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Module Structure of Constant Linear Systems and Its Applications to Controllability*

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We shall introduce a new module structure to a large class of continuous-time constant linear systems. This is done as a natural extension of the classical $k[z]$-module structure of finite-dimensional constant linear systems. This module action is used to investigate the relationship between reachability and controllability of linear systems. After introducing the notion of K-controllability due to Kamen [12], we give the following result in Section 5: If a constant linear system is described by a functional differential equation $\dot{x} = Fx + Gu$, where x and G belong to a Banach space X, and if G is K-controllable to zero, then every reachable state is reachable and controllable in bounded time. (The result given in Section 5 is a little more general than this.) We also give a simple example in Section 6 to illustrate this result.

1. INTRODUCTION

It is a common understanding that the theory of finite-dimensional linear systems is adequately algebraized and many problems, if not all, can be solved via algebraic methods in this framework (see, for example, Kalman *et al.* [9]). Among all, module theory has proved very successful in realization theory for this class of systems. On the other hand, infinite-dimensional linear systems have been studied mainly via functional–analytic methods: Baras and Brockett [1]; Brockett and Fuhrmann [4]; Fuhrmann [7]; Manitius and Triggiani [14]; Triggiani [20, 21]; Yamamoto [23a, b]; etc.

There is, however, another trend which attempts to study infinite-dimensional linear systems with various algebraic techniques. Kalman and Hautus [10] may be the first to consider a module structure of continuous-time linear systems for the use in realization theory. Kamen [12] introduced a slightly different module structure for an even larger class of systems and used this successfully to derive interesting results (for example, a criterion for bounded-time reachability of a system). The same type of approach is

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pursued also in Denham and Yamashita [5] for a class of delay-differential systems.

While Kamen's work is of definite fundamental importance, consideration of continuity of the module action is left out. But continuity is important especially when we deal with infinite-dimensional systems because one might draw quite a wrong conclusion due to a small error in parameters of the system if the underlying operation were discontinuous.

The present study follows essentially the same spirit of Kamen's work, but also attempts to unite it with functional–analytic aspects of the problem. As a result, we can define a module action on systems without relying directly on the convolution structure of the input space. (A similar approach has been carried out in Fuhrmann [8] for those systems whose state transition is generated by a self-adjoint operator in a Hilbert space.)

In Section 2, we fix our terminology and notations. Section 3 is devoted to the study of the extension of a module action of linear combinations of δ-measures to that of measures of compact support. As a by-product, we obtain some properties of convolution in the space of measures of compact support contained in $(-\infty, 0]$. Our central result in this section is Theorem 3.28 which states, loosely speaking, that constant linear systems admit $M_c(\mathbf{R}^-)$-module structure, where $M_c(\mathbf{R}^-)$ denotes the set of measures of compact support in $(-\infty, 0]$. In Section 4, we shall further investigate convolution in the space of measures whose support is bounded on the left. Results in this section along with those in Section 3 enable us to study reachability and controllability algebraically. After defining the notion of K-controllability due to Kamen [12], we prove the following result in Section 5: *If a single element* (*called G*) *of the state space is K-controllable to zero, then every reachable state is reachable and controllable in bounded time.* We also give a simple example in Section 6 to illustrate this result.

2. Mathematical Preliminaries and Basic Framework

Throughout this work k denotes a fixed field, either $\mathbf{R}$, the field of real numbers, or $\mathbf{C}$, the field of complex numbers. We are mainly concerned with locally convex Hausdorff topological vector spaces over k. For results and notations on locally convex spaces we shall mainly rely on Bourbaki [3], Schaefer [16], or Treves [19].

Let $\langle X, Y\rangle$ denote a *duality* between locally convex spaces X and Y, i.e., there exists a separately continuous bilinear mapping

$$\langle \cdot, \cdot \rangle : X \times Y \to k$$

such that $\langle x, y \rangle = 0$ for all $y \in Y$ implies $x = 0$, and $\langle x, y \rangle = 0$ for all $x \in X$ implies $y = 0$. For a subset B of X, B° denotes the *absolute polar* of B, i.e.,

$$B^\circ := \{y \in Y : |\langle x, y \rangle| \leqslant 1 \quad \text{for all} \quad x \in B\}.$$

If C is a subset of Y, then C° is the subset of X:

$$C^\circ := \{x \in X : |\langle x, y \rangle| \leqslant 1 \quad \text{for all} \quad y \in C\}.$$

Note that X and X' (the dual space of X) always form a duality with respect to the standard duality $\langle x, x^* \rangle$, $x \in X$, $x^* \in X'$, where $\langle x, x^* \rangle$ denotes the value of x^* at x. Unless otherwise stated, duality is always understood in this sense. So for a subset B in X, B° is taken with respect to this duality and is a subset of X' even if the duality $\langle X, X' \rangle$ is not explicitly specified.

The following notational conventions will be adopted in the sequel:

$\mathbf{R}^- := (-\infty, 0]$.

$C(-\infty, 0]$ (or $C(\mathbf{R}^-)$) := the space of continuous functions on $(-\infty, 0]$; this space is endowed with the topology of uniform convergence on each compact subset of $(-\infty, 0]$.

$C[0, \infty)$:= the space of continuous functions on $[0, \infty)$ whose topology is given by the uniform convergence on each compact subset of $[0, \infty)$.

$M_c(-\infty, 0]$ (or $M_c(\mathbf{R}^-)) := (C(-\infty, 0])'$, i.e., the dual space of $C(-\infty, 0]$; it is easy to see that this is the space of Radon measures with compact support contained in $(-\infty, 0]$.

$L^2[a, b]$:= the space of all k-valued Lebesgue square integrable functions on $[a, b]$; similarly for $L^2[a, b)$, etc.

$\Omega := \bigcup_{n=1}^{\infty} L^2[-n, 0]$; we endow this space with the topology of strict inductive limit of the sequence $\{L^2[-n, 0]\}_{n=1}^{\infty}$, i.e., $\Omega = \lim_{\rightarrow} L^2[-n, 0]$.

The basic object we study is a constant (time-invariant) linear continuous-time system, which is rigorously defined as follows:

DEFINITION 2.1. A (single-input) *constant linear* (*continuous-time*) *system* (without output) is a pair $\Sigma = (X, \varphi)$ such that

(i) X is a complete locally convex Hausdorff space;

(ii) for each $t \geqslant 0$, $\varphi(t, \cdot, \cdot): X \times L^2[0, t) \rightarrow X$ is a (jointly) continuous linear map (for $t = 0$, $\varphi(0, x, \cdot) = x$ for all x, is required).

(iii) for every $t, s \geqslant 0$, φ satisfies

$$\varphi(t + s, x, u) = \varphi(t, \varphi(s, x, u|_{[0,s)}), \sigma_s^l u|_{[0,t)})$$

for all x in X, u in $L^2[0, t + s)$, where $(\sigma_s^l u)(t) := u(t + s)$.

(iv) $\lim_{t \to t_0} \varphi(t, x, 0) = \varphi(t_0, x, 0)$ for every $t_0 \geqslant 0$ and $x \in X$.

X is called the *state space*, φ the *state-transition map* of Σ.

