojciechowski theorem to the case of families, to prove (3.2).

Recall that $P_{P_{\partial M}(L)}$ denotes the orthogonal projection from $L^2(S(TM) \otimes E \otimes \mathbf{C}^N)$ to the space of the direct sum of eigenspaces of non-negative eigenvalues of $D^E_{P_{\partial M}(L)}$. It is obviously a generalized spectral section of $D^{\psi,g}_{P_{\partial M}(L)}(u)$ in the sense of [8]. Let $P_{P_{\partial M}(L)}(1)$ denote the orthogonal projection from $L^2(S(TM) \otimes E \otimes \mathbf{C}^N)$ to the space of the direct sum of eigenspaces of non-negative eigenvalues of $D^{\psi,g}_{P_{\partial M}(L)}(1)$.

As in [8, (1.11)], let $T(P_{P_{\partial M}(L)}, P_{P_{\partial M}(L)}(1))$ be the Fredholm operator

$$T\big(P_{P_{\partial M}(L)}, P_{P_{\partial M}(L)}(1)\big) = P_{P_{\partial M}(L)}(1)P_{P_{\partial M}(L)} : \mathrm{Im}(P_{P_{\partial M}(L)}) \to \mathrm{Im}\big(P_{P_{\partial M}(L)}(1)\big). \tag{3.3}$$

Now we observe that the argument in the proof of [8, Theorem 4.4] still works in our present situation, and we obtain,

$$-\,\mathrm{sf}\big(D^{\psi,g}_{P_{\partial M}(L)}(u),\ 0 \leqslant u \leqslant 1\big) = \operatorname{ind} T\big(P_{P_{\partial M}(L)}, P_{P_{\partial M}(L)}(1)\big). \tag{3.4}$$

From (2.7), (3.1), (3.3) and (3.4), one deduces that

$$-\,\mathrm{sf}\big(D^{\psi,g}_{P_{\partial M}(L)}(u),\ 0 \leqslant u \leqslant 1\big) = \operatorname{ind}\big(g^{-1}P^{\psi}_{gP_{\partial M}(L)g^{-1}}gP_{P_{\partial M}(L)}\big) = \operatorname{ind} T^E_{g,\psi}(L). \tag{3.5}$$

The proof of Theorem 3.1 is completed. □

3.2. η-Invariants and the spectral flow

As usual, by [2], for any $u \in [0,1]$, one can define the η-invariant $\eta(D^{\psi,g}_{P_{\partial M}(L)}(u))$ as well as the corresponding reduced η-invariant

$$\bar{\eta}\big(D^{\psi,g}_{P_{\partial M}(L)}(u)\big) = \frac{\dim\ker(D^{\psi,g}_{P_{\partial M}(L)}(u)) + \eta(D^{\psi,g}_{P_{\partial M}(L)}(u))}{2}. \tag{3.6}$$

As was mentioned in [12], it follows from the work of Grubb [11] that when mod $\mathbf{Z}$, the reduced η-invariants $\bar{\eta}(D^{\psi,g}_{P_{\partial M}(L)}(u))$ vary smoothly with respect to $u \in [0,1]$. And we denote by $\frac{d}{du}(\bar{\eta}(D^{\psi,g}_{P_{\partial M}(L)}(u))$ the smooth function on $[0,1]$ of the local variation (after mod $\mathbf{Z}$) of these reduced η-invariants.

By [12, Lemma 3.4] and (3.1), one then has

$$\mathrm{sf}\big(D^{\psi,g}_{P_{\partial M}(L)}(u),\ 0 \leqslant u \leqslant 1\big) = \bar{\eta}\big(D^{\psi,g}_{P_{\partial M}(L)}(1)\big) - \bar{\eta}\big(D^{E}_{P_{\partial M}(L)}\big) - \int_0^1 \frac{d}{du}\bar{\eta}\big(D^{\psi,g}_{P_{\partial M}(L)}(u)\big)\,du. \tag{3.7}$$

By (3.1) and an obvious conjugation, one sees directly that

$$\bar{\eta}\big(D^{\psi,g}_{P_{\partial M}(L)}(1)\big) = \bar{\eta}\big(D^{\psi}_{gP_{\partial M}(L)g^{-1}}\big). \tag{3.8}$$

Set $M_- = M \setminus ([0,1] \times \partial M)$. On the boundary $\partial M_- = \{1\} \times \partial M$ of M_-, we use the boundary condition $P_{\partial M}(L)$ and denote it by $P^E_{\partial M_-}(L)$. By [14, Proposition 2.16] one has

$$\bar{\eta}\big(D^{E}_{P_{\partial M}(L)}\big) = \bar{\eta}\big(D^{E}_{P_{\partial M_-}(L)}\big). \tag{3.9}$$

From (2.8), (3.8), (3.9) and using [12, Theorem 7.7], one deduces that

$$\begin{aligned}\bar{\eta}\big(D^{\psi,g}_{P_{\partial M}(L)}(1)\big) - \bar{\eta}\big(D^{E}_{P_{\partial M}(L)}\big) &= \bar{\eta}\big(D^{\psi}_{gP_{\partial M}(L)g^{-1}}\big) - \bar{\eta}\big(D^{E}_{P_{\partial M_-}(L)}\big)\\ &= \bar{\eta}\big(D^{\psi,g}_{[0,1]}\big) - \tau_\mu\big(\mathcal{P}^{\psi}_{[0,1]}, P_{\partial M}(L), \mathcal{P}^{E}_{M_-}\big),\end{aligned} \tag{3.10}$$

where $\mathcal{P}^E_{M_-}$ is the Calderón projection operator associated to D^E on M_-, $\mathcal{P}^{\psi}_{[0,1]}$ the Calderón projection operator associated to D^{ψ} on $[0,1] \times \partial M$ with the boundary condition $gP_{\partial M}(L)g^{-1}$ at $\{0\} \times \partial M$, and $\tau_\mu(\mathcal{P}^{\psi}_{[0,1]}, P_{\partial M}(L), \mathcal{P}^E_{M_-})$ is the Maslov triple index in the sense of Kirk, Lesch [12, Definition 6.8].

From (3.2), (3.7) and (3.10), one sees that in order to establish an index formula for $T^E_{g,\psi}$, one needs only to compute

$$\int_0^1 \frac{d}{du}\bar{\eta}\big(D^{\psi,g}_{P_{\partial M}(L)}(u)\big)\,du. \tag{3.11}$$

From (3.1), one verifies that

$$\frac{d}{du} D^{\psi,g}_{P_{\partial M}(L)}(u) = (1-\psi)g^{-1}\big[D^E, g\big] \tag{3.12}$$

is a bounded operator.

By the main result in [11], when $t \to 0^+$, one has the asymptotic expansion

$$\begin{aligned} &\operatorname{Tr}\big[(1-\psi)g^{-1}\big[D^E, g\big]\exp\big(-tD^{\psi,g}_{P_{\partial M}(L)}(u)^2\big)\big] \\ &\quad = \sum_{-\dim M \leqslant k < 0} c_k(u)t^{k/2} + c_0(u)\log t + c_0'(u) + o(1). \end{aligned} \tag{3.13}$$

From (3.12), (3.13) and by proceeding as in [14, Section 2], one deduces easily that

$$\frac{d}{du}\bar{\eta}\big(D^{\psi,g}_{P_{\partial M}(L)}(u)\big) = -\frac{c_{-1}(u)}{\sqrt{\pi}}. \tag{3.14}$$

The following result gives the explicit value of each $c_{-1}(u)$.

