Understanding nuclear shape phase transitions at the nucleon level with a boson mapping approach

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A. Introduction

Nuclei have been found for a long time to possess interesting modes of collective motion and geometric shapes, such as vibrating spheroid, rotational ellipsoid, and exotically deformed shapes. There exist phase transitions among these shapes, which have attracted great attention in recent years. Most of the investigations were carried out in the interacting boson model, in which a nucleus is regarded as a N-boson system and holds the SU(5) symmetry in the simplest SU(6) model. Taking the coherent state method, one has shown that the SU(5), SU(3), and O(6) symmetries correspond to the shape phase of a spheroid, axially prolate rotor, γ-soft rotor, respectively. There is also a SU(3) symmetry corresponding to an axially oblate deformed shape phase.

The IBM is a phenomenological model of nuclear structure which has a deep connection with the microscopic shell model. A long-standing significant question is to identify directly the shape phase structure, especially, the shape phase transition, in fermion space or at nucleon level. Recently there have been studies on nuclear shape phase transitions and their critical point symmetries in the framework of shell model, density functional approach, and relativistic mean field approach. In particular, the monopole-pairing and the quadrupole–quadrupole (QQ) interaction were respectively taken to represent the superconducting and rotational phase, or the SU(5) and SU(3) symmetries of IBM.

However, the effects of microscopic interactions on the shape phases and their transitions are still unclear, especially in a more general shell model that incorporates not only monopole-pair and quadrupole–quadrupole interactions but also one-body term and quadrupole-pair interaction. In this Letter, we take a general shell model that incorporates not only monopole-pair and quadrupole–quadrupole interactions but also one-body term that we study the shape phases of nuclei and their transitions, with the Dyson boson mapping approach. The investigation concentrates on the influence of each interaction on the nuclear shape phases. A correspondence between the strength of each of the interactions and the nuclear shape phases is obtained. The investigation also indicates that increasing the quadrupole-pair interaction strength can induce the vibrational to the axially prolate rotational shape phase transition and enhancing the quadrupole–quadrupole interaction can drive the phase transition from the axially oblate rotational to the axially prolate rotational, with the γ-soft rotational being the critical point.
mately reproduce the properties of low-lying nuclear states given by the direct shell model calculations very well (with relative error less than 20%). The DBM had been used in many areas of nuclear physics [33], especially in describing the spectroscopic properties of low-lying states [34–40]. For instance, the low-lying levels of even–even nuclei $^{114}$Cd [34], $^{70–75}$Ge, $^{72–80}$Se [35], $^{156}$Pt [36], $^{46–50}$Ca [37], $^{116}$Cd, $^{122–118}$Sn, $^{148–156}$Sm [38], $^{156–168}$Er [39] and odd-A nuclei $^{101–109}$Rh [40] are qualitatively described very well with the DBM.

The main purpose of this Letter is to study the dependence of nuclear shape phases on the monopole-pairing, the quadrupole-pairing and the quadrupole–quadrupole interactions between nucleons. Since the DBM has been shown to be a quite good approximation of shell model and can describe the observed low-lying nuclear states well, as mentioned above, we implement the DBM to avoid the extremely large space difficulty in shell model. Our study gives such a correspondence explicitly and reveals some novel features. The investigation also shows that there exist shape phase transitions driven by these interactions. For example, the transition from the vibrational to the axially prolate rotational can be induced by the quadrupole-pair interaction, and the transition from the axially oblate to the axially prolate deformed shape, with the electric quadrupole (E2) transition rate, which is written as

$$B(E2, L_1^\pi \rightarrow L_2^\pi) = \frac{1}{2L_1^\pi + 1} |\langle L_1^\pi \parallel \hat{T}(E2) \parallel L_2^\pi \rangle|^2,$$

and the quadrupole moment defined as

$$Q(L^\pi) = \left[L^\pi \parallel \frac{16\pi}{5} \hat{T}(E2) \parallel L^\pi \right].$$

where $\hat{T}(E2)$ is just $\hat{Q}_2$ multiplied by an effective charge.