It is useful to keep the following dynamical interpretation in mind: The elements of the state space X serve as a memory device of the system, and when a new input comes in, the state is accordingly modified to reflect the change. The rule for this is governed by the state-transition map φ. Of course, every system must produce outputs (and outputs are given corresponding to the present state), but since we shall be concerned only with reachability and controllability of systems, the output equation is somewhat immaterial for our present purpose, and hence is dropped from the definition. For a more detailed discussion on the output equation, see Yamamoto [23a].

It is plain to see that to a given system $\Sigma = (X, \varphi)$ there is associated a unique strongly continuous semigroup $\{\Phi(t)\}_{t \geqslant 0}$ of X such that $\varphi(t, x, u) = \Phi(t)x + \varphi(t, 0, u)$ (indeed, put $\Phi(t)x := \varphi(t, x, 0)$ for each $t \geqslant 0$). We call this semigroup the *associated semigroup* of the system Σ. In what follows, an associated semigroup $\{\Phi(t)\}_{t \geqslant 0}$ is usually assumed to be *locally equicontinuous*, i.e., for each $T > 0$ and a continuous seminorm p of X, there exists a continuous seminorm q of X such that

$$p(\Phi(t)x) \leqslant q(x) \qquad \text{for all} \quad x \in X \qquad \text{and} \qquad t \in [0, T].$$

Remark 2.2. This class of semigroups is not overly special. Indeed, it is known (Kōmura [3]) that when X is a barrelled space (which in particular includes Banach spaces), every strongly continuous semigroup is always locally equicontinuous.

The following immediate consequence of Definition 2.1 will be extensively used in the subsequent sections.

PROPOSITION 2.3. *Let $\Sigma = (X, \varphi)$ be a constant linear system. Then there exists a continuous linear map $g \colon \Omega \to X$ such that*

$$\varphi(t, x, u) = \Phi(t)x + g(\sigma_t^l u) \tag{2.4}$$

for all $x \in X$ and $u \in L^2[0, t)$.

Proof. See Yamamoto [23a]. (Put $g(\omega) := \varphi(t, 0, \sigma_t^r \omega)$ for $\omega \in L^2[-t, 0]$, where σ_t^r denotes the right shift operator: $(\sigma_t^r \omega)(s) := \omega(s - t)$.) ∎

Loosely speaking, the above-defined g gives the correspondence: past inputs $\mapsto$ present states, assuming that the state of the system was 0 at time $-\infty$. In this sense the present state is a memory of the past history of an input. Ω is called the *input space* (the *space of past inputs*) and g the *reachability map* of the system Σ (g may be denoted as g^{Σ} if it is necessary to explicitly show the dependence on Σ).

3. Module Structure of Constant Linear Systems

Let $\Sigma = (X, \varphi)$ be a constant linear system whose associated semigroup $\{\Phi(t)\}_{t \geq 0}$ is locally equicontinuous. Our aim here is to endow Σ with a natural module action so that it is compatible with ordinary convolution.

First notice that for every $t \leqslant 0$, the Dirac point mass δ_t acts naturally on X via the formula

$$\delta_t \cdot x := \Phi(-t)x, \tag{3.1}$$

where Φ is the associated semigroup of Σ. It is then only natural to extend (3.1) to the case of linear combinations of δ_t's as

$$\left(\sum_{j=1}^{n} a_j \delta_{t_j}\right) \cdot x := \sum_{j=1}^{n} a_j \Phi(-t_j)x, \qquad t_j \leqslant 0, a_j \in k. \tag{3.2}$$

Looking at this formula, one is tempted to proceed as follows: "Since the set of all linear combinations of δ_t's approximates *any* measure, we can define $\mu \cdot x$ ($\mu \in M_c(\mathbf{R}^-)$) as a limit of (3.2)." In order that this argument is justified it is necessary (and sufficient) that (i) $A_c(\mathbf{R}^-) := \{\sum_{j \in J} a_j \delta_{t_j} : t_j \leqslant 0, J = \text{finite}\}$ be dense in $M_c(\mathbf{R}^-)$ with respect to a certain "nice" topology and (ii) the action (2.2) be continuous with respect to the same topology. Fortunately, this path can be traced, and we proceed according to the following program: We first give an easy characterization of relatively compact sets in $C(-\infty, 0]$. Then we show that for every $\mu \in M_c(\mathbf{R}^-)$, a fixed relatively compact $K \subset C(-\infty, 0]$ and $\varepsilon > 0$, there exists an element $\sum a_j \delta_{t_j}$ in $A_c(\mathbf{R}^-)$ such that

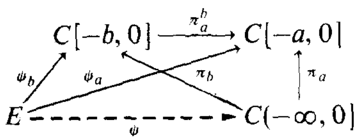
$$\sup_{\psi \in K} \left| \left\langle \sum a_j \delta_{t_j} - \mu, \psi \right\rangle \right| < \varepsilon,$$

i.e., μ can be approximated by elements of $A_c(\mathbf{R}^-)$.

Let $\pi_a : C(-\infty, 0] \to C[-a, 0]$ be the projection which sends each $\psi \in C(-\infty, 0]$ to $\psi|_{[-a, 0]}$. Clearly π_a is a continuous linear map. Further, if $a < b$, there is associated a continuous linear projection $\pi_a^b : C[-b, 0] \to C[-a, 0]$. The following two properties are readily seen: $\pi_a^b \pi_b^c = \pi_a^c$ if $a < b < c$, and $\pi_a^b \pi_b = \pi_a$. As regards this family of spaces and continuous linear maps, we have the following lemma:

Lemma 3.3. *The space $C(-\infty, 0]$ is the projective limit of the family $\{C[-a, 0], \pi_a^b\}_{a, b > 0}$.*

Proof. Suppose that we are given a locally convex space E and continuous linear maps $\psi_a: E \to C[-a, 0]$ for all $a > 0$ such that $\pi_a^b \psi_b = \psi_a$ if $a < b$.



It suffices to show (see Schaefer [16, II.5]) that there exists a unique continuous linear map $\psi: E \to C(-\infty, 0]$ such that $\pi_a \psi = \psi_a$ for all $a > 0$. To see this, note that we must define $\psi(x)|_{[-a,0]} := \psi_a(x)$ in conformity with the requirement $\pi_a \psi = \psi_a$. Due to the condition $\pi_a^b \psi_b = \psi_a$, this defines a unique element in $C(-\infty, 0]$. Now the linearity and the uniqueness of ψ are obvious. We have only to prove the continuity of ψ. But this follows readily from the definition of the topology of $C(-\infty, 0]$ (cf. Section 2). ∎

This lemma now enables us to prove the following:

LEMMA 3.4. *A subset $K \subset C(-\infty, 0]$ is relatively compact if and only if $\pi_a(K)$ is relatively compact in $C[-a, 0]$ for each $a > 0$.*

Proof. Suppose that K is relatively compact, i.e., the closure $\bar{K}$ is compact. Then $\pi_a(\bar{K})$ is compact due to the continuity of π_a. Since $\pi_a(\bar{K})$ is closed, $\overline{\pi_a(K)}$ is contained in $\pi_a(\bar{K})$. Hence $\overline{\pi_a(K)}$ is compact as a closed subset of a compact set, i.e., $\pi_a(K)$ is relatively compact.

