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Normally contracting Lie group actions

Takashi Inaba[a,*,1], Shigenori Matsumoto[b,2], Yoshihiko Mitsumatsu[c,3]

[a] Department of Mathematics and Informatics, Graduate School of Science, Chiba University, Chiba 263-8522, Japan
[b] Department of Mathematics, College of Science and Technology, Nihon University, 1-8-14 Kanda, Surugadai, Chiyoda-ku, Tokyo 101-8308, Japan
[c] Department of Mathematics, Faculty of Science and Engineering, Chuo University, 1-13-27, Kasuga, Bunkyo-ku, Tokyo 112-8551, Japan

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ABSTRACT

We show that there are no normally contracting actions of unimodular Lie groups on closed manifolds.

1. Introduction

Let $\varphi : M \times G \to M$ be a locally free right C^∞ action of a connected Lie group G of dimension n on a closed C^∞ manifold M of dimension $n+s+u$, with orbit foliation $\mathcal{F}$. Endow M with a Riemannian metric. For a linear map $A : V \to W$ of normed linear spaces, we denote by $\|A\|$ the operator norm of A, and by $m(A)$ the conorm of A, i.e.

$$m(A) = \inf\bigl\{\|Av\| \bigm| v \in V,\ \|v\| = 1\bigr\}.$$

Denote by $T\mathcal{F}$ the tangent bundle of $\mathcal{F}$.

According to [2], the action φ is called an *Anosov action* if there is an element $g \in G$, called an *Anosov element*, and continuous subbundles E^s and E^u of the tangent bundle TM of M such that

(1) $E^s \oplus E^u \oplus T\mathcal{F} = TM$,
(2) E^s and E^u are invariant by the tangent map g_* of the action of g, and

* Corresponding author.
E-mail addresses: inaba@math.s.chiba-u.ac.jp (T. Inaba), matsumo@math.cst.nihon-u.ac.jp (S. Matsumoto), yoshi@math.chuo-u.ac.jp (Y. Mitsumatsu).

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(3) there exist $C > 0$ and $\lambda > 0$ such that for any $k > 0$ and $x \in M$, we have

$$\left\| g_*^k\big|_{E_x^s}\right\|, \left\| g_*^{-k}\big|_{E_x^u}\right\| \leqslant C\exp(-k\lambda), \tag{1.1}$$

$$\frac{\|g_*^k|_{E_x^s}\|}{m(g_*^k|T_x\mathcal{F})}, \frac{\|g_*^{-k}|_{E_s^u}\|}{m(g_*^{-k}|T_x\mathcal{F})} \leqslant C\exp(-k\lambda). \tag{1.2}$$

As is well known, conditions (1.1) and (1.2) do not depend upon a particular choice of the Riemannian metric. The case where $G = \mathbb{R}^n$ has drawn attention of many authors. In that case, condition (1.2) follows from (1.1) and is redundant. But for general G, (1.2) is natural and turns out to be useful for example to get the topological transitivity of the action under a suitable condition [2].

This paper arises from an effort of the authors to understand the proof of Corollary 7 in [2]. Instead we found an argument to show the following theorems, which improve Corollaries 7, 8 and Theorem 9 in [2]. An Anosov action φ is called a *normally contracting action* if the dimension u of the bundle E^u is 0 and $s > 0$. Examples of such actions are given in Section 2.

Theorem 1.1. *If there exists a normally contracting action of a Lie group G on a closed manifold, then G is not unimodular.*

In [2], a normally contracting action is called *central* if an Anosov element g is in the center of G, and the nonexistence for $s = 1$ and $n = 2$ and the minimality for $s = 1$ of such actions are shown. As a by-product of our argument we obtain:

Theorem 1.2. *There are no central normally contracting actions of any Lie groups on closed manifolds.*

In fact in these theorems we do not use condition (1.2).

Section 3 is devoted to the proof of these theorems. In Section 4, we raise a further problem.