2.1. Shell model and boson mapping method

2.1. Shell model

A general and widely-used shell model Hamiltonian, especially when describing the properties of low-lying states of nuclei with spherical shape and/or quadrupole deformation (for example, Refs. [28–30]), can be written as

$$\hat{H}_F = \hat{H}_0 - \frac{1}{2} \sum_{\alpha \beta} \hat{b}_\alpha^\dagger \hat{b}_\beta \hat{p}_\alpha - \frac{1}{2} g_2 \hat{p}_2 \cdot \hat{p}_2 - \frac{1}{2} k \cdot \hat{Q}_2 \cdot \hat{Q}_2,$$

(1)

where $\hat{H}_0 = \sum_{jm} \epsilon_j a_j^\dagger a_j$ is the single-body interaction, with $\epsilon_j$ being the single-particle energy, $g_0, g_2$ and $k$ are the strength of monopole-pair, quadrupole-pair, quadrupole–quadrupole interaction, respectively. And $\cdots$ denotes the normal product of fermion operators. $\hat{p}_\alpha^\dagger, \hat{p}_\alpha$ are the fermion creation and annihilation operators, respectively, and expressed as

$$\hat{p}_\alpha^\dagger = \sum_{jm} a_j^\dagger a_j,$$

(2)

$$\hat{p}_2^\dagger_{\mu} = \sum_{jm} \langle jm1 | \hat{q}_{2\mu} | jm2 \rangle a_j^\dagger a_j,$$

(3)

$$\hat{Q}_2^\dagger_{\mu} = \sum_{jm} \langle jm1 | \hat{q}_{2\mu} | jm2 \rangle a_j^\dagger a_j,$$

(4)

where $\hat{q}_{2\mu} = (-1)^{l-m} \hat{q}_{2\mu}$ and $\hat{q}_{2\mu} = \hat{r}_2 Y_{2\mu}$.

To investigate the shape phases and their transitions of the nuclei with spherical shape and/or quadrupole deformation, besides the energy spectrum, other particularly interested properties are the electric quadrupole (E2) transition rate, which is written as

$$B(E2, L_1^\pi \rightarrow L_2^\pi) = \frac{1}{2L_1^\pi + 1} |\langle L_1^\pi \parallel \hat{T}(E2) \parallel L_2^\pi \rangle|^2,$$

(5)

and the quadrupole moment defined as

$$Q(L^\pi) = \left[L^\pi \parallel \frac{16\pi}{5} \hat{T}(E2) \parallel L^\pi \right].$$

(6)

where $\hat{T}(E2)$ is just $\hat{Q}_2$ multiplied by an effective charge.

2.2. Dyson boson mapping method

To investigate the properties of the nuclei in the $A \sim 130$ and $A \sim 150$ mass regions, which involve quite rich shape phases and shape phase transitions, a quite large valence nucleon space should be taken. Since shell model calculation is difficult in very large space, one usually employ the Dyson boson mapping (DBM) method [31,32] or other approaches (for example, the Otsuka-Arima-Iachello (OAI) mapping procedure [20] or the projected shell model [28]) to simplify the calculation. In this letter we take the DBM approach and present briefly the main formula in the following (more details of the DBM and its applications can be found in Ref. [33]).

Let $a_\alpha^\dagger$ and $a_\alpha$ be the fermion creation and annihilation operators, respectively, and express the fermion basis of the system as $|m\rangle$. Here, we take the symbol $\alpha$ to denote the set of quantum numbers $|n_a, l_a, j_a, m_a\rangle$ specifying a nuclear state. The corresponding boson state $|m\rangle$ is constructed by the so-called ideal boson operators $b_{\alpha \beta}^\dagger$ which satisfies the relations:

$$b_{\alpha \beta}^\dagger = -b_{\beta \alpha}^\dagger,$$

(7)

$$[b_{\alpha \beta}, b_{\gamma \delta}^\dagger] = \delta_{\alpha \gamma} \delta_{\beta \delta} - \delta_{\alpha \delta} \delta_{\beta \gamma},$$

(8)

$$[b_{\alpha \beta}, b_{\gamma \delta}] = [b_{\alpha \beta}^\dagger, b_{\gamma \delta}^\dagger] = 0.$$  

(9)