Conversely, suppose that $\pi_a(K)$ is relatively compact for every $a > 0$. We cite the standard result on projective limits from Schaefer [16, II.5] that the projective limit $C(-\infty, 0]$ is a closed linear subspace of the product space $\prod_{a>0} C[-a, 0]$. Identifying K with the image under the inclusion: $C(-\infty, 0] \to \prod_{a>0} C[-a, 0]$, we know that $\bar{K}$ may be identified with $\prod_{a>0} \pi_a(K)$, and hence contained in $\prod_{a>0} \overline{\pi_a(K)}$, which is compact by Tychonoff's theorem. Therefore, K is relatively compact. ∎

Remark 3.5. By the Ascoli-Arzelà theorem we know that a subset of $C[-a, 0]$ is relatively compact if and only if it is uniformly bounded and equicontinuous. Hence by the lemma above, a subset $K \subset C(-\infty, 0]$ is compact if and only if it is *locally* uniformly bounded and equicontinuous. We are now ready to prove the following:

PROPOSITION 3.6. *$A_c(\mathbf{R}^-) = \{\sum_{i \in I} a_i \delta_{t_i} : t_i \leqslant 0, I$ is finite$\}$ is dense in $M_c(\mathbf{R}^-)$.*

Proof. Without loss of generality we may assume that our field k is the reals. (If $k = \mathbf{C}$, decompose any measure μ as $\mu = \mu_1 + i\mu_2$ for some real measures μ_1 and μ_2, and reduce this case to the real one.)

Take any relatively compact K in $C(-\infty, 0]$. Since every real measure is the difference of two positive measures, it suffices to prove that every positive measure can be approximated uniformly on K. So take any positive measure μ and an $\varepsilon > 0$. Let J be the support of μ, and a a positive number such that $J \subsetneq [-a, 0]$ (note J is compact). Let ψ be an element of $C(-\infty, 0]$. Then

$$\langle \mu, \psi \rangle = \int_{-a}^{0} \psi(t)\, d\mu(t) = \int_{-a}^{0} \psi^{+}(t)\, d\mu(t) - \int_{-a}^{0} \psi^{-}(t)\, d\mu(t), \qquad (3.7)$$

where $\psi^{+}(t) := \max\{\psi(t), 0\}$ and $\psi^{-}(t) := \max\{-\psi(t), 0\}$. Put $K^{+} := \{\psi^{+} : \psi \in K\}$, and similarly for K^{-}. Then K^{+} and K^{-} are also relatively compact. Then in view of (3.7) it is clearly enough to prove that μ is approximated uniformly on K^{+} (and K^{-}). So we may again assume, without loss of generality, that every ψ in K is positive.

Since $\pi_a(K)$ is relatively compact in $C[-a, 0]$ by Lemma 3.4, there exists $\eta > 0$ such that $|t - t'| < \eta$ implies $|\psi(t) - \psi(t')| < \varepsilon$ for every ψ in K due to the equicontinuity of $\pi_a(K)$ (Remark 3.5). Fix n such that $a/2^n < \eta$. Let Δ_n be the partition of the interval $[-a, 0]$ into 2^n subintervals whose length are all equal to $a/2^n$. Let $\Delta_n : a = t_0 < t_1 < \cdots < t_m = 0$. Define M_j and m_j by

$$\begin{aligned} M_j &:= \sup\{\psi(t) : t_j < t \leqslant t_{j+1}, \psi \in K\}, \\ m_j &:= \inf\{\psi(t) : t_j < t \leqslant t_{j+1}, \psi \in K\}. \end{aligned} \qquad (3.8)$$

Then $|M_j - m_j| < \varepsilon$ for all $j = 0, \ldots, 2^n - 1$. Let $\chi_{(a,b]}$ be the indicator (characteristic function) of $(a, b]$. Then, by definition,

$$\sum_{j=0}^{m-1} m_j \chi_{(t_j, t_{j+1}]} \leqslant \psi \leqslant \sum_{j=0}^{m-1} M_j \chi_{(t_j, t_{j+1}]}$$

for every $\psi \in K$. This, along with the positivity of μ, implies

$$\sum_{j=0}^{m-1} m_j \mu(t_j, t_{j+1}] \leqslant \langle \mu, \psi \rangle \leqslant \sum_{j=0}^{m-1} M_j \mu(t_j, t_{j+1}].$$

Pick any $s_j \in (t_j, t_{j+1}]$ and set $\hat{\mu} := \sum_{j=0}^{m-1} \mu_{(t_j, t_{j+1}]} \delta_{s_j}$. Then $\hat{\mu}$ belongs to $A_c(\mathbf{R}^{-})$ and approximates μ uniformly on K. Indeed,

$$\begin{aligned} |\langle \hat{\mu}, \psi \rangle - \langle \mu, \psi \rangle| &= \left| \sum_{j=0}^{m-1} \psi(s_j) \mu(t_j, t_{j+1}] - \langle \mu, \psi \rangle \right| \\ &\leqslant \left| \sum_{j=0}^{m-1} M_j \mu(t_j, t_{j+1}] - \sum_{j=0}^{m-1} m_j \mu(t_j, t_{j+1}] \right| \end{aligned}$$

$$= \sum_{j=0}^{m-1} (M_j - m_j)\mu(t_j, t_{j+1}]$$

$$\leqslant \varepsilon \sum_{j=0}^{m-1} \mu(t_j, t_{j+1}] = \varepsilon \cdot \mu(-a, 0]$$

for all $\psi \in K$. Since $\hat{\mu}$ depends only on K but not on each ψ, the proof is complete. ∎

We may now proceed to the main theme of this section: the proof of the extendability of the module action: $(\sum a_j \delta_{t_j}, x) \mapsto \sum a_j \Phi(-t_j)x$. But before going into the actual proof, we first give a brief sketch. Define a bilinear map $\mathscr{B}: A_c(\mathbf{R}^-) \times X \to X$ by

$$\mathscr{B}(\sum a_j \delta_{t_j}, x) := \sum a_j \delta_{t_j} \cdot x = \sum a_j \Phi(-t_j)x. \tag{3.9}$$