Theorem 3.2. *We have*

$$\begin{aligned} \frac{c_{-1}(u)}{\sqrt{\pi}} = \left(\frac{1}{2\pi\sqrt{-1}}\right)^{(\dim M+1)/2} &\int_M \widehat{A}\big(R^{TM}\big)\operatorname{Tr}\big[\exp\big(-R^E\big)\big] \\ &\times \operatorname{Tr}\big[g^{-1}\,dg\exp\big((1-u)u\big(g^{-1}\,dg\big)^2\big)\big]. \end{aligned} \tag{3.15}$$

Therefore,

$$\begin{aligned} \operatorname{ind} T^E_{g,\psi} = -\left(\frac{1}{2\pi\sqrt{-1}}\right)^{(\dim M+1)/2} &\int_M \widehat{A}\big(R^{TM}\big)\operatorname{Tr}\big[\exp\big(-R^E\big)\big]\operatorname{ch}(g,d) \\ &- \bar{\eta}\big(D^{\psi,g}_{[0,1]}\big) + \tau_\mu\big(\mathcal{P}^{\psi}_{[0,1]}, P_{\partial M}(L), \mathcal{P}^E_{M_-}\big). \end{aligned} \tag{3.16}$$

Remark 3.3. When $g|_{\partial M} = \mathrm{Id}$, Theorem 3.2 was proved by Douglas and Wojciechowski [9]. Thus, our main concern will be the case where $g|_{\partial M}$ is not identity. Note that in this case, $gP_{\partial M}(L)g^{-1} - P_{\partial M}(L)$ is in general not a smoothing operator.

3.3. A Proof of Theorem 3.2

Here we prove Theorem 3.2. The contribution to the left-hand side of (3.15) splits into the interior and boundary parts. The interior contribution can be handled by the standard local index theory techniques and the boundary contribution can be easily seen to be zero.

Recall that our purpose is to study the asymptotic behavior when $t \to 0^+$ of the following quantity:

$$\operatorname{Tr}\big[(1-\psi)g^{-1}\big[D^E, g\big]\exp\big(-t\big(D^{\psi,g}_{P_{\partial M}(L)}(u)\big)^2\big)\big], \tag{3.17}$$

for $0 \leqslant u \leqslant 1$, where $D^{\psi,g}_{P_{\partial M}(L)}(u)$ is defined by (3.1):

$$D_\psi(u) = (1-u)D^E + ug^{-1}D^\psi g = D^E + u(1-\psi)g^{-1}\big[D^E, g\big]. \tag{3.18}$$

From (3.18) one verifies that

$$\big(D_\psi(u)\big)^2 = \big(D^E\big)^2 + B(u), \tag{3.19}$$

where

$$B(u) = u\big[D^E, (1-\psi)g^{-1}\big[D^E, g\big]\big] + u^2(1-\psi)^2\big(g^{-1}\big[D^E, g\big]\big)^2. \tag{3.20}$$

Since

$$\begin{aligned}\big[D^E, g^{-1}\big[D^E, g\big]\big] &= \big[D^E, g^{-1}\big]\big[D^E, g\big] + g^{-1}\big[D^E, \big[D^E, g\big]\big] \\ &= -\big(g^{-1}\big[D^E, g\big]\big)^2 + g^{-1}\big[D^E, \big[D^E, g\big]\big],\end{aligned} \tag{3.21}$$

one finds that

$$\begin{aligned}B(u) = {} & \big(u^2(1-\psi)^2 - u(1-\psi)\big)\big(g^{-1}\big[D^E, g\big]\big)^2 + u(1-\psi)g^{-1}\big[D^E, \big[D^E, g\big]\big] \\ & - u\psi' c\left(\frac{\partial}{\partial x}\right)g^{-1}\big[D^E, g\big].\end{aligned} \tag{3.22}$$

Let $e_1, \ldots, e_{\dim M}$ be an orthonormal basis of TM. Then by (2.3), one verifies that

$$\begin{aligned}\big[D^E, \big[D^E, g\big]\big] &= \sum_{i,j=1}^{\dim M} \big[c(e_i)\nabla_{e_i}, c(e_j)(\nabla_{e_j} g)\big] \\ &= -\sum_{i=1}^{\dim M} \big(\nabla_{e_i}^2 - \nabla_{\nabla_{e_i}^{TM} e_i}\big)g - 2\sum_{i=1}^{\dim M} (\nabla_{e_i} g)\nabla_{e_i}.\end{aligned} \tag{3.23}$$

Therefore, $B(u)$ is a differential operator of order one.

For brevity of notation, from now on we will denote the elliptic operator $D^{\psi,g}_{P_{\partial M}(L)}(u)$ simply by $D_\psi(u)$ (with the boundary condition $P_{\partial M}(L)$ understood).

Also, denote $\dim M = 2n+1$. We first show that the study of the limit as $t \to 0^+$ of the term in (3.13) can be reduced to separate computations in the interior and near the boundary. We fix the ϵ which defines the cut-off function ψ (for example, we can take $\epsilon = \frac{1}{4}$).

Let $\overline{D}_\psi(u)$ be the double of the Dirac type operator $D_\psi(u)$, which lives on the double of M. Let $E_I(t)$ denote the heat kernel associated to $e^{-t(\overline{D}_\psi(u))^2}$. Let $E_{L,b}(t)$ denote the heat kernel of $e^{-t(D^\psi_{gP_{\partial M}(L)g^{-1}})^2}$ on the half-cylinder $[0, +\infty) \times \partial M$, where we extend everything from $[0, \epsilon] \times \partial M$ canonically. By our assumption, this is simply the conjugation of the heat kernel $e^{-t(D^E_{P_{\partial M}(L)})^2}$ on the half-cylinder:

$$e^{-t(D^\psi_{gP_{\partial M}(L)g^{-1}})^2} = g e^{-t(D^E_{P_{\partial M}(L)})^2} g^{-1}, \tag{3.24}$$

where D^E assumes the product form (2.4).

Following [2], we use $\rho(a,b)$ to denote an increasing C^∞ function of the real variable x such that

$$\rho = 0 \quad \text{for } x \leqslant a, \qquad \rho = 1 \quad \text{for } x \geqslant b. \tag{3.25}$$

Define four C^∞ functions by

$$\begin{aligned} \phi_1 &= 1 - \rho\left(\frac{5}{6}\epsilon, \epsilon\right), \qquad \psi_1 = 1 - \rho\left(\frac{3}{6}\epsilon, \frac{4}{6}\epsilon\right), \\ \phi_2 &= \rho\left(\frac{1}{6}\epsilon, \frac{2}{6}\epsilon\right), \qquad \psi_2 = \rho\left(\frac{3}{6}\epsilon, \frac{4}{6}\epsilon\right). \end{aligned} \tag{3.26}$$

Lemma 3.4. *There exists $C > 0$ such that as $t \to 0^+$, one has*

$$\begin{aligned} &\operatorname{Tr}\left[(1-\psi)g^{-1}\left[D^E, g\right]\exp\left(-t\left(D_\psi(u)\right)^2\right)\right] \\ &\quad = \int_M \operatorname{Tr}\left[(1-\psi)g^{-1}\left[D^E, g\right]E_I(t)(x,x)\right]\psi_2(x)\,d\mathrm{vol} \\ &\qquad + \int_{[0,+\infty)\times\partial M} \operatorname{Tr}\left[(1-\psi)g^{-1}\left[D^E, g\right]E_{L,b}(t)(x,x)\right]\psi_1(x)\,d\mathrm{vol} + O\left(e^{-C/t}\right). \end{aligned} \tag{3.27}$$

Proof. We construct a parametrix for $\exp(-t(D_\psi(u))^2)$ by patching:

$$E(t) = \phi_1 E_{L,b}(t)\psi_1 + \phi_2 E_I(t)\psi_2.$$

By the standard theory the interior heat kernel is exponentially small as $t \to 0^+$ for $x \neq y$. That is, there exists $C_1 > 0$ such that for $0 < t \leqslant 1$ (say),

$$\left|E_I(t)(x,y)\right| \leqslant C_1 t^{-n-\frac{1}{2}} e^{-\frac{d(x,y)^2}{4t}}. \tag{3.28}$$

Moreover, the same estimate holds for derivatives of $E_I(t)(x,y)$ if we replace $t^{-n-\frac{1}{2}}$ by $t^{-n-\frac{1}{2}-l_1-\frac{l_2}{2}}$ where l_1 is the number of time differentiation and l_2 is the number of spatial differentiation.