The Dyson boson image of an arbitrary fermion operator $\hat{O}_F$ is obtained by substituting all the fermion operators $[a_\alpha^\dagger, a_\alpha, a_\alpha^\dagger a_\beta, a_\alpha a_\beta]$ in their boson correspondences which are written as:

$$\langle a_\alpha^\dagger a_\beta \rangle_{DBM} = b_{\alpha \beta}^\dagger = b_{\alpha \beta} - \sum_{\gamma \delta} b_{\alpha \gamma}^\dagger b_{\beta \delta}^\dagger b_{\gamma \delta},$$

(10)

$$\langle a_\alpha a_\beta \rangle_{DBM} = b_{\beta \alpha},$$

(11)

$$\langle a_\alpha a_\beta \rangle_{DBM} = \rho_{\beta \alpha} = \sum_{\gamma} b_{\alpha \gamma}^\dagger b_{\beta \gamma}.$$

(12)

With the DBM procedure, one can obtain the boson image of the shell model Hamiltonian $\hat{H}_F$, which can be simply written as $\hat{H}_F = \hat{H}_0 + \hat{V}$, with $\hat{V}$ being a two-body operator. And so do all the other operators. Then all the calculations in fermion space can be correspondingly carried out in the ideal boson space with boson degrees of freedom.

To concerning explicitly the property of collective motion and the fact that each nuclear state holds a definite angular momentum in practical calculation, one usually take the collective boson $b_{JM}^\dagger$, which can be expressed as

$$b_{JM}^\dagger = \sum_{j_1 j_2} \chi(j_1 j_2) b_{JM}^\dagger(j_1 j_2),$$

(13)

where $b_{JM}^\dagger(j_1 j_2)$ is a combination of the ideal boson with relation

$$b_{JM}^\dagger(j_1 j_2) = (1 + \delta_{j_1 j_2})^{1/2} \sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | JM \rangle b_{j_1 m_1 j_2 m_2}^\dagger,$$

(14)
and $\chi(j_1 j_2)$ are structural coefficients which can be obtained by solving the secular equation of the single-body ideal boson Hamiltonian $H_0^{(IB)}$ in the collective boson space $b_{JM}^\dagger b_{JM}(0)$.

$$H_0^{(IB)}b_{JM}^\dagger b_{JM}(0) = E_J b_{JM}(0).$$

(15)

Eventually, the nucleon space and all the operators in shell model can be expressed in the collective boson space. It should be noted that the mapped single-body ideal boson Hamiltonian $H_0^{(IB)}$ (which is not explicitly shown because of its complexity) includes the contributions not only from the single-body fermion term but also from all the other two-body fermion terms. Therefore, the collective boson images of the operators reflect the information of single nucleon and the strengths of the interactions between nucleons, as well as the effective charges, i.e., all the input information of the shell model. The obtained results are then the manifestation of those in shell model with quite good approximation.

Furthermore, one also needs proper truncation for the collective boson space by truncating the $f$ space in practical calculation. In our present investigation, we take the $SD$ pair truncation, i.e., keep the operators with angular momentum $J = 0$ and $J = 2$, which has been shown to be quite successful in describing the properties of low-lying nuclear states with spherical shape and/or quadrupole deformation [34–36,38–40,26,41].

3. Identification of shape phases in shell model

Nuclear shape phases are the manifestations of the collective motion modes of nuclei. Every characteristic quantity takes the same value no matter the calculation is carried out in fermion space or boson space. Then, to identify the nuclear shape phases in the calculation at nucleon level, it is helpful to examine the correspondence between the calculated results in the shell model with DBM approximation and those in the dynamical symmetries of the IBM, since it has been shown that the vibration, the axial rotation, the $\gamma$-soft rotation corresponds to the symmetry $U(5)$, SU(3), O(6) in the IBM, respectively [3–5].