We shall first prove that $\mathscr{B}$ is separately continuous. Since $A_c(\mathbf{R}^-)$ is dense in $M_c(\mathbf{R}^-)$ by Proposition 3.6, it then becomes possible to define $\mu \cdot x$ as the limit of a net $\mu_i \cdot x$ where $\{\mu_i\}_{i \in I}$ is a net in $A_c(\mathbf{R}^-)$ converging to $\mu \in M_c(\mathbf{R}^-)$. The resulting extension $\bar{\mathscr{B}}$ of $\mathscr{B}$, which is a map of $M_c(\mathbf{R}^-) \times X$ to X, is clearly bilinear. However, the separate continuity of $\mathscr{B}$ is not enough to guarantee even the separate continuity of $\bar{\mathscr{B}}$. Of course, if $\mathscr{B}$ were jointly continuous, $\bar{\mathscr{B}}$ would also be continuous. But when X is infinite dimensional, it is not so, in general. This is where a slightly stronger notion of separate continuity, namely, *hypocontinuity*, becomes a useful tool. We first recall the definition: Let X, Y and Z be locally convex Hausdorff spaces, and $\mathscr{B}$ a bilinear map of $X \times Y$ into Z. Also let $\mathscr{S}$ be a family of bounded subsets of X. The bilinear map $\mathscr{B}$ is *$\mathscr{S}$-hypocontinuous* if $\mathscr{B}$ is separately continuous and if, for each $S \in \mathscr{S}$ and each 0-neighborhood W in Z, there exists a 0-neighborhood V in Y such that $\mathscr{B}(S \times V) \subset W$. This amounts to saying that the "rate" of continuity of $\mathscr{B}$ with respect to the variable in Y is uniform on each $S \in \mathscr{S}$; in other words, for each $S \in \mathscr{S}$ the family of linear maps $\{\mathscr{B}_x: Y \to Z; x \in S\}$, where $\mathscr{B}_x(y) := \mathscr{B}(x, y)$, is equicontinuous. The $\mathscr{E}$-hypocontinuity of $\mathscr{B}$ is analogously defined for a family of bounded subsets $\mathscr{E}$ of Y. Finally, a bilinear map is $(\mathscr{S}, \mathscr{E})$-hypocontinuous if it is both $\mathscr{S}$-hypocontinuous and $\mathscr{E}$-hypocontinuous.

Remark 3.10. Note that if $\mathscr{S}$ is the family of all finite subsets of X, then $\mathscr{S}$-hypocontinuity coincides with separate continuity in the first variable.

The following well-known result on hypocontinuous bilinear mappings is the key to the extension of the module action of $A_c(\mathbf{R}^-)$ to that of $M_c(\mathbf{R}^-)$.

THEOREM 3.11. *Let X_1, X, Y_1, Y be locally convex spaces such that X_1 and Y_1 are dense subspaces of X and Y, respectively. Suppose that $\mathscr{S}$ ($\mathscr{E}$) is a family of bounded subsets of X_1 (Y_1) with the property that $\bar{\mathscr{S}}$ ($\bar{\mathscr{E}}$) covers*

X (Y), where $\bar{\mathscr{S}}$ ($\bar{\mathscr{T}}$) denotes the family of the closures taken in X (Y) of all $S \in \mathscr{S}$ ($T \in \mathscr{T}$), respectively. Finally, let Z be a quasi-complete locally convex Hausdorff space. Then every $(\mathscr{S}, \mathscr{T})$-hypocontinuous bilinear mapping of $X_1 \times Y_1$ into Z admits a unique extension to $X \times Y$ which is bilinear and $(\bar{\mathscr{S}}, \bar{\mathscr{T}})$-hypocontinuous.

For a proof, see Bourbaki [3] or Schaefer [16, III.5.4].

Let us return to the original problem. If we can find a family of bounded subsets $\mathscr{S}$ of $A_c(\mathbf{R}^-)$ such that $\bar{\mathscr{S}}$ covers $M_c(\mathbf{R}^-)$ and if our bilinear map $\mathscr{B}: A_c(\mathbf{R}^-) \times X \to X$: $(\sum a_j \delta_{t_j}, x) \mapsto \sum a_j \delta_{t_j} \cdot x$ is $\mathscr{S}$-hypocontinuous, then we can apply Theorem 3.11 by setting $X_1 := A_c(\mathbf{R}^-)$, $X := M_c(\mathbf{R}^-)$, $Y_1 = Y := X$, and $\mathscr{T} =$ the family of all finite subsets of X (cf. Remark 3.10). Thus, our task breaks into the following three steps:

(i) Show that $\mathscr{B}$ is separately continuous.

(ii) Find a suitable $\mathscr{S}$ so that the condition of Theorem 3.11 is satisfied.

(iii) Show that $\mathscr{B}$ is $\mathscr{S}$-hypocontinuous.

We need the following lemmas to prove the claim (i).

LEMMA 3.12. *Let $\Sigma = (X, \varphi)$ be a constant linear system with the associated semigroup Φ. For each $x \in X$, $x^* \in X'$, and $\tau \leqslant 0$,*

$$\langle \Phi(-\tau)x, x^* \rangle = \langle \delta_{-\tau}, \langle\langle \Phi(\cdot)x, x^* \rangle\rangle \rangle.$$

Proof. Triviality. ∎

LEMMA 3.13. *Let Φ be a semigroup in a locally convex space X. For each $x \in X$ and $a > 0$, $C_a := \{\Phi(t)x: 0 \leqslant t \leqslant a\}$ is compact in X.*

Proof. Let $\bar{\phi}: [0, a] \to X$ be the map given by $\bar{\phi}(t) := \Phi(t)x$. Then $\bar{\phi}$ is continuous due to the strong continuity of Φ. Since $[0, a]$ is compact, $C_a = \operatorname{im} \bar{\phi}$ is compact as the image of a compact set under a continuous map. ∎

Now let p be any continuous seminorm of X. By the Mackey–Arens theorem on duality of locally convex spaces (see, e.g., Bourbaki [3], Schaefer [16, IV.3], or Treves [19, II.36]) there exists a weakly ($\sigma(X', X)$–) compact, balanced, convex subset B of X' such that

$$p(x) = p_B(x) := \sup\{|\langle x, x^* \rangle| : x^* \in B\},$$

or, equivalently, $B^\circ = \{x \in X: p(x) \leqslant 1\}$, where B° denotes the absolute polar of B with respect to the duality $\langle X, X' \rangle$ (see Section 2). Fixing such p and B, we have

LEMMA 3.14. *Let* $K := \{\langle \Phi(\cdot)x, x^* \rangle : x^* \in B\}$. *Then* K *is relatively compact in* $C[0, \infty)$.

Proof. Take any $a > 0$ and let K_a be the restriction of K to the interval $[0, a]$, i.e., $K_a = \{\langle \Phi(\cdot)x, x^* \rangle|_{[0,a]} : x^* \in B\}$. By Lemma 3.4 (since $C[0, \infty)$ and $C(-\infty, 0]$ are clearly isomorphic), it suffices to prove that K_a is relatively compact in $C[0, a]$ for each $a > 0$.

Let us first prove that K_a is uniformly bounded. By Lemma 3.13, $C_a = \{\Phi(t)x : 0 \leqslant t \leqslant a\}$ is compact and B° is a 0-neighborhood of X. Since compact sets are bounded, there exists $\alpha > 0$ such that $C_a \subset \alpha B^\circ$. Then it follows that

$$\begin{aligned} \sup_{\substack{0 \leqslant t \leqslant a \\ x^* \in B}} |\langle \Phi(t)x, x^* \rangle| &= \sup_{\substack{y \in C_a \\ x^* \in B}} |\langle y, x^* \rangle| \\ &\leqslant \sup_{\substack{z \in B^\circ \\ x^* \in B}} |\alpha z, x^* \rangle| \leqslant |\alpha| \end{aligned} \tag{3.15}$$

because $|\langle z, x^* \rangle| \leqslant 1$ for all $z \in B^\circ$ and $x \in B$ (see Section 2). Hence, K_a is uniformly bounded.