One has the same estimate for $E_{L,b}(t)$ as shown in [2, Proposition 2.21]:

$$\left|E_{L,b}(t)(x,y)\right| \leqslant C_2 t^{-n-\frac{1}{2}} e^{-\frac{d(x,y)^2}{4t}}. \tag{3.29}$$

Furthermore, similar estimates for derivatives continue to hold as in the case of interior heat kernel.

By our construction, the distance of the support of ϕ_i', $i = 1,2$, to the support of ψ_i, $i = 1,2$, is at least $\frac{1}{6}\epsilon$. Therefore the estimates above give

$$\left(\frac{\partial}{\partial t} + \left(D_\psi(u)\right)^2\right)E(t) = O\left(e^{-\frac{C}{t}}\right) \tag{3.30}$$

for some $C = C(\epsilon) > 0$, and the derivatives of $(\frac{\partial}{\partial t} + (D_\psi(u))^2)E(t)$ decays exponentially as well (with a smaller C). Hence by Duhamel principle, one deduces that

$$\begin{aligned}\left[(1-\psi)g^{-1}\left[D^E, g\right]\exp\left(-\left(D_\psi(u)\right)^2\right)\right] &= \phi_1\left[(1-\psi)g^{-1}\left[D^E, g\right]E_{L,b}(t)\right]\psi_1 \\ &\quad + \phi_2\left[(1-\psi)g^{-1}\left[D^E, g\right]E_I(t)\right]\psi_1 + O\left(e^{-\frac{C}{t}}\right).\end{aligned}$$

Our result follows. □

Clearly,

$$\operatorname{Tr}\left[(1-\psi)g^{-1}\left[D^E, g\right]E_{L,b}(t)(x,x)\right]\psi_2(x) = 0. \tag{3.31}$$

We therefore turn our attention to the interior contribution.

Lemma 3.5. *We have, $t \to 0^+$,*

$$\begin{aligned}&\int_M \operatorname{Tr}\left[(1-\psi)g^{-1}\left[D^E, g\right]E_I(t)(x,x)\right]\psi_2\, d\mathrm{vol} \\ &\quad \to \left(\frac{1}{2\pi\sqrt{-1}}\right)^{n+1}\int_M \widehat{A}\left(R^{TM}\right)\operatorname{Tr}\left[\exp\left(-R^E\right)\right]\operatorname{ch}(g,d)\, d\mathrm{vol}.\end{aligned} \tag{3.32}$$

Proof. By (3.22) and applying by now the standard local index techniques analogous to [10] and [8, Section 4(e)], we obtain that as $t \to 0^+$,

$$\begin{aligned}&\operatorname{Tr}\left[(1-\psi)g^{-1}\left[D^E, g\right]E_I(t)(x,x)\right] \\ &\quad \to \left(\frac{1}{2\pi\sqrt{-1}}\right)^{n+1}\widehat{A}\left(R^{TM}\right)\operatorname{Tr}\left[\exp\left(-R^E\right)\right] \\ &\quad\quad \times \int_0^1 \operatorname{Tr}\left[(1-\psi)g^{-1}\,dg\exp\left(\left(u(1-\psi)-u^2(1-\psi)^2\right)\left(g^{-1}\,dg\right)^2 + u\,d\psi\, g^{-1}\,dg\right)\right]du.\end{aligned} \tag{3.33}$$

It follows from the nilpotency of $d\psi$ that

$$\begin{aligned}&\operatorname{Tr}\left[(1-\psi)g^{-1}\,dg\exp\left(\left(u(1-\psi)-u^2(1-\psi)^2\right)\left(g^{-1}\,dg\right)^2 + u\,d\psi\, g^{-1}\,dg\right)\right] \\ &\quad = \operatorname{Tr}\left[(1-\psi)g^{-1}\,dg\exp\left(\left(u(1-\psi)-u^2(1-\psi)^2\right)\left(g^{-1}\,dg\right)^2\right)\right] \\ &\quad\quad + \operatorname{Tr}\left[(1-\psi)u\,d\psi\left(g^{-1}\,dg\right)^2\exp\left(\left(u(1-\psi)-u^2(1-\psi)^2\right)\left(g^{-1}\,dg\right)^2\right)\right].\end{aligned} \tag{3.34}$$

Since

$$\operatorname{Tr}\left[\left(g^{-1}\,dg\right)^{2k}\right] = 0, \tag{3.35}$$

for any positive integer k, the second term on the right-hand side of (3.34) is zero. On the other hand, the first term on the right-hand side of (3.34), when restricted to the cylindrical part of M, contains no form in the normal direction x by our product structure assumption on g. Thus its integration over the cylindrical part of M, where the cut-off function ψ may not be zero, is zero. This is true even when integrated together with $\widehat{A}(R^{TM})\operatorname{Tr}[\exp(-R^E)]$ as we also have product structure assumption on these geometric data. It follows then that the right-hand side of (3.33) equals

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^{n+1}\int_M \widehat{A}\big(R^{TM}\big)\operatorname{Tr}\big[\exp\big(-R^E\big)\big]\int_0^1 \operatorname{Tr}\big[g^{-1}\,dg\exp\big((1-u)u\big(g^{-1}\,dg\big)^2\big)\big]\,du. \tag{3.36}$$

Lemma 3.5 follows. □

By Lemmas 3.4, 3.5, one gets (3.15). Then (3.16) follows from (3.4), (3.5), (3.7), (3.10), (3.14) and (3.15).

The proof of Theorem 3.2 is now complete.

4. Spectral flow, Maslov indices and the index of the Toeplitz operator

In this section, we prove the index formula for the Toeplitz operator T_g^E, as stated in Theorem 2.3. As we have seen in the previous section, an index formula (3.16) for the perturbed Toeplitz operator $T_{g,\psi}^E$ has been established. To go from the perturbed Toeplitz operator to the original Toeplitz operator, we make use of the spectral flow, reformulated in [8] in terms of generalized spectral sections, and the theory of Maslov indices, as developed by [12].

4.1. Comparison of indices of Toeplitz and perturbed Toeplitz operators

Here we show that the difference of the index of the Toeplitz operator and that of the perturbed Toeplitz operator can be expressed in terms of a spectral flow by using the formulation of [8] via generalized spectral sections.

Lemma 4.1. *We have*

$$\operatorname{ind} T_g^E - \operatorname{ind} T_{g,\psi}^E = \operatorname{sf}\big(D^{\psi}(s),\ 0\leqslant s\leqslant 1\big), \tag{4.1}$$

where

$$D^{\psi}(s) = gD^{\psi,g}(s)g^{-1} = g\big(D^E + (1-s\psi)g^{-1}\big[D^E, g\big]\big)g^{-1} \tag{4.2}$$

is equipped with the boundary condition $gP_{\partial M}(L)g^{-1}$.