Quantities of interest are the normalized low-lying levels’ energies and the electric quadrupole transition rates. The low-lying levels’ energies are normalized to $E_1$, and the $B(E2)$s are normalized to $B(E2; 2_1^+ \rightarrow 2_1^-)$. For the convenience of expression, we denote the characteristic quantities as $R_{41} = \frac{E_{41}}{E_{21}}$, $R_{61} = \frac{E_{61}}{E_{21}}$, $R_{02} = \frac{E_{02}}{E_{21}}$, $R_{22} = \frac{E_{22}}{E_{21}}$, $B_{41,1} \equiv \frac{B(E2; 2_1^+ \rightarrow 2_2^-)}{B(E2; 2_1^+ \rightarrow 2_1^-)}$, $B_{61,1} \equiv \frac{B(E2; 6_1^+ \rightarrow 4_1^+)}{B(E2; 2_1^+ \rightarrow 0_1^-)}$, $B_{02,2} \equiv \frac{B(E2; 0_2^- \rightarrow 2_2^-)}{B(E2; 2_1^+ \rightarrow 0_1^-)}$, and $B_{22,2} \equiv \frac{B(E2; 2_2^\pm \rightarrow 2_2^-)}{B(E2; 2_1^+ \rightarrow 0_1^-)}$. It is known that these quantities can characterize the shape phase structure and transition, and some of them can even distinguish the first order from the second order phase transition (see, for example, Ref. [18]). Table 1 lists the values of these quantities in the dynamical symmetries of IBM (with total boson number $N = 10$), which are to be compared with those obtained in the microscopic calculations.

Besides, quadrupole moment $Q(2_1^-)$ is needed to discriminate the axially prolate rotational phase (with SU(3) symmetry in IBM) from the axially oblate rotational phase (with SU(3) symmetry in IBM), since all the quantities listed in Table 1 take the respective same value in the two phases. With an effective charge $1$, it has been shown that the value of $Q(2_1^-)$ is negative, positive in the prolate rotational phase, oblate rotational phase, respectively, and equates zero in the vibrational and $\gamma$-soft rotational phases (see, for example, Ref. [7]).

4. Calculation and numerical results

It has been known that the nuclei in the $A \sim 130$ and $A \sim 150$ mass region exhibit quite rich shape phases and their transitions. To investigate the influences of some modes of the nucleon–nucleon correlations (or effective interactions) on the shape phases, we take the shell model Hamiltonian in Eq. (1) and 20 nucleons (corresponding to the boson number 10 mentioned in last section, as an example) in the valence space $2d_{5/2}$, $1g_{7/2}$, $3s_{1/2}$, $1h_{11/2}$, $2d_{3/2}$, with energy 0.0, 0.8, 1.3, 2.5, 2.8 MeV, respectively, quite close to those used in Ref. [42]. The single-particle wave functions are taken to be the harmonic oscillator’s with oscillation constant $b^2 = 1.0A^{1/3} \text{ fm}^2$. In addition, for simplicity, protons and neutrons are not differentiated, and the single-particle orbits’ energies are set to be invariant in the whole calculations. The effective charge is set to be 1.0 in the electric properties’ calculations.

Since it is difficult to carry out the shell model calculation in such a large valence nucleon space directly, we employ the DBM approach. As mentioned in Section 2.2, although the final diagonalization is performed in boson space, all the information of the shell model Hamiltonian in Eq. (1) has been embodied in the mapped boson Hamiltonian by the mapping procedure (the mapped boson Hamiltonian is not given because of its complexity). By carrying out a series of calculations with various values of the interaction strengths and looking over the variation behaviors of the quantities mentioned in last section, We can analyze the effects of the interactions on the nuclear shape phases and their transitions.

4.1. Effect of monopole-pairing

We first look at the effect of monopole-pairing with parameters $g_0 \in (0, 0.50]$ and $g_2 = k = 0$. In Ref. [26], the monopole-pair interaction was directly used to correspond to the $U(5)$ phase. However, there is not such a direct equivalence from their algebraic structure. Furthermore, the one-body term in the Hamiltonian may also influence the shape phase of the system.

Fig. 1 illustrates the calculated results of the dependence of the energy levels and the normalized ones on the monopole-pair interaction strength $g_0$. Fig. 2 displays the $g_0$ dependence of the normalized $E2$ transition rates. From the two figures, one can notice easily that the degenerate level structure of the vibrational states ($U(5)$ symmetric states in the IBM), such as the $2_2^-$–$4_2^-$, and the $0_2^-$–$2_1^+$–$4_1^+$, are reproduced excellently as $g_0 > 0.15$, and the normalized energies and $B(E2)$s are apparently $U(5)$-symmetrically valued. So a large $g_0$ favors a spherical phase. By the way, such a critical value of $g_0$ agrees well with the empirical value [43] $g_0 \sim 0.20/A$, which means that the monopole-pair interaction upholds the spherical phase in realistic nuclei.