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2. Examples

Example 2.1. Let H be either $PSL(2,\mathbb{R})$ or the 3-dimensional solvable Lie group Solv_3 which admits a cocompact lattice. Then H contains

$$\mathrm{Aff}_+(\mathbb{R}) = \left\{ \begin{pmatrix} e^{-t/2} & x \\ 0 & e^{t/2} \end{pmatrix} \,\Big|\, t, x \in \mathbb{R} \right\},$$

a closed subgroup isomorphic to the group of the orientation preserving affine transformations of the real line. For any cocompact lattice Γ of H, consider the right action φ of $G = \mathrm{Aff}_+(\mathbb{R})$ on the quotient manifold $M = \Gamma \setminus H$. The flow given by the action of

$$g^t = \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix}$$

is a classical example of Anosov flows, and the orbit foliation of the G-action φ is the weak stable foliation of the Anosov flow $\{g^t\}$. Therefore the G-action φ is normally contracting, with an Anosov element g.

Example 2.2. M. Asaoka [1] classified all the locally free actions of $G = \mathrm{Aff}_+(\mathbb{R})$ on closed 3-manifolds up to C^∞-conjugacy. Notably there are actions which do not preserve a smooth volume form. But they all have the same dynamical properties as in Example 2.1, and especially the G-actions are normally contracting.

Example 2.3. Let $A \in SL(n+s-1,\mathbb{Z})$, $n \geqslant 2$, $s \geqslant 1$, be a matrix with $n-1$ distinct simple eigenvalues bigger than 1 and all the others in $(0,1)$. We regard A as an automorphism of the torus T^{n+s-1}. Then the suspension of A, denoted by M, is an $(n+s)$-dimensional manifold which admits an Anosov flow. The weak unstable foliation of the flow is the orbit foliation of an action of a metabelian Lie group G of dimension n. The action of G on M is normally contracting.

Example 2.4. Let Γ be a cocompact lattice of $H = PSL(2,\mathbb{C})$. The Lie group H contains a closed subgroup $G = \mathrm{Aff}(\mathbb{C})$, isomorphic to the group of the complex affine transformations of the complex plane. The right action of G on the quotient space $M = \Gamma \setminus H$ is normally contracting.

Example 2.5. Suppose there is a normally contracting action of a Lie group G_1 on a closed manifold N, with an Anosov element g. Let G_2 be a Lie group which admits a cocompact lattice Γ. Then the product right action of $G = G_1 \times G_2$ on $M = N \times (\Gamma \setminus G_2)$ is normally contracting, with an Anosov element (g, e_{G_2}), where e_{G_2} denotes the unit element of G_2.

Example 2.6. Let G_1 be any semisimple Lie group and let H_2 be either $PSL(2,\mathbb{R})$ or $PSL(2,\mathbb{C})$. Let G_2 be either $\mathrm{Aff}_+(\mathbb{R})$ or $\mathrm{Aff}(\mathbb{C})$ accordingly, and choose an irreducible cocompact lattice Γ of $G_1 \times H_2$. Then the right action of $G = G_1 \times G_2$ on $M = \Gamma \setminus (G_1 \times H_2)$ is normally contracting.

3. Proofs of the main theorems

First let us outline the proof of Theorem 1.1. Let φ be a normally contracting action of a unimodular n-dimensional Lie group G on a closed manifold M of dimension $n+s$, where s is the dimension of the stable bundle E^s. Since G is unimodular, G admits a bi-invariant volume form Ω. Transmitting Ω by the action φ, one obtains a continuous φ-invariant n-form ω on M such that ω yields a volume form on $T\mathcal{F}$ and that $\iota_v\omega = 0$ for any $v \in E^s$. (Notice that this can be done only when Ω is bi-invariant.) Next we construct a continuous s-form α such that $\iota_v\alpha = 0$ for any $v \in T\mathcal{F}$ and that $\alpha \wedge \omega$ is nonvanishing. Then the φ-invariance of ω and the normally contracting property of φ imply that the volume form $\alpha \wedge \omega$ is contracting by the action of an Anosov element g, contradicting the compactness of M.