Proof. First, we note that

$$\operatorname{ind} T_g^E(L) = \operatorname{ind}\big(P_{gP_{\partial M}(L)g^{-1}}\,g\,P_{P_{\partial M}(L)}\big) = \operatorname{ind}\big(P^g_{P_{\partial M}(L)}P_{P_{\partial M}(L)}\big),$$

where $P^{g}_{P_{\partial M}(L)}$ is the orthogonal projection onto the eigenspaces of $(g^{-1}D^E g, P_{\partial M})$ with non-negative eigenvalues. Similarly,

$$\operatorname{ind} T^E_{g,\psi}(L) = \operatorname{ind}\bigl(P^{\psi}_{gP_{\partial M}(L)g^{-1}} g P_{P_{\partial M}(L)}\bigr) = \operatorname{ind}\bigl(P^{g,\psi}_{P_{\partial M}(L)} P_{P_{\partial M}(L)}\bigr),$$

where $P^{g,\psi}_{P_{\partial M}(L)}$ is the orthogonal projection onto the eigenspaces of $(g^{-1}D^{\psi} g, P_{\partial M})$ with non-negative eigenvalues.

Thus,

$$\operatorname{ind} T^E_g - \operatorname{ind} T^E_{g,\psi} = \operatorname{ind}\bigl(P^{g}_{P_{\partial M}(L)} P_{P_{\partial M}(L)}\bigr) - \operatorname{ind}\bigl(P^{g,\psi}_{P_{\partial M}(L)} P_{P_{\partial M}(L)}\bigr).$$

Noting that $P_{P_{\partial M}(L)}$ is again a generalized spectral section of $D^{\psi,g}(s)$, we have by the argument in [8] that

$$\operatorname{ind} T^E_g - \operatorname{ind} T^E_{g,\psi} = \operatorname{sf}\bigl(D^{\psi,g}(s),\ 0 \leqslant s \leqslant 1\bigr) = \operatorname{sf}\bigl(D^{\psi}(s),\ 0 \leqslant s \leqslant 1\bigr). \qquad \square$$

4.2. Maslov indices and the splitting of spectral flow

Already from Theorem 3.2 and Lemma 4.1 we obtain an index formula for T^E_g. To put this formula into the (much better) form as stated in our main result, Theorem 2.3, we need to make use of Maslov indices as developed in [12].

The (double) Maslov index is an integer invariant for Fredholm pairs of paths of Lagrangian subspaces. It is an algebraic count of how many times these Lagrangian subspaces intersect along the path. We will follow the treatment of [12] closely.

Let H be an Hermitian symplectic Hilbert space, i.e., there is an unitary map $J : H \to H$ such that $J^2 = -1$ and the eigenspaces with eigenvalues $\pm\sqrt{-1}$ have equal dimension. The Lagrangian subspaces in H can be identified with their orthogonal projections, the space of which is

$$\operatorname{Gr}(H) = \bigl\{P \in B(H) \mid P = P^*,\ P^2 = P,\ JPJ^* = I - P\bigr\}.$$

A pair (P, Q), $P, Q \in \operatorname{Gr}(H)$, is called Fredholm if

$$T(Q, P) = PQ : \operatorname{Im} Q \to \operatorname{Im} P$$

is Fredholm.

For a (continuous) path $(P(t), Q(t))$, $0 \leqslant t \leqslant 1$, of Fredholm pairs, $P(t), Q(t) \in \operatorname{Gr}(H)$, the Maslov index associates an integer $\operatorname{Mas}(P(t), Q(t))$ [12].

On the other hand, for a triple $P, Q, R \in \operatorname{Gr}(H)$ such that (P, Q), (Q, R), (P, R) are Fredholm and at least one of the differences $P - Q$, $Q - R$, $P - R$ is compact, an integer $\tau_\mu(P, Q, R)$ can be defined [12], which is called the Maslov triple index. They satisfy the following important relation [12, (6.24)]:[3]

[3] Note the sign correction on the left-hand side of (4.3).

$$\tau_\mu\big(P(1), Q(1), R(1)\big) - \tau_\mu\big(P(0), Q(0), R(0)\big)$$
$$= \operatorname{Mas}\big(P(t), Q(t)\big) + \operatorname{Mas}\big(Q(t), R(t)\big) - \operatorname{Mas}\big(P(t), R(t)\big). \tag{4.3}$$

The following theorem is a slight generalization of [12, Theorem 7.6], which itself is a generalization of a result of Nicolaescu. It follows from the same argument.

Theorem 4.2. *Let M be a manifold with boundary and H a separating hypersurface in M such that $H \cap \partial M = \emptyset$ and that $M = M^+ \cup_H M^-$. Let $D(t)$, $0 \leqslant t \leqslant 1$, be a smooth path of Dirac type operators equipped with self-adjoint elliptic boundary conditions of APS type on the boundary of M. If $D(t)$ is of product type near the separating hypersurface H, then*

$$\operatorname{sf}\big(D(t),\ 0 \leqslant t \leqslant 1\big) = \operatorname{Mas}(\mathcal{P}_{M^-}, \mathcal{P}_{M^+}), \tag{4.4}$$

where $\mathcal{P}_{M^-}$ ($\mathcal{P}_{M^+}$) denotes the Calderón projections on M^- (M^+) with the boundary conditions on $M^- \cap \partial M$ ($M^+ \cap \partial M$) coming from those on ∂M.

4.3. Proof of Theorem 2.3

We are now in position to prove our main result. By Lemma 4.1 we have

$$\operatorname{ind} T_g^E - \operatorname{ind} T_{g,\psi}^E = \operatorname{sf}\big(D^\psi(s),\ 0 \leqslant s \leqslant 1\big).$$

Applying Theorem 4.2 to $M = [0,1] \times \partial M \cup M_-$ with the boundary condition $g P_{\partial M}(L) g^{-1}$ on ∂M, we obtain

$$\operatorname{sf}\big(D^\psi(s),\ 0 \leqslant s \leqslant 1\big) = \operatorname{Mas}\big(\mathcal{P}^\psi_{[0,1]}(s),\ \mathcal{P}_{M_-}\big). \tag{4.5}$$

Here $\mathcal{P}^\psi_{[0,1]}(s)$ denotes the Calderón projection operator associated to $D^\psi(s)$ on $[0,1] \times \partial M$ with the boundary condition $g P_{\partial M}(L) g^{-1}$ at $\{0\} \times \partial M$.

Hence by Theorem 3.2 and (4.5), we have

$$\operatorname{ind} T_g^E = -\left(\frac{1}{2\pi\sqrt{-1}}\right)^{(\dim M+1)/2} \int_M \widehat{A}\big(R^{TM}\big) \operatorname{Tr}\big[\exp\big(-R^E\big)\big] \operatorname{ch}(g)$$
$$- \bar{\eta}\big(D^{\psi,g}_{[0,1]}\big) + \tau_\mu\big(\mathcal{P}^\psi_{[0,1]}, P_{\partial M}(L), \mathcal{P}_{M_-}\big) + \operatorname{Mas}\big(\mathcal{P}^\psi_{[0,1]}(s), \mathcal{P}_{M_-}\big). \tag{4.6}$$