One can also see from Figs. 1 and 2 that, as $g_0$ is very small, the calculated results show approximately the feature of the axially rotational phase. To differentiate the axially prolate from the axially oblate shape, we display the calculated variation behavior of the quadrupole moment $Q(2_2^-)$ with respect to the $g_0$ in Fig. 3. The figure shows evidently that the value of $Q(2_2^-)$ is positive definite when $g_0$ is very small. It indicates that a nucleon system with very weak monopole-pair interaction may present an axially oblate deformed shape. Here, it should be noted that the point $g_0 = 0$ is not included in the figures, since $g_0 = 0$ means only the single-body term remains and there would then not exist any collective state.

1 We display here and in the next two subsections only the results of the nuclei with $A = 130$ as examples. Those of the nuclei with $A = 150$ exhibit very similar behaviors (the concrete data are available if required).
Table 1
Values of interested quantities in several collective modes of the nucleus with 10 pairs of nucleons (10 bosons in the IBM). Those marked with a star depend on additional parameters in the Hamiltonian (as given in Ref. [12]).

<table>
<thead>
<tr>
<th>Mode</th>
<th>$R_{11}$</th>
<th>$R_{61}$</th>
<th>$R_{22}$</th>
<th>$R_{42}$</th>
<th>$B_{41}$</th>
<th>$B_{02}$</th>
<th>$B_{62}$</th>
<th>$B_{22}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vibration (U(5))</td>
<td>2.00</td>
<td>3.00</td>
<td>2.00</td>
<td>2.00</td>
<td>1.80</td>
<td>2.40</td>
<td>1.80</td>
<td>1.80</td>
</tr>
<tr>
<td>$\gamma$-soft rotation (O(6))</td>
<td>2.50</td>
<td>4.50</td>
<td>4.50</td>
<td>2.50</td>
<td>1.38</td>
<td>1.52</td>
<td>0.00</td>
<td>1.38</td>
</tr>
<tr>
<td>Axial rotation (SU(3))</td>
<td>3.33</td>
<td>7.00</td>
<td>23.7*</td>
<td>24.7*</td>
<td>1.40</td>
<td>1.48</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

The single-body term, which can’t be omitted in the calculation, may affect the phases by providing a background field, although its precise effect is still unclear.

It should also be mentioned that, in the region $g_0 > 0.15$, the $Q(2^+_1)$ has small positive values, rather than exact zero. These relative small positive values of $Q(2^+_1)$ may be the contribution of the sing-body interaction and needs further investigation. Anyway, the calculated result displayed in Fig. 3 manifests that, with an arbitrary positive effective charge (it is set to be 1.0 in our calculation), the values of $Q(2^+_1)$ in $g_0 > 0.15$ region are much smaller than those in the region with very small $g_0$. The claim that $g_0 > 0.15$ region corresponds to the spherical phase and very small $g_0$ to the axially oblate deformed phase is based on such a drastical changing behavior.

Fig. 1. (Color online.) Calculated result of the dependence of the low-lying levels’ energies and the normalized energies $R_{41}$, $R_{61}$, $R_{02}$, and $R_{22}$ on the monopole-pair interaction strength $g_0$ when $g_2 = k = 0$.

Fig. 2. (Color online.) Calculated result of the dependence of the normalized $B(E2)$s $B_{412}$, $B_{614}$, $B_{022}$, and $B_{222}$ on the monopole-pair interaction strength $g_0$ when $g_2 = k = 0$.

Fig. 3. Calculated result of the dependence of quadrupole moment $Q(2^+_1)$ on the $g_0$ when $g_2 = k = 0$.

Fig. 4. (Color online.) Calculated result of the dependence of the low-lying levels’ energies and the normalized ones $R_{41}$, $R_{61}$, $R_{02}$, and $R_{22}$ on the quadrupole-pair interaction strength $g_2$ when $g_0 = 0.15$ and $k = 0$.