We now prove that K_a is equicontinuous. According to the following lemma (Lemma 3.16), for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|t - t'| < \delta$ ($t, t' \in [0, a]$) implies $(\Phi(t) - \Phi(t'))x \in \varepsilon B^\circ$. It now follows that

$$\begin{aligned} |\langle (\Phi(t) - \Phi(t'))x, x^* \rangle| &= |\langle \varepsilon y, x^* \rangle| \qquad \text{for some} \quad y \in B^\circ \\ &= \varepsilon |\langle y, x^* \rangle| \leqslant \varepsilon \end{aligned}$$

because $y \in B^\circ$ and $x^* \in B$. Thus, K_a is equicontinuous. In view of Remark 3.5, this completes the proof. ∎

It now remains only to prove

LEMMA 3.16. *Let* Φ, x, B *and* K_a *be as above. For every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $(\Phi(t) - \Phi(t'))x \in \varepsilon B^\circ$ *for all* t, $t' \in [0, a]$ *with* $|t - t'| < \delta$.

Proof. Take any $s \in [0, a]$. Since εB° is a 0-neighborhood of X, there exists $\delta(s) > 0$ such that $|t - s| < \delta(s)$ implies $(\Phi(t) - \Phi(s))x \in (\varepsilon/2)B^\circ$ because of the strong continuity of Φ. Define $J(s) := \{t \in [0, a] : |t - s| < (1/2)\delta(s)\}$. Then $[0, a] = \bigcup_{s \in [0,a]} J(s)$, which is clearly an open cover of $[0, a]$. Since $[0, a]$ is compact, there exist finitely many points $s_1, \ldots, s_r \in [0, a]$ such that

$$[0, a] = \bigcup_{j=1}^{r} J(s_j).$$

Let $\delta := (1/2)\min\{\delta(s_1),..., \delta(s_r)\}$. Now suppose $|t-t'| < \delta$. There exists i such that $t' \in J(s_i)$. Hence $|t'-s_i| < (1/2)\,\delta(s_i)$. Also,

$$\begin{aligned}|t-s_i| &\leqslant |t-t'| + |t'-s_i| \\ &\leqslant \delta + \tfrac{1}{2}\delta(s_i) \leqslant \tfrac{1}{2}\delta(s_i) + \tfrac{1}{2}\delta(s_i) = \delta(s_i).\end{aligned}$$

This implies $(\Phi(t) - \Phi(s_i))x$, $(\Phi(s_i) - \Phi(t'))x \in (\varepsilon/2)B°$. Hence $(\Phi(t) - \Phi(t'))x = (\Phi(t) - \Phi(s_i))x + (\Phi(s_i) - \Phi(t'))x \in \varepsilon B°$. ∎

We are now ready to prove the following proposition.

PROPOSITION 3.17. *The bilinear map*

$$\mathscr{B}: A_c(\mathbf{R}^-) \times X \to X: (\textstyle\sum \alpha_j \delta_{t_j}, x) \mapsto \sum \alpha_j \Phi(-t_j)x$$

is separately continuous.

Proof. If we fix an element $\sum \alpha_j \delta_{t_j} \in A_c(\mathbf{R}^-)$, the continuity of the correspondence: $x \mapsto (\sum \alpha_j \delta_{t_j}) \cdot x$ is obvious from the continuity of each $\Phi(-t_j)$.

Now fix an element $x \in X$, and take any continuous seminorm p of X. As before (see the discussion before Lemma 3.14)), there exists a weakly compact, convex, balanced set B in X' whose absolute polar coincides with the neighborhood $V := \{x \in X: p(x) \leqslant 1\}$, i.e.,

$$p(x) = p_B(x) := \sup_{x^* \in B} |\langle x, x^*\rangle|.$$

From this and Lemma 3.12 the following equalities follow:

$$\begin{aligned}p((\textstyle\sum \alpha_j \delta_{t_j}) \cdot x) &= \sup_{x^* \in B} |\langle \textstyle\sum \alpha_j \delta_{t_j} \cdot x, x^*\rangle| \\ &= \sup_{x^* \in B} |\langle \textstyle\sum \alpha_j \Phi(-t_j)x, x^*\rangle| \qquad (3.18) \\ &= \sup_{x^* \in B} |\langle \textstyle\sum \alpha_j \delta_{-t_j}, \langle \Phi(\cdot\,)x, x^*\rangle\rangle|.\end{aligned}$$

For every $\varepsilon > 0$, let

$$U := \{\nu \in A_c(\mathbf{R}^-): \sup_{\check{y} \in K} |\langle \nu, y\rangle| \leqslant \varepsilon\},$$

where $\check{y}(t) := y(-t)$ and $K = \{\langle \Phi(\cdot\,)x, x^*\rangle: x^* \in B\}$. Since K is relatively compact in $C[0, \infty)$ by Lemma 3.14, U is a 0-neighborhood of $A_c(\boldsymbol{R}^-)$. If $\sum \alpha_j \delta_{t_j}$ belongs to U, then

$$\sup_{x^* \in B} |\langle \textstyle\sum \alpha_j \delta_{-t_j}, \langle \Phi(\cdot\,)x, x^*\rangle\rangle| = \sup_{\check{y} \in K} |\langle \sum \alpha_j \delta_{t_j}, y\rangle| \leqslant \varepsilon.$$

Hence by (3.18) the correspondence $\sum \alpha_j \delta_{t_j} \mapsto \sum \alpha_j \delta_{t_j} \cdot x$ is continuous. ∎

We may now proceed to the second part of our program. Let $\mathscr{S}$ denote the family of all equicontinuous subsets of $A_c(\mathbf{R}^-)$. What we need to prove is that every element of $M_c(\mathbf{R}^-)$ belongs to the closure of $\mathscr{S}$. We need the following lemma.

LEMMA 3.19. *For every equicontinuous set E of $A_c(\mathbf{R}^-)$, there exist $a, C > 0$ depending only on E such that $\sum_{j \in J} \alpha_j \delta_{t_j} \in E$ implies $t_j \in [-a, 0]$ and $\sum |\alpha_j| \leqslant C$. Conversely, every subset of $A_c(\mathbf{R}^-)$ of the form*

$$E = \{\textstyle\sum_{j \in J} \alpha_j \delta_{t_j} : J = \text{finite},\ t_j \in [-a, 0], \sum |\alpha_j| \leqslant C\} \tag{3.20}$$

is equicontinuous.