Using (2.13), we rewrite (4.6) as

$$\operatorname{ind} T_g^E = -\left(\frac{1}{2\pi\sqrt{-1}}\right)^{(\dim M+1)/2} \int_M \widehat{A}\big(R^{TM}\big) \operatorname{Tr}\big[\exp\big(-R^E\big)\big] \operatorname{ch}(g) - \bar{\eta}(\partial M, g)$$
$$- \operatorname{sf}\big\{D^{\psi,g}_{[0,1]}(s),\ 0 \leqslant s \leqslant 1\big\} + \tau_\mu\big(\mathcal{P}^\psi_{[0,1]}, P_{\partial M}(L), \mathcal{P}_{M_-}\big) + \operatorname{Mas}\big(\mathcal{P}^\psi_{[0,1]}(s), \mathcal{P}_{M_-}\big). \tag{4.7}$$

On the other hand, by (4.3),

$$\tau_\mu\big(\mathcal{P}^{\psi}_{[0,1]}, P_{\partial M}(L), \mathcal{P}_{M_-}\big) - \tau_\mu\big(g P_{\partial M}(L) g^{-1}, P_{\partial M}(L), \mathcal{P}_{M_-}\big)$$
$$= \operatorname{Mas}\big(\mathcal{P}^{\psi}_{[0,1]}(s), P_{\partial M}(L)\big) - \operatorname{Mas}\big(\mathcal{P}^{\psi}_{[0,1]}(s), \mathcal{P}_{M_-}\big). \tag{4.8}$$

And finally, by using [12, Theorem 7.5],

$$\operatorname{sf}\big\{D^{\psi,g}_{[0,1]}(s),\ 0 \leqslant s \leqslant 1\big\} = \operatorname{sf}\big\{D^{\psi}_{[0,1]}(s),\ 0 \leqslant s \leqslant 1\big\} = \operatorname{Mas}\big(\mathcal{P}^{\psi}_{[0,1]}(s), P_{\partial M}(L)\big). \tag{4.9}$$

From (4.7)–(4.9), one gets

$$\operatorname{ind} T^E_g = -\left(\frac{1}{2\pi\sqrt{-1}}\right)^{(\dim M+1)/2} \int_M \widehat{A}\big(R^{TM}\big) \operatorname{Tr}\big[\exp\big(-R^E\big)\big] \operatorname{ch}(g) - \bar\eta(\partial M, g)$$
$$+ \tau_\mu\big(g P_{\partial M}(L) g^{-1}, P_{\partial M}(L), \mathcal{P}_{M_-}\big). \tag{4.10}$$

This completes the proof of Theorem 2.3.

5. Generalizations and some further results

In this section, we first show that for any even-dimensional closed spin manifold X and any K^1 representative $g : X \to U(N)$, the invariant $\bar\eta(X, g)$ defined in (2.13) is independent of the cut-off function. Then we generalize Theorem 2.3 to the case where one no longer assumes that g is of the product type near ∂M. Finally we take a further look at the η-type invariant $\bar\eta(X, g)$ and study some of its basic properties.

This section is organized as follows. In Section 5.1, we study the variation of $\bar\eta(X, g)$ in the cut-off function which gives us the desired independence. We also make a conjecture about what this eta invariant really should be. In Section 5.2, we take a look at the variations of the odd Chern character forms. In Section 5.3, we prove an extension of Theorem 2.3 to the case where we no longer assume g is of product structure near ∂M. In Section 5.4, we make a further study of the eta invariant $\bar\eta(X, g)$.

5.1. The invariant $\bar\eta(X, g)$

Recall that the invariant of η type associated to a Dirac operator on an even-dimensional manifold X with vanishing index and the K^1 representative g over X is defined in (2.13) as (here we have inserted ψ in the notation to indicate that, a priori, it depends on the cut-off function ψ)

$$\bar\eta(X, g, \psi) = \bar\eta\big(D^{\psi,g}_{[0,1]}\big) - \operatorname{sf}\big\{D^{\psi,g}_{[0,1]}(s),\ 0 \leqslant s \leqslant 1\big\},$$

where $D^{\psi,g}_{[0,1]}(s)$ is a path connecting $g^{-1} D^E g$ with $D^{\psi,g}_{[0,1]}$ defined by

$$D^{\psi,g}(s) = D^E + (1 - s\psi) g^{-1}\big[D^E, g\big]$$

on $[0, 1] \times X$, with the boundary condition $P_X(L)$ on $\{0\} \times X$ and the boundary condition $\mathrm{Id} - g^{-1} P_X(L) g$ at $\{1\} \times X$.

Proposition 5.1. *The invariant $\bar\eta(X, g, \psi)$ is independent of the cut-off function ψ.*

Proof. Let ψ_1, ψ_2 be two cut-off functions and

$$\psi_t = (2-t)\psi_1 + (t-1)\psi_2, \quad 1 \leqslant t \leqslant 2, \tag{5.1}$$

be the smooth path of cut-off functions connecting the two. Then

$$\bar{\eta}\big(D^{\psi_2,g}_{[0,1]}\big) - \bar{\eta}\big(D^{\psi_1,g}_{[0,1]}\big) = \int_1^2 \frac{\partial}{\partial t}\bar{\eta}\big(D^{\psi_t,g}_{[0,1]}\big)\,dt + \mathrm{sf}\big\{D^{\psi_t,g}_{[0,1]},\ 1 \leqslant t \leqslant 2\big\}. \tag{5.2}$$

As before, we can compute $\frac{\partial}{\partial t}\bar{\eta}(D^{\psi_t,g}_{[0,1]})$ via heat kernel and local index theorem technique (cf. Section 3.3) and find

$$\frac{\partial}{\partial t}\bar{\eta}\big(D^{\psi_t,g}_{[0,1]}\big) \equiv 0. \tag{5.3}$$

Here we have once again used the fact that g is constantly extended along the radial direction. Therefore

$$\begin{aligned}\bar{\eta}(X,g,\psi_2) - \bar{\eta}(X,g,\psi_1) &= \mathrm{sf}\big\{D^{\psi_t,g}_{[0,1]},\ 1 \leqslant t \leqslant 2\big\} - \mathrm{sf}\big\{D^{\psi_2,g}_{[0,1]}(s),\ 0 \leqslant s \leqslant 1\big\} \\ &\quad + \mathrm{sf}\big\{D^{\psi_1,g}_{[0,1]}(s),\ 0 \leqslant s \leqslant 1\big\} = 0\end{aligned}$$

by the additivity and the homotopy invariance of spectral flow. □

Thus, the eta type invariant $\bar{\eta}(X,g)$, which we introduced using a cut-off function, is, in fact, independent of the cut-off function. This leads naturally to the question of whether $\bar{\eta}(X,g)$ can actually be defined directly. We now state a conjecture for this question.

Let $D^{[0,1]}$ be the Dirac operator on $[0,1] \times X$. We equip the boundary condition $gP_X(L)g^{-1}$ at $\{0\} \times X$ and the boundary condition $\mathrm{Id} - P_X(L)$ at $\{1\} \times X$.

Then $(D^{[0,1]}, gP_X(L)g^{-1}, \mathrm{Id} - P_X(L))$ forms a self-adjoint elliptic boundary problem. We denote the corresponding elliptic self-adjoint operator by $D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)}$.

Let $\eta(D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)}, s)$ be the η-function of $D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)}$. By [12, Theorem 3.1], which goes back to [11], one knows that the η-function $\eta(D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)}, s)$ admits a meromorphic extension to $\mathbf{C}$ with poles of order at most 2. One then defines, as in [12, Definition 3.2], the η-invariant of $D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)}$, denoted by $\eta(D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)})$, to be the constant term in the Laurent expansion of $\eta(D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)}, s)$ at $s=0$.

Let $\bar{\eta}(D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)})$ be the associated reduced η-invariant.