4.2. Effect of quadrupole-pairing

Next we study how the quadrupole-pair interaction affects the shape phase structure, by taking parameter $g_2 \in [0, 0.15]$ with $g_0 = 0.15$ and $k = 0$. Note that $g_2 < g_0$ holds true for most realistic depictions of nuclei [21]. Since $g_0 = 0.15$ with $g_2 = k = 0$ corresponds to the spherical phase, our calculation here shows in fact the effect of the quadrupole-pair interaction on the spherical phase.

The calculated results of the effects of the quadrupole-pair interaction strength $g_2$ on the low-lying energy levels (together with the normalized ones) and the normalized E2 transition rates are shown in Figs. 4, 5, respectively. The two figures show apparently that, for the quadrupole-pair interaction strength $g_2 \in (0, 0.03)$, the system is in a vibrational phase approximately. For $g_2 \in (0.06, 0.10)$, the characteristic quantities take values corresponding to those of the axially rotational phase listed in Table 1. And Fig. 6 shows that the value of quadrupole moment $Q(2^+_1)$...
is negative as \( g_2 \in (0.06, 0.10) \) region. It manifests that, if \( g_2 \in (0.06, 0.10) \), the nucleon system is in the axially prolate rotational phase. The increase of the quadrupole-pair interaction strength in the region \( g_2 \in (0.03, 0.06) \) induces a shape phase transition from the vibrational to the axially prolate rotational. Furthermore, when \( g_2 \) is close to the upper limit of the calculation (0.15), the parts (b) and (c) of Fig. 4 and Fig. 5 show that the normalized energies and \( B(E2) \)s of the low-lying states do not take any corresponding characteristic value listed in Table 1. It indicates that the nucleon system in such cases is not in the axially rotational phase but holds any other pure mode of collective motion (the dynamical symmetry in the IBM). Therefore, increasing the quadrupole-pair interaction to the \( g_2 > 0.10 \) region will generate the mixture of the three modes of collective motions.

### 4.3. Effect of quadrupole–quadrupole interaction

Finally, we analyze the influence of the quadrupole–quadrupole interaction on the nuclear shape phases. The quadrupole–quadrupole interaction coincides with Elliott’s SU(3) model [44], where the \( Q_2 \) is a quadrupole tensor in the SU(3) symmetry. However, whether the system maintains definitely the axially rotational phase (SU(3) symmetric phase) in a more realistic model including single-body term is still necessary to be checked. We then take such an examination. The calculated results of the variation behaviors of the low-lying states’ energies together with the normalized ones, the normalized \( B(E2) \)s, and the quadrupole moment \( Q(2^+_1) \) with respect to the quadrupole–quadrupole interaction strength \( k \) (with parameters \( k \in (0, 0.5) \) and \( g_0 = g_2 = 0 \)) are displayed in

![Fig. 5](image-url)  
**Fig. 5.** The same as Fig. 4 but for the normalized \( B(E2) \)s \( B_{4121}, B_{6411}, B_{0221} \) and \( B_{2221} \).

![Fig. 6](image-url)  
**Fig. 6.** The same as Fig. 4 but for the quadrupole moment \( Q(2^+_1) \).

Figs. 7, 8 and 9. It should be mentioned that the \( k = 0 \) point is not included in these figures and \( k \) is limited to below 0.20 in part (a) of Fig. 7 for a better view of the energy levels. From these figures, one can recognize apparently that there is a special point \( k \approx 0.10 \) at which all the quantities listed in Table 1 are approximately symmetric. Specifically, this point corresponds to the \( \gamma \)-soft rotational phase since all the characteristic quantities agree excellently with the corresponding values listed in Table 1, and the \( 02-31-42-61 \) quartet appears evidently (it’s \( 03-31-42-61 \) quartet in vibrational (U(5) symmetric) phase). Moreover, the quadrupole moment \( Q(2^+_1) \) takes the value zero.