Proof. Suppose that E is equicontinuous. By definition, there exists a 0-neighborhood V of $C(-\infty, 0]$ such that

$$\sup\{|\langle x^*, v\rangle| : x^* \in E, v \in V\} \leqslant 1. \tag{3.21}$$

Without loss of generality we may assume that V is of the form

$$V = \{\psi \in C(-\infty, 0] : \sup_{-k \leqslant t \leqslant 0} |\psi(t)| \leqslant \varepsilon\}$$

for some $\varepsilon > 0$ and integer $k > 0$. Take $a := k + 1$. Suppose that there exists an element $x^* = \sum \alpha_j \delta_{t_j} \in E$ such that $\alpha_{j_0} \neq 0$ and $t_{j_0} < -a$ for some j_0. For each n, let χ_n be a continuous function such that $\chi_n(t_{j_0}) = n\bar{\alpha}_{j_0}$ and $\chi_n(t) = 0$ on $[-a, 0]$ and at all t_j's except t_{j_0}. Then clearly $\chi_n \in V$ for every n, and hence

$$\langle x^*, \chi_n\rangle = \alpha_{j_0} n\bar{\alpha}_{j_0} = n|\alpha_{j_0}|^2$$

must be smaller than or equal to 1 according to (3.21). But this is obviously a contradiction. Hence, if $x^* = \sum \alpha_j \delta_{t_j}$ belongs to E, then $t_j \in [-a, 0]$.

Now take any $x^* = \sum \alpha_j \delta_{t_j} \in E$. Let $\chi(t)$ be an element of $C(-\infty, 0]$ such that $|\chi(t)| \leqslant \varepsilon$ for all t and $\chi(t_j) = (\bar{\alpha}_j / |\alpha_j|)\varepsilon$ for all t_j's. Then χ belongs to V, and hence $|\langle x^*, \chi\rangle| \leqslant 1$. This implies that

$$|\langle x^*, \chi\rangle| = \left|\sum_{j \in J} \alpha_j \bar{\alpha}_j \frac{\varepsilon}{|\alpha_j|}\right| = \sum_{j \in J} |\alpha_j| \varepsilon \leqslant 1.$$

Thus, $\sum_{j \in J} |\alpha_j| \leqslant 1/\varepsilon$ for every $x^* \in E$. Since ε does not depend on each x^*, but only on E, the first half is proved.

Conversely, assume E to be the set given by (3.20). Take any $\varepsilon > 0$ and define V by

$$V := \{\psi \in C(-\infty, 0] : \sup_{-a \leqslant t \leqslant 0} |\psi(t)| \leqslant \varepsilon / C\}.$$

Then V is a neighborhood of 0 in $C(-\infty, 0]$. It now follows that

$$\begin{aligned}
&\sup\{|\langle\mu, \psi\rangle| : \mu \in E, \psi \in V\} \\
&\quad = \sup\{|\langle\sum \alpha_j\delta_{t_j}, \psi\rangle| : \sum|\alpha_j| \leqslant C, t_j \in [-a, 0], \psi \in V\} \\
&\quad = \sup\{|\sum \alpha_j \psi(t_j)| : \sum|\alpha_j| \leqslant C, t_j \in [-a, 0], \psi \in V\} \\
&\quad \leqslant \sup\{\sum|\alpha_j|(\varepsilon/C) : \sum|\alpha_j| \leqslant C\} \\
&\quad = \varepsilon.
\end{aligned}$$

Thus, E is equicontinuous. ∎

LEMMA 3.22. *For every $\mu \in M_c(\mathbf{R}^-)$, there exists $E \in \mathscr{S}$ such that $\mu \in \bar{E}$, where the closure is taken in $M_c(\mathbf{R}^-)$.*

Proof. Let $a > 0$ be such that $\operatorname{supp}\mu \subsetneq [-a, 0]$. Then let $C := \mu(-a, 0]$. Define E by

$$E := \{\sum \alpha_j\delta_{t_j} : \sum|\alpha_j| \leqslant C, t_j \in [-a, 0]\}.$$

According to Lemma 3.19, E is equicontinuous. Then the proof of Proposition 3.6 shows that μ can be approximated by elements of E. ∎

We are now ready to prove the following:

PROPOSITION 3.23. *Suppose that $\{\Phi(t)\}_{t \geqslant 0}$ is a locally equicontinuous semigroup on X. Then the bilinear map*

$$\mathscr{B} : A_c(\mathbf{R}^-) \times X \to X : (\sum \alpha_j\delta_{t_j}, x) \mapsto \sum \alpha_j \Phi(-t_j)x$$

is $\mathscr{S}$-hypocontinuous, where $\mathscr{S}$ denotes the family of all equicontinuous subsets of $A_c(\mathbf{R}^-)$.

Proof. Take any $E \in \mathscr{S}$ and a continuous seminorm p of X. By Lemma 3.19 there exist $a, C > 0$ such that if $\sum \alpha_j\delta_{t_j} \in E$, then $t_j \in [-a, 0]$ and $\sum|\alpha_j| \leqslant C$. Since $\{\Phi(t)\}_{t \geqslant 0}$ is locally equicontinuous, there exists a continuous seminorm q of X such that $p(\Phi(t)x) \leqslant q(x)$ for all $t \in [0, a]$. Then we have that

$$\begin{aligned}
p\left(\sum \alpha_j\delta_{t_j} \cdot x\right) = p\left(\sum \alpha_j \Phi(-t_j)\, x\right) &\leqslant \sum|\alpha_j| p(\Phi(-t_j)\, x) \\
&\leqslant \left(\sum|\alpha_j|\right) q(x) \leqslant Cq(x).
\end{aligned}$$

Therefore, $\mathscr{B}$ is $\mathscr{S}$-hypocontinuous. ∎

We can now prove the following central result of this section.

THEOREM 3.24. *Let $\{\Phi(t)\}_{t\geqslant 0}$ be a locally equicontinuous semigroup on a complete locally convex space X. Then the bilinear map*

$$\mathscr{B}: A_c(\mathbf{R}^-) \times X \to X: \left(\sum \alpha_j \delta_{t_j}, x\right) \mapsto \sum \alpha_j \Phi(-t_j)x$$

admits a unique extension

$$\mathscr{B}: M_c(\mathbf{R}^-) \times X \to X$$

which is $\bar{\mathscr{S}}$-hypocontinuous where $\bar{\mathscr{S}}$ denotes the closure of all equicontinuous subsets of $A_c(\mathbf{R}^-)$.

Proof. Let $\mathscr{C}$ be the family of all finite sets of X. Since $\mathscr{B}$ is $\mathscr{S}$-hypocontinuous by Proposition 3.23, it is $(\mathscr{S}, \mathscr{C})$-hypocontinuous in view of the separate continuity of $\mathscr{B}$. By Theorem 3.11, $\mathscr{B}$ can be uniquely extended to $M_c(\mathbf{R}^-) \times X$ and is $(\bar{\mathscr{S}}, \mathscr{C})$-hypocontinuous. ∎

We list some direct consequences of the theorem.

For each $t \geqslant 0$, let σ_t^r be the *right shift operator* of $C(-\infty, 0]$ defined by $(\sigma_t^r \psi)(\tau) = \psi(\tau - t)$. The *left shift operator* σ_t (or σ_t^l) is then defined as the adjoint of σ_t^r, i.e.,

$$\langle \sigma_t \mu, \psi \rangle := \langle \mu, \sigma_t^r \psi \rangle, \qquad \mu \in M_c(\mathbf{R}^-), \psi \in C(-\infty, 0]. \tag{3.25}$$

It is easy to ensure the continuity of each σ_t. Further, it is also easy to directly prove that the family $\{\sigma_t\}_{t\geqslant 0}$ constitutes a locally equicontinuous semigroup. Let us just prove the local equicontinuity.