Conjecture 5.2.

$$\bar{\eta}(X,g) = \bar{\eta}\big(D^{[0,1]}_{gP_X(L)g^{-1},P_X(L)}\big).$$

If this conjecture is correct, then the result stated in [17, Theorem 5.2] is also correct. A previous version of the current article was devoted to a proof of [17, Theorem 5.2], and a referee pointed out a gap in that version. This is why we now introduce a new η-type invariant, which makes the picture clearer.

5.2. A Chern–Weil type theorem for odd Chern character forms

In this subsection, we assume that there is a smooth family g_t, $0 \leqslant t \leqslant 1$, of the automorphisms of the trivial complex vector bundle $\mathbf{C}^N \to M$, and study the variations of the odd Chern character forms $\mathrm{ch}(g_t, d)$, when $t \in [0, 1]$ changes.

The following lemma is taken from [10, Proposition 1.3] (cf. [16, Lemma 1.17]).

Lemma 5.3. *For any positive odd integer n, the following identity holds,*

$$\frac{\partial}{\partial t}\mathrm{Tr}\big[\big(g_t^{-1}\,dg_t\big)^n\big] = n\,d\,\mathrm{Tr}\left[g_t^{-1}\frac{\partial g_t}{\partial t}\big(g_t^{-1}\,dg_t\big)^{n-1}\right]. \tag{5.4}$$

Proof. First of all, from the identity $g_t g_t^{-1} = \mathrm{Id}$, one verifies by differentiation that

$$\frac{\partial g_t^{-1}}{\partial t} = -g_t^{-1}\left(\frac{\partial g_t}{\partial t}\right)g_t^{-1}. \tag{5.5}$$

One then computes that

$$\begin{aligned}\frac{\partial}{\partial t}\big(g_t^{-1}\,dg_t\big) &= \frac{\partial g_t^{-1}}{\partial t}\,dg_t + g_t^{-1}\,d\frac{\partial g_t}{\partial t} = -\left(g_t^{-1}\frac{\partial g_t}{\partial t}\right)g_t^{-1}\,dg_t + g_t^{-1}\,d\frac{\partial g_t}{\partial t}\\ &= -\left(g_t^{-1}\frac{\partial g_t}{\partial t}\right)g_t^{-1}\,dg_t + \big(g_t^{-1}\,dg_t\big)\left(g_t^{-1}\frac{\partial g_t}{\partial t}\right) + d\left(g_t^{-1}\frac{\partial g_t}{\partial t}\right).\end{aligned} \tag{5.6}$$

One also verifies that

$$d\big(g_t^{-1}\,dg_t\big)^2 = d\big(g_t^{-1}\,dg_t\big)g_t^{-1}\,dg_t - g_t^{-1}\,dg_t\,d\big(g_t^{-1}\,dg_t\big) = 0,$$

from which one deduces that for any positive even integer k,

$$d\big(g_t^{-1}\,dg_t\big)^k = 0. \tag{5.7}$$

From (5.5)–(5.7), one verifies that

$$\begin{aligned}&\frac{\partial}{\partial t}\mathrm{Tr}\big[\big(g_t^{-1}\,dg_t\big)^n\big]\\ &\quad= n\,\mathrm{Tr}\left[\frac{\partial}{\partial t}\big(g_t^{-1}\,dg_t\big)\big(g_t^{-1}\,dg_t\big)^{n-1}\right]\\ &\quad= n\,\mathrm{Tr}\left[\left[g_t^{-1}\,dg_t, g_t^{-1}\frac{\partial g_t}{\partial t}\right]\big(g_t^{-1}\,dg_t\big)^{n-1}\right] + n\,\mathrm{Tr}\left[d\left(g_t^{-1}\frac{\partial g_t}{\partial t}\right)\big(g_t^{-1}\,dg_t\big)^{n-1}\right]\end{aligned}$$

$$= n \operatorname{Tr}\left[\left[g_t^{-1}\,dg_t,\, g_t^{-1}\frac{\partial g_t}{\partial t}\left(g_t^{-1}\,dg_t\right)^{n-1}\right]\right] + n \operatorname{Tr}\left[d\left(g_t^{-1}\frac{\partial g_t}{\partial t}\left(g_t^{-1}\,dg_t\right)^{n-1}\right)\right]$$

$$= n\,d \operatorname{Tr}\left[g_t^{-1}\frac{\partial g_t}{\partial t}\left(g_t^{-1}\,dg_t\right)^{n-1}\right]. \tag{5.8}$$

The proof of Lemma 5.3 is completed. □

For any $t \in [0, 1]$, set

$$\widetilde{\mathrm{ch}}(g_t, d,\ 0 \leqslant t \leqslant 1) = \sum_{n=0}^{(\dim M-1)/2} \frac{n!}{(2n)!} \int_0^1 \operatorname{Tr}\left[g_t^{-1}\frac{\partial g_t}{\partial t}\left(g_t^{-1}\,dg_t\right)^{2n}\right] dt. \tag{5.9}$$

By Lemma 5.3 and (5.9), one gets

Theorem 5.4. (Cf. [10, Proposition 1.3].) *The following identity holds,*

$$\mathrm{ch}(g_1, d) - \mathrm{ch}(g_0, d) = d\widetilde{\mathrm{ch}}(g_t, d,\ 0 \leqslant t \leqslant 1). \tag{5.10}$$

5.3. An index theorem for the case of non-product structure near boundary

In this section, we no longer assume the product structure of $g : M \to U(N)$ near the boundary ∂M. Then, clearly, the Toeplitz operator $T_g^E(L)$ is still well defined. Moreover, by an easy deformation argument, we can construct a smooth one parameter family of maps $g_t : M \to U(N)$, $0 \leqslant t \leqslant 1$, with $g_0 = g$, $g_1 = g'$ such that for any $t \in [0, 1]$, $g_t|_{\partial M} = g_0|_{\partial M}$, and that $g_1 = g'$ is of product structure near ∂M.

By the homotopy invariance of the index of Fredholm operators, one has

$$\operatorname{ind} T_g^E(L) = \operatorname{ind} T_{g'}^E(L). \tag{5.11}$$

Now by Theorem 2.3, one has

$$\operatorname{ind} T_{g'}^E(L) = -\left(\frac{1}{2\pi\sqrt{-1}}\right)^{(\dim M+1)/2} \int_M \widehat{A}\left(R^{TM}\right) \operatorname{Tr}\left[\exp\left(-R^E\right)\right] \mathrm{ch}(g', d)$$
$$- \bar{\eta}(\partial M, g) + \tau_\mu\left(g P_{\partial M}(L) g^{-1}, P_{\partial M}(L), \mathcal{P}_M\right). \tag{5.12}$$

Since $g_t|_{\partial M}$ is constant in t, from (5.9) and (5.10) one deduces that

$$\int_M \widehat{A}\left(R^{TM}\right) \operatorname{Tr}\left[\exp\left(-R^E\right)\right] \mathrm{ch}(g', d) - \int_M \widehat{A}\left(R^{TM}\right) \operatorname{Tr}\left[\exp\left(-R^E\right)\right] \mathrm{ch}(g, d)$$
$$= \int_{\partial M} \widehat{A}\left(R^{TM}\right) \operatorname{Tr}\left[\exp\left(-R^E\right)\right] \widetilde{\mathrm{ch}}(g_t, d,\ 0 \leqslant t \leqslant 1) = 0. \tag{5.13}$$

From (5.11)–(5.13), one deduces

Theorem 5.5. *Formula* (2.16) *still holds if one drops the condition that g is of product structure near ∂M.*

5.4. Some results concerning the η-invariant associated to g

We start with Corollary 2.6 which says that the following number,

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^{(\dim M+1)/2}\int_M \widehat{A}\left(R^{TM}\right)\operatorname{Tr}\left[\exp\left(-R^E\right)\right]\operatorname{ch}(g,d)+\bar{\eta}(\partial M,g), \tag{5.14}$$

is an integer.