Figs. 7 and 8 also manifest that, when \( k \) takes a value much smaller or larger than 0.1, the axially rotational phases become prevailing, since the normalized energies and the normalized \( B(E2) \)s of the low-lying states in axially rotational phase listed in Table 1 are well reproduced. Furthermore, Fig. 9 shows that, as the quadrupole–quadrupole interaction strength increases from below to above 0.1, the value of \( Q(2^+_1) \) decreases from positive to zero and then to negative. It indicates that a transition from the axially oblate rotational to the \( \gamma \)-soft rotational and further to the axially prolate rotational occurs. In the IBM, the transition from axially oblate to axially prolate shape is identified as a first-order phase transition, and the \( \gamma \)-soft rotational phase is shown to be the critical point [7]. Our microscopic calculation confirms the existence of such a shape phase transition (although the order of the phase
4.4. Correspondence between microscopic parameter settings and shape phases

Our calculations and above discussions manifest that there exists evidently a correspondence between the strength parameter space of each interaction and the nuclear shape phases. Such a correspondence can be summarized in Table 2. The results listed in the table indicates that, as the monopole-pair interaction is dominant (the quadrupole-pair interaction strength is less than 20% of the monopole one and the quadrupole–quadrupole interaction can be neglected), the nuclei appear in a spherical shape. If the quadrupole-pair interaction strength is in the region about 40% to 60% of the monopole-pair interaction strength or the quadrupole–quadrupole interaction is sufficiently strong, the nuclei would be in axially prolate deformed shape. If the quadrupole–quadrupole interaction strength takes a special value and the monopole-pair and quadrupole-pair interactions can be ignored, the nuclei can be in γ-soft rotational phase. If the monopole-pair interaction or the quadrupole–quadrupole interaction is very weak, the nuclei are in axially oblate shape. It is remarkable here that, the concrete values listed in Table 2 and mentioned above would be varied with calculations in different configurations and parameters of single nucleon states, but the general dependence of the shape phases on each of the interactions would not be changed.

By the way, seeing from Table 2, one can notice that the axially oblate shape appears only as the monopole-pair interaction and the quadrupole–quadrupole interaction are very weak, and the γ-soft rotational phase emerges only if the quadrupole–quadrupole interaction strength takes a special value and the other two modes of interactions can be neglected. So small parameter spaces for the axially oblate rotational and the γ-soft rotational phases to appear can help us understand the fact that the nuclei possessing purely each of these two modes of collective motions are quite rare in nature.

5. Summary and remarks

In summary, we have presented a microscopic analysis on the shape phases of nuclei and their transitions, as well as their dependence on the basic interactions. A correspondence between the strength parameter space of each interaction and various nuclear shape phases is obtained. It shows that the vibrational phase is mainly sustained by the monopole-pair interaction. The axially prolate deformation can be induced by quadrupole-pairing and/or quadrupole–quadrupole interactions. The γ-soft rotational phase can arise from a special quadrupole–quadrupole interaction. And the axially oblate rotational phase can appear only as the monopole-pairing and the quadrupole–quadrupole interactions are very weak and there isn’t a quadrupole-pair interaction.

Moreover, our calculation manifests that varying the interaction strengths can induce shape phase transitions. Concretely, the quadrupole-pairing interaction can induce the transition from the vibrational to the axially prolate rotational phase, the quadrupole–quadrupole interaction can drive the transition from the axially oblate to the axially prolate deformed shape phase, with the γ-soft rotational phase being the critical point.

Finally, it should be mentioned that the present microscopic calculation is carried out with the help of the Dyson boson mapping approach, in which two approximations are applied compared with direct shell model calculation. One is the bosonization of the nucleon pairs, and the other is the truncation of the model space. Although our description of the mapping procedure and analyses of the numerical results show that the results stem from the effective nucleon–nucleon interactions, whether some of the conclusions are specifically related to the approximations (especially the bosonization of the nucleon pairs) still needs to be checked in full shell model calculations. A practical examination at present stage can be carried out in the projected shell model [28] since the numerical task is not so heavy as that in full shell model in the case of very large nucleus space as taken in this Letter. It should also be remarked that, only the dependence of shape phases on each of the interactions is analyzed in the present Letter. It would be very interesting to explore the integrative effects of two or three interactions and investigate the complete nuclear shape phase structure in boson mapping approximation of shell model. The related investigations are in progress.

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