LEMMA 3.26. *The family of continuous linear maps $\{\sigma_t\}_{t\geqslant 0}$ is locally equicontinuous.*

Sketch of Proof. Take any $a > 0$ and any relatively compact $K \subset C(-\infty, 0]$. In view of the equality

$$\sup_{\psi \in K} |\langle \sigma_t \mu, \psi \rangle| = \sup_{\psi \in K} |\langle \mu, \sigma_t^r \psi \rangle| = \sup_{\chi \in \sigma_t^r K} |\langle \mu, \chi \rangle|,$$

it suffices to prove that $\bigcup_{0 \leqslant t \leqslant a} \sigma_t^r K$ is a relatively compact set of $C(-\infty, 0]$. In order to show this it is enough to check the uniform boundedness and equicontinuity of this set on every interval $[-b, 0]$ (see Remark 3.5). However, the uniform boundedness and equicontinuity follow immediately from those of K on $[-b-a, 0]$, whence the proof. ∎

Specializing Theorem 3.24 to the case $X = M_c(\mathbf{R}^-)$ and $\Phi(t) = \sigma_t$, we know that the "*convolution*" *is well defined and belongs to* $M_c(\mathbf{R}^-)$. To see that our definition indeed agrees with the usual convolution, just notice that $\delta_t * \nu$ (we write $\mu * \nu$ instead of $\mu \cdot \nu$ in this case) is $\sigma_{-t}\nu$ ($t \leqslant 0$) according to both our definition and the usual definition (recall (3.1)). Then the two definitions agree on $A_c(\mathbf{R}^-) \times M_c(\mathbf{R}^-)$. Since $A_c(\mathbf{R}^-)$ is dense in $M_c(\mathbf{R}^-)$ (Proposition 3.6), these two definitions agree on $M_c(\mathbf{R}^-) \times M_c(\mathbf{R}^-)$ by the separate continuity of convolution.

Remark 3.27. Note that in this case convolution $\mu * \nu$ is $\mathscr{S}$-hypocontinuous in both variables due to the symmetry of $\mu * \nu$, i.e., $(\mu, \nu) \mapsto \mu * \nu$ is $(\mathscr{S}, \mathscr{S})$-hypocontinuous.

The space of measures with compact support $M_c(\mathbf{R}^-)$ thus *admits a structure of k-algebra* with convolution as product. Its identity is δ_0, the Dirac measure at 0. We can now state the main result of this section.

THEOREM 3.28. *Let $\Sigma = (X, \varphi)$ be a constant linear system whose associated semigroup $\{\Phi(t)\}_{t \geqslant 0}$ is locally equicontinuous. Then Σ admits an $M_c(\mathbf{R}^-)$-module structure via the definition*

$$\mu \cdot x := \mathscr{B}(\mu, x), \tag{3.29}$$

where $\mathscr{B}$ is the bilinear map given in Theorem 3.24.

Proof. We need only to check $(\mu * \nu) \cdot x = \mu \cdot (\nu \cdot x)$; the rest follows from the bilinearity of $\mathscr{B}$.

First of all, when both μ and ν belong to $A_c(\mathbf{R}^-)$, this equality is obvious. Now let μ belong to $A_c(\mathbf{R}^-)$, and let $\{\nu_\beta\}_{\beta \in B}$ be a net in $A_c(\mathbf{R}^-)$ converging to $\nu \in M_c(\mathbf{R}^-)$. Then by the separate continuity of convolution and of $\mu \cdot x$, we have

$$\begin{aligned}(\mu * \nu) \cdot x &= \left(\lim_\beta (\mu * \nu_\beta)\right) \cdot x = \lim_\beta ((\mu * \nu_\beta) \cdot x) \\ &= \lim_\beta (\mu \cdot (\nu_\beta \cdot x)) = \mu \cdot \left(\lim_\beta (\nu_\beta \cdot x)\right) \\ &= \mu \cdot \left(\left(\lim_\beta \nu_\beta\right) \cdot x\right) = \mu \cdot (\nu \cdot x).\end{aligned}$$

Hence $(\mu * \nu) \cdot x = \mu \cdot (\nu \cdot x)$ follows for $\mu \in A_c(\mathbf{R}^-)$, $\nu \in M_c(\mathbf{R}^-)$, and $x \in X$. Repeating the same argument for μ, we have the desired equality $(\mu * \nu) \cdot x = \mu \cdot (\nu \cdot x)$. ∎

Now restrict $\{\sigma_t\}_{t \geqslant 0}$ to the input space $\Omega = \lim_{\to} L^2[-n, 0]$. It is easy to check that $\{\sigma_t\}_{t \geqslant 0}$ constitutes a locally equicontinuous semigroup, and its definition coincides with

$$(\sigma_t \omega)(\tau) := \omega(\tau + t), \qquad \tau \leqslant -t$$
$$= 0, \qquad -t < \tau \leqslant 0.$$

We may then specialize Theorem 3.28 to this case and conclude that Ω is an $M_c(\mathbf{R}^-)$-module. It is easy to see that $\mu \cdot \omega$ ($\mu \in M_c(\mathbf{R}^-)$, $\omega \in \Omega$) agrees with the usual convolution because Ω can be viewed as a subspace of $M_c(\mathbf{R}^-)$. So denote $\mu \cdot \omega$ by $\mu * \omega$ again in this case. We can now prove the following

PROPOSITION 3.30. *Let $\Sigma = (X, \varphi)$ be a constant linear system whose associated semigroup is locally equicontinuous, and g its reachability map as defined in Proposition* 2.3. *Then g is an $M_c(\mathbf{R}^-)$-module homomorphism.*

Proof. In view of the linearity of g, it suffices to check $g(\mu * \omega) = \mu \cdot g(\omega)$ for every $\mu \in M_c(\mathbf{R}^-)$ and $\omega \in \Omega$. First of all, $g(\delta_t * \omega) = \delta_t \cdot g(\omega)$ is valid for every $t \leqslant 0$. For, if $\operatorname{supp} \omega \subset [-s, 0]$, then by definition it follows that (note: $\operatorname{supp} \sigma_{-t}\omega \subset [-s+t, t]$)

$$\begin{aligned} g(\delta_t * \omega) = g(\sigma_{-t}\omega) &= \varphi(s-t, 0, \sigma^r_{s-t}(\sigma_{-t}\omega)) \\ &= \varphi(-t, \varphi(s, 0, \sigma^r_s \omega), 0) \qquad (\sigma^r_{s-t}(\sigma_{-t}\omega) = \sigma^r_s \omega) \\ &= \Phi(-t) g(\omega) \qquad \text{(by (2.4))} \\ &= \delta_t \cdot g(\omega). \end{aligned}$$

Hence $g(\mu * \omega) = \mu \cdot g(\omega)$ is valid for every $\mu \in A_c(\mathbf{R}^-)$ and $\omega \in \Omega$. Then the separate continuity of the module action and convolution, along with the facts that g is continuous and $A_c(\mathbf{R}^-)$ is dense in $M_c(\mathbf{R}^-)$, implies the desired equality $g(\mu * \omega) = \mu \cdot g(\omega)$. ∎

Remark 3.31. The convolution in $M_c(\mathbf{R}^-)$ is not, unfortunately, jointly continuous. So in the current literature on topological algebras this kind of algebra is not of utmost concern; sometimes, it is not even called a topological algebra. For more detail, see Beckenstein *et al.* [2].