It is not difficult to find example such that $\bar{\eta}(\partial M,g)$ is not an integer. So it is not a trivial invariant and deserves further study.

Our first result will show that the number in (5.14) is still an integer if $P_{\partial M}(L)$ is changed to a $Cl(1)$-spectral section in the sense of Melrose and Piazza [13].

Thus let P be a $Cl(1)$-spectral section associated to $P_{\partial M}(L)$. That is, P defers from $P_{\partial M}(L)$ only by a finite-dimensional subspace. Then P, as well as $g^{-1}Pg$, is still a self-adjoint elliptic boundary condition for D^E and in view of [7], all our previous discussion carries over. We can thus define the corresponding η-invariant $\bar{\eta}(\partial M,g,P)$ similarly (we inserted P in the notation to emphasize its dependence).

Proposition 5.6. *The following identity holds for any two $Cl(1)$-spectral sections P, Q associated to $P_{\partial M}(L)$,*

$$\bar{\eta}(\partial M,g,P)\equiv\bar{\eta}(\partial M,g,Q)\quad \bmod \mathbf{Z}. \tag{5.15}$$

Proof. Let $D^{[0,1],\psi}_{P,g^{-1}Pg}$ denote the elliptic self-adjoint operator defined by $D^{\psi,g}$ on $[0,1]\times\partial M$, with the boundary condition P on $\{0\}\times\partial M$ and the boundary condition $\mathrm{Id}-g^{-1}Pg$ at $\{1\}\times\partial M$. Then we have

$$\bar{\eta}(\partial M,g,P)-\bar{\eta}(\partial M,g,Q)\equiv\bar{\eta}\left(D^{[0,1],\psi}_{P,g^{-1}Pg}\right)-\bar{\eta}\left(D^{[0,1],\psi}_{Q,g^{-1}Qg}\right)\quad \bmod \mathbf{Z}. \tag{5.16}$$

In view of the definition of the operator $D^{[0,1],\psi}$ and using the mod $\mathbf{Z}$ version of [12, Theorem 7.7] repeatedly, one deduces that

$$\begin{aligned}
&\bar{\eta}\left(D^{[0,1],\psi}_{P,g^{-1}Pg}\right)-\bar{\eta}\left(D^{[0,1],\psi}_{Q,g^{-1}Qg}\right)\\
&\quad=\bar{\eta}\left(D^{[0,1],\psi}_{P,g^{-1}Pg}\right)-\bar{\eta}\left(D^{[0,1],\psi}_{Q,g^{-1}Pg}\right)+\bar{\eta}\left(D^{[0,1],\psi}_{Q,g^{-1}Pg}\right)-\bar{\eta}\left(D^{[0,1],\psi}_{Q,g^{-1}Qg}\right)\\
&\quad\equiv\bar{\eta}\left(D^{[0,1]}_{P,Q}\right)-\bar{\eta}\left(\left(g^{-1}Dg\right)^{[0,1]}_{g^{-1}Pg,g^{-1}Qg}\right)\quad \bmod \mathbf{Z}.
\end{aligned}\tag{5.17}$$

On the other hand, one clearly has

$$\bar{\eta}\left(D^{[0,1]}_{P,Q}\right)\equiv\bar{\eta}\left(\left(g^{-1}Dg\right)^{[0,1]}_{g^{-1}Pg,g^{-1}Qg}\right)\quad \bmod \mathbf{Z}. \tag{5.18}$$

From (5.17) and (5.18), we obtain (5.16). □

By Proposition 5.6, when mod $\mathbf{Z}$, $\bar{\eta}(\partial M, g, P)$ depends only on $D^E_{\partial M}$ and $g|_{\partial M}$. From now on we denote this $\mathbf{R}/\mathbf{Z}$-valued function by $\bar{\eta}(D^E_{\partial M}, g)$.

Remark 5.7. In fact, for any closed spin manifold X of even dimension, if the canonical Dirac operator D^E_X has vanishing index (then by [13] there exist the associated $Cl(1)$-spectral sections), one can define $\bar{\eta}(D^E_X, g)$ for $g : X \to U(N)$.

The next result describes the dependence of $\bar{\eta}(D^E_{\partial M}, g)$ on $g|_{\partial M}$.

Theorem 5.8. *If $\{g_t\}_{0\leqslant t\leqslant 1}$ is a smooth family of maps from M to $U(N)$, then*

$$\begin{aligned}&\bar{\eta}\big(D^E_{\partial M}, g_1\big) - \bar{\eta}\big(D^E_{\partial M}, g_0\big)\\&\quad\equiv -\left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{\dim M+1}{2}} \int_{\partial M} \widehat{A}\big(R^{TM}\big)\operatorname{Tr}\big[\exp\big(-R^E\big)\big]\widetilde{\operatorname{ch}}(g_t, d,\ 0\leqslant t\leqslant 1) \mod \mathbf{Z}. \qquad (5.19)\end{aligned}$$

In particular, if $g_0 = \mathrm{Id}$, that is, $g = g_1$ is homotopic to the identity map, then

$$\begin{aligned}&\bar{\eta}\big(D^E_{\partial M}, g\big)\\&\quad\equiv -\left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{\dim M+1}{2}} \int_{\partial M} \widehat{A}\big(R^{TM}\big)\operatorname{Tr}\big[\exp\big(-R^E\big)\big]\widetilde{\operatorname{ch}}(g_t, d,\ 0\leqslant t\leqslant 1) \mod \mathbf{Z}. \qquad (5.20)\end{aligned}$$

Proof. By the integrality of the number in (5.14), Proposition 5.6 and the definition of $\bar{\eta}(D^E_{\partial M}, g_t)$, one finds

$$\bar{\eta}\big(D^E_{\partial M}, g_t\big) \equiv -\left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{\dim M+1}{2}} \int_{M} \widehat{A}\big(R^{TM}\big)\operatorname{Tr}\big[\exp\big(-R^E\big)\big]\operatorname{ch}(g_t, d) \mod \mathbf{Z}. \qquad (5.21)$$

By Theorem 5.4 and (5.21) one deduces that

$$\begin{aligned}&\bar{\eta}\big(D^E_{\partial M}, g_1\big) - \bar{\eta}\big(D^E_{\partial M}, g_0\big)\\&\quad\equiv -\left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{\dim M+1}{2}} \int_{M} \widehat{A}\big(R^{TM}\big)\operatorname{Tr}\big[\exp\big(-R^E\big)\big]\,d\widetilde{\operatorname{ch}}(g_t, d,\ 0\leqslant t\leqslant 1) \mod \mathbf{Z}\\&\quad\equiv -\left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{\dim M+1}{2}} \int_{\partial M} \widehat{A}\big(R^{TM}\big)\operatorname{Tr}\big[\exp\big(-R^E\big)\big]\widetilde{\operatorname{ch}}(g_t, d,\ 0\leqslant t\leqslant 1) \mod \mathbf{Z}, \qquad (5.22)\end{aligned}$$

which is exactly (5.19).

(5.20) follows from (5.19) immediately. □

Remark 5.9. As we mentioned in Remark 2.5, $\bar{\eta}(\partial M, g)$ gives an intrinsic interpretation of the Wess–Zumino term in the WZW theory. When $\partial M = S^2$, the Bott periodicity tells us that every K^1 element g on S^2 can be deformed to the identity (adding a trivial bundle if necessary). Hence, (5.20) gives another intrinsic form of the Wess–Zumino term, which is purely local on S^2.