For $h \in G$, denote by $L_h : G \to G$ (resp. $R_h : G \to G$) the left (resp. right) translation by h. Let $X_1, \ldots, X_n$ be left invariant vector fields on G which form a linear basis of the tangent space at each point of G. Thus we have for any $h \in G$,

$$(L_h)_* X_i = X_i.$$

Now assume for contradiction that G is unimodular. Consider the volume form Ω on G uniquely defined by

$$\Omega(X_1, \ldots, X_n) = 1.$$

Since G is unimodular, the form Ω is not only left invariant, but also right invariant. That is, for any element $h \in G$,

$$\Omega\big((R_h)_* X_1, \ldots, (R_h)_* X_n\big) = 1.$$

Let $\varphi : M \times G \to M$ be a normally contracting action. Thus we have

$$TM = E^s \oplus T\mathcal{F}.$$

For any point $x \in M$, consider a map $\psi_x : G \to M$ defined by

$$\psi_x(h) = xh,$$

where xh is a customary abbreviation of $\varphi(x,h)$. Define vector fields $X_1^*, \ldots, X_n^*$ of M tangent to the orbit foliation $\mathcal{F}$ by

$$\big(X_i^*\big)_x = (\psi_x)_*(X_i)_e,$$

where e denotes the identity element of G.

Define a continuous n-form ω of M so as to satisfy

$$\omega\big(X_1^*, \ldots, X_n^*\big) = 1 \quad \text{and} \quad \iota_v\omega = 0, \quad \forall v \in E^s.$$

Notice that ω is uniquely defined and that ω_x $(x \in M)$ satisfies

$$\omega_x\big((\psi_x)_* Y_1, \ldots, (\psi_x)_* Y_n\big) = \Omega_e(Y_1, \ldots, Y_n)$$

for any $x \in M$ and vector fields Y_i on G. Still denote by $g : M \to M$ the right action of an Anosov element $g \in G$. The derivative g_* preserves the subbundles E^s and $T\mathcal{F}$. Notice also that

$$g\psi_x = \psi_{xg} R_g L_g^{-1}.$$

Lemma 3.1. *The n-form ω is invariant by the action of an Anosov element g.*

Proof. Since $g^*\omega$ satisfies

$$\iota_v\big(g^*\omega\big) = \iota_{g_*v}\omega = 0, \quad \forall v \in E^s,$$

it suffices to show

$$\big(g^*\omega\big)\big(X_1^*, \ldots, X_n^*\big) = 1.$$

But we have

$$\begin{aligned}
\big(g^*\omega\big)_x\big(X_1^*, \ldots, X_n^*\big) &= \omega_{xg}\big(g_*X_1^*, \ldots, g_*X_n^*\big) \\
&= \omega_{xg}\big(g_*(\psi_x)_*X_1, \ldots, g_*(\psi_x)_*X_n\big) \\
&= \omega_{xg}\big((\psi_{xg})_*(R_g)_*(L_g)_*^{-1}X_1, \ldots, (\psi_{xg})_*(R_g)_*(L_g)_*^{-1}X_n\big) \\
&= \Omega_e\big((R_g)_*(L_g)_*^{-1}X_1, \ldots, (R_g)_*(L_g)_*^{-1}X_n\big) \\
&= \Omega_e\big((R_g)_*X_1, \ldots, (R_g)_*X_n\big) \\
&= \Omega_e(X_1, \ldots, X_n) = 1,
\end{aligned}$$

as is required. □

Assume the bundle E^s is nonoriented. There is a double covering of M such that the lift of E^s is oriented. The action φ can also be lifted to the double covering, and it is again a normally contracting action. Since we are going to deduce a contradiction from the hypothesis that G is unimodular, we can very well assume that the bundle E^s is oriented. Thus M is oriented as well. Then one can choose a nonvanishing s-form α of M which satisfies

$$\iota_{X_1^*}\alpha = \cdots = \iota_{X_n^*}\alpha = 0 \quad \text{and} \quad \alpha \wedge \omega > 0.$$

Notice that α is unique up to a multiple of positive function. For any $x \in M$ define $\|\alpha_x\|$ by

$$\|\alpha_x\| = \sup\{|\alpha_x(v_1, \ldots, v_s)| \mid v_i \in T_xM,\ \|v_i\| = 1,\ 1 \leqslant \forall i \leqslant s\}.$$

Lemma 3.2. *We have*

$$\|(g^k)^*(\alpha_x)\| \leqslant C_1 \exp(-s\lambda k)\|\alpha_x\|,$$

for some $C_1 > 0$.