Now let $\widetilde{g}^{TM}$ (respectively $(\widetilde{g}^E, \widetilde{\nabla}^E)$) be another Riemannian metric (respectively another couple of Hermitian metric and connection) on TM (respectively E). Let $\widetilde{R}^{TM}$ (respectively $\widetilde{R}^E$) be the curvature of $\widetilde{\nabla}^{TM}$ (respectively $\widetilde{\nabla}^E$), the Levi-Civita connection of $\widetilde{g}^{TM}$. Let $\widetilde{D}^E$ be the corresponding (twisted) Dirac operator.

Let ω be the Chern–Simons form which transgresses the $\widehat{A} \wedge \mathrm{ch}$ forms:

$$d\omega = \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\frac{\dim M+1}{2}} \left(\widehat{A}\left(\widetilde{R}^{TM}\right)\left[\exp\left(-\widetilde{R}^E\right)\right] - \widehat{A}\left(R^{TM}\right)\left[\exp\left(-R^E\right)\right]\right). \tag{5.23}$$

One then has the following formula describing the variation of $\bar{\eta}(D^E_{\partial M}, g)$, when $g^{TM}|_{\partial M}$, $g^E|_{\partial M}$ and $\nabla^E|_{\partial M}$ change.

Theorem 5.10. *The following identity holds,*

$$\bar{\eta}\left(\widetilde{D}^E_{\partial M}, g\right) - \bar{\eta}\left(D^E_{\partial M}, g\right) \equiv -\int_{\partial M} \omega\, \mathrm{ch}(g, d) \mod \mathbf{Z}. \tag{5.24}$$

Proof. The proof of (5.24) follows directly from (5.23) and the integrality of the numbers of the form (5.14). $\square$

As the last result of this subsection, we prove an additivity formula for $\bar{\eta}(D^E_{\partial M}, g)$.

Theorem 5.11. *Given $f, g : M \to U(N)$, the following identity holds in* $\mathbf{R}/\mathbf{Z}$,

$$\bar{\eta}\left(D^E_{\partial M}, fg\right) = \bar{\eta}\left(D^E_{\partial M}, f\right) + \bar{\eta}\left(D^E_{\partial M}, g\right). \tag{5.25}$$

Proof. Let P be a $Cl(1)$-spectral section for $D^E_{\partial M}$ in the sense of [13]. By [12, Theorem 7.7], one deduces that in $\mathbf{R}/\mathbf{Z}$,

$$\bar{\eta}\left(D^E_{\partial M}, fg\right) = \bar{\eta}\left(D^E_{\partial M}, f\right) + \bar{\eta}\left(\left(f^{-1} D^E f\right)_{\partial M}, fg\right). \tag{5.26}$$

On the other hand, by proceeding as in (5.19), one deduces that the following formula holds in $\mathbf{R}/\mathbf{Z}$,

$$\bar{\eta}\left(\left(f^{-1} D^E f\right)_{\partial M}, fg\right) = \bar{\eta}\left(D^E_{\partial M}, g\right). \tag{5.27}$$

From (5.26) and (5.27), (5.25) follows. $\square$

Remark 5.12. Formulas (5.19), (5.20), (5.24) and (5.25) still hold if ∂M is replaced by a closed even-dimensional spin manifold X on which the Dirac operator D^E_X has vanishing index, and $g : X \to U(N)$ is defined only on X.

Remark 5.13. It might be interesting to note the duality that $\bar{\eta}(D^E_{\partial M}, g)$ is a spectral invariant associated to a K^1 representative on an *even*-dimensional manifold, while the usual Atiyah–Patodi–Singer η-invariant [2] is a spectral invariant associated to a K^0 representative on an *odd*-dimensional manifold.

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Appendix A. Toeplitz index and the Atiyah–Patodi–Singer index theorem

In this appendix we outline a new proof of (1.1), which computes the index of Toeplitz operators on closed manifolds. We use the notation in Section 2, but we assume instead that the odd-dimensional manifold M has no boundary.

We form the cylinder $[0,1]\times M$ and pull back everything to it from M. We also identify $S(TM)$ with $S_+(T([0,1]\times M))|_{\{i\}\times M}$, $i=0,1$. Let $\widetilde{D}^E$ now be the twisted Dirac operator on $[0,1]\times M$ acting on $\Gamma(S_+(T([0,1]\times M))\otimes E\otimes \mathbf{C}^N)$. Then

$$P^E : L^2\big(S(TM)\otimes E\otimes \mathbf{C}^N\big)\to L^2_{\geqslant 0}\big(S(TM)\otimes E\otimes \mathbf{C}^N\big)$$

is the Atiyah–Patodi–Singer boundary condition for $\widetilde{D}^E$ at $\{0\}\times M$. We equip the generalized Atiyah–Patodi–Singer boundary condition $\mathrm{Id} - gP^Eg^{-1}$ at $\{1\}\times M$.

Let $\widetilde{D}^E_{P^E,gP^Eg^{-1}}$ denote the elliptic operator with the Atiyah–Patodi–Singer boundary condition at $\{0\}\times M$ and with the boundary condition $\mathrm{Id} - gP^Eg^{-1}$ at $\{1\}\times M$.

By using the standard variation formula for the index of elliptic boundary problems of Dirac type operators (cf. [6]), one deduces directly that

Theorem A.1. *The following identity holds,*

$$\operatorname{ind} T^E_g = \operatorname{ind} \widetilde{D}^E_{P^E,gP^Eg^{-1}}. \tag{A.1}$$

Now let $\phi:[0,1]\to[0,1]$ be an increasing function such that $\phi(u)=0$ for $0\leqslant u\leqslant \frac14$ and $\phi(u)=1$ for $\frac34\leqslant u\leqslant 1$. Moreover, let $\widehat{D}^E$ be the Dirac type operator on $[0,1]\times M$ such that for any $0\leqslant u\leqslant 1$,

$$\widehat{D}^E(u) = \big(1-\phi(u)\big)\widetilde{D}^E + \phi(u) g\widetilde{D}^E g^{-1}. \tag{A.2}$$

Let $\widehat{D}^E_{P^E, gP^Eg^{-1}}$ denote the elliptic boundary value problem for $\widehat{D}^E$ with the boundary condition P^E at $\{0\} \times M$ and with the boundary condition $\mathrm{Id} - gP^Eg^{-1}$ at $\{1\} \times M$. Then by the homotopy invariance of the index of Fredholm operators, one has directly that

$$\operatorname{ind} \widetilde{D}^E_{P^E, gP^Eg^{-1}} = \operatorname{ind} \widehat{D}^E_{P^E, gP^Eg^{-1}}. \tag{A.3}$$

Now one can apply the Atiyah–Patodi–Singer index theorem [2], combined with the local index computation involving the Dirac type operator $\widehat{D}^E$, to get that

$$\begin{aligned} \operatorname{ind} \widehat{D}^E_{P^E, gP^Eg^{-1}} &= -\big\langle \widehat{A}(TM)\operatorname{ch}(E)\operatorname{ch}(g), [M]\big\rangle - \bar{\eta}\big(D^E\big) + \bar{\eta}\big(gD^Eg^{-1}\big) \\ &= -\big\langle \widehat{A}(TM)\operatorname{ch}(E)\operatorname{ch}(g), [M]\big\rangle. \end{aligned} \tag{A.4}$$

From (A.1), (A.3) and (A.4), one gets (1.1).

Remark A.2. In view of the above proof of (1.1), one may think of Theorem 2.3 as an index theorem on manifolds with corners.

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