Proof. Of course the validity of the lemma does not depend upon a choice of a Riemannian metric. So let us choose a Riemannian metric for which the two subbundles $T\mathcal{F}$ and E^s are orthogonal at each point of M. Then since

$$\iota_w((g^k)^*\alpha_x) = 0 \quad \text{and} \quad \|v + w\| \geqslant \|v\|, \quad \forall v \in E^s_{xg^{-k}},\ \forall w \in T_{xg^{-k}}\mathcal{F},$$

there are nonzero vectors $v_1, \ldots, v_s \in E^s_{xg^{-k}}$ such that

$$|((g^k)^*\alpha_x)(v_1, \ldots, v_s)| = \|(g^k)^*\alpha_x\| \cdot \|v_1\| \cdots \|v_s\|.$$

On the other hand we have

$$\begin{aligned} |((g^k)^*\alpha_x)(v_1, \ldots, v_s)| &= |\alpha_x((g^k)_* v_1, \ldots, (g^k)_* v_s)| \\ &\leqslant \|\alpha_x\| \cdot \|(g^k)_* v_1\| \cdots \|(g^k)_* v_s\| \\ &\leqslant C^s \exp(-sk\lambda)\|\alpha_x\| \cdot \|v_1\| \cdots \|v_s\|, \end{aligned}$$

showing the lemma. □

Now Lemmata 3.1 and 3.2 imply that the volume form $\alpha \wedge \omega$ is everywhere contracting by $(g^k)^*$ for large k. The contradiction shows Theorem 1.1.

Even if the Lie group is not unimodular, the above argument works without any change if an Anosov element g belongs to the kernel of the modular function, that is, if $(R_g)^*\Omega = \Omega$. Therefore we have:

Theorem 3.3. *An Anosov element of a normally contracting action does not belong to the kernel of the modular function.*

Theorem 1.2 follows from this.

4. A further problem

Among examples in Section 2, genuine vanillas are only Examples 2.1–2.4, in which the Lie group G is not a product of two Lie groups. If G is not a product, it must be either solvable or simple by a theorem of Levi. If further there is a normally contracting action of G, then by Theorem 1.1, G must be solvable.

In Examples 2.1–2.3, the one parameter subgroup $\{g^t\}$ of G which contains an Anosov element g defines an Anosov flow on the ambient manifold M. This is not the case with Example 2.4 since there is a compact subgroup K, isomorphic to $U(1)$, which commutes with g^t. However if one takes a quotient M/K of M by the action of K, then $\{g^t\}$ defines an Anosov flow on M/K. These examples lead us to the following question.

Question 4.1. Let G be a Lie group which is not a product of two Lie groups. Assume there is a normally contracting G-action on a closed manifold M. Does there exist a compact subgroup K of G commuting with an Anosov element g such that the one parameter subgroup $\{g^t\}$ defines an Anosov flow on the quotient space M/K?

In fact this is the case with the Lie group $\mathrm{Aff}_+(\mathbb{R})$. For, any element which does not belong to the kernel of the modular function is conjugate to

$$\begin{pmatrix} e^{-t_0/2} & 0 \\ 0 & e^{t_0/2} \end{pmatrix},$$

for some $t_0 \neq 0$. Now the flow given by the action of the one parameter subgroup

$$\left\{ \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$$

is Anosov since it is "normally expanding" along the orbit foliation.

Likewise for $G = \mathrm{Aff}(\mathbb{C})$, Question 4.1 is affirmative. To mention one more example, let G be a simply connected Lie group whose Lie algebra is generated by X and $U_1, \ldots, U_{n-1}$ such that for any $1 \leqslant i \leqslant j \leqslant n-1$,

$$[X, U_i] = -\alpha_i U_i \quad (\alpha_i > 0) \quad \text{and} \quad [U_i, U_j] = 0.$$

Then Question 4.1 is in the affirmative for any normally contracting action of G.

The Lie group G in Question 4.1 is solvable and does not admit a lattice. Question 4.1 is more about the structure of the Lie group than about the dynamics of the action.

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