4. Convolution of Measures and Functions

In the previous section we have investigated the convolution in $M_c(\mathbf{R}^-)$ and proved, in particular, that the convolution of $\mu \in M_c(\mathbf{R}^-)$ and $\omega \in \Omega$ is well defined and belongs to Ω. This result is yet not enough for later developments; in Section 5 we need to consider convolution of measures and functions *whose support need not be bounded on the right.*

Let us begin by introducing some function spaces. Let $C_r(-\infty, n]$ be the space of k-valued continuous functions on $\mathbf{R}$ whose support is contained in $(-\infty, n]$. Introduce the following countable seminorms to $C_r(-\infty, n]$:

$$q_m^n(\psi) := \sup_{m \leqslant t \leqslant n} |\psi(t)|. \tag{4.1}$$

$C_r(-\infty, n]$ then becomes a Fréchet space.

Now let $C_r(\mathbf{R}) := \bigcup_{-\infty < n < \infty} C_r(-\infty, n]$. Clearly, $C_r(\mathbf{R})$ consists of continuous functions on $\mathbf{R}$ whose support is bounded on the right. Introduce the strict inductive limit topology on $C_r(\mathbf{R})$, i.e., the finest locally convex topology which makes each inclusion $j_n: C_r(-\infty, n] \to C_r(\mathbf{R})$ continuous. Now define $M_l(\mathbf{R})$ to be the dual space of $C_r(\mathbf{R})$. We have the following proposition.

PROPOSITION 4.2. *$M_l(\mathbf{R})$ is precisely the space of (Radon) measures on $\mathbf{R}$ with support bounded on the left.*

Proof. Let μ be a Radon measure with support bounded on the left, say $\operatorname{supp} \mu \subset [a, \infty)$. We need to prove that μ is a continuous linear form on $C_r(\mathbf{R})$. In view of the inductive limit topology, this amounts to showing that μ is a continuous linear form on each $C_r(-\infty, n]$.

Take a continuous function α such that $\alpha(t) \equiv 1$ for $t \geqslant a$, $\alpha(t) \equiv 0$ for $t \leqslant a-1$, and $0 \leqslant \alpha(t) \leqslant 1$ for $a-1 < t < a$. For every $\psi \in C_r(-\infty, n]$ define

$$\langle \mu, \psi \rangle := \langle \mu, \alpha\psi \rangle. \tag{4.3}$$

Since $\alpha\psi$ belongs to $C_0[a-1, n]$, the right-hand side makes sense. Furthermore, it is plain to see that the right-hand side does not depend on the choice of α. Indeed, if β is another such function, then the support of $(\alpha - \beta)\psi$ is contained in $(-\infty, a)$. But since $\operatorname{supp} \mu \subset [a, \infty)$, $\langle \mu, (\alpha - \beta)\psi \rangle = 0$, so $\langle \mu, \alpha\psi \rangle = \langle \mu, \beta\psi \rangle$. Thus, $\langle \mu, \psi \rangle$ is well defined by (4.3).

As regards the continuity of μ, we have

$$\begin{aligned} |\langle \mu, \psi \rangle| = |\langle \mu, \alpha\psi \rangle| &\leqslant C_{\mu,n} \cdot \sup_{a-1 \leqslant t \leqslant n} |\alpha(t)\psi(t)| \\ &\leqslant C_{\mu,n} \cdot \sup_{a-1 \leqslant t \leqslant n} |\psi(t)| \end{aligned}$$

for some constant $C_{\mu,n} > 0$ because μ is a measure. Thus, μ is a continuous linear form on each $C_r(-\infty, n]$, and hence $\mu \in (C_r(\mathbf{R}))'$.

Conversely, suppose $\mu \in (C_r(\mathbf{R}))'$. One can easily see the inclusion

$C_0(\mathbf{R}) \subset C_r(\mathbf{R})$, which implies that μ is a measure. We need to prove that the support of μ is bounded on the left. For every n, there exist a and $C > 0$ such that

$$|\langle \mu, \psi \rangle| \leqslant C \sup_{a \leqslant t \leqslant n} |\psi(t)|, \qquad \psi \in C_r(-\infty, n] \tag{4.4}$$

because μ is continuous on each $C_r(-\infty, n]$. This inequality (4.4) readily implies that $\langle \mu, \psi \rangle = 0$ for all ψ whose support is contained in $(-\infty, a)$. Hence the support of μ is bounded on the left. ∎

We may now proceed to investigate the convolution structure of $M_l(\mathbf{R})$. Our program is as follows: We shall begin by defining the convolution of two measures belonging to $M_l(\mathbf{R})$. For this we follow the standard procedure given by Schwartz [17]. We shall then specialize this case to convolution of elements in $M_l(\mathbf{R})$ and $L^2_+(\mathbf{R})$. Here $L^2_+(\mathbf{R})$ denotes the set of all k-valued locally L^2-functions on $\mathbf{R}$ with support bounded on the left. Naturally, $L^2_+(\mathbf{R}) \subset M_l(\mathbf{R})$. Our first objective is to show the following:

PROPOSITION 4.5. *Every pair $(\mu, \nu) \in M_l(\mathbf{R}) \times M_l(\mathbf{R})$ is convolvable.*

Proof. We must define, for every $\psi \in C_r(\mathbf{R})$,

$$\langle \mu * \nu, \psi \rangle := \langle \mu_x \otimes \nu_y, \psi(x+y) \rangle = \int \psi(x+y)\, d\mu(x)\, d\nu(y). \tag{4.6}$$

Note that there exist a, b, $c \in \mathbf{R}$ such that $\operatorname{supp} \psi \subset (-\infty, a]$, $\operatorname{supp} \mu \subset [b, \infty)$, and $\operatorname{supp} \nu \subset [c, \infty)$. It now follows that

$$\begin{aligned}
\int_{\mathbf{R}^2} \psi(x+y)\, d\mu(x)\, d\nu(y) &= \int_c^\infty \int_b^\infty \psi(x+y)\, d\mu(x)\, d\nu(y) \\
&= \int_c^\infty \int_b^{a-c} \psi(x+y)\, d\mu(x)\, d\nu(y) \qquad (4.7)\\
&= \int_c^{a-b} \int_b^{a-c} \psi(x+y)\, d\mu(x)\, d\nu(y).
\end{aligned}$$

Furthermore, the correspondence

$$y \mapsto \int_b^{a-c} \psi(x+y)\, d\mu(x)$$

is a continuous function of y because of the local uniform continuity of ψ. Moreover, the support of this function is contained in $(-\infty, a-b]$. Therefore, (4.6) possesses a definite value.

Now if ψ belongs to $C_r(-\infty, a]$, then

$$\begin{aligned}
|\langle \mu * \nu, \psi \rangle| &= \left| \int_c^{a-b} \int_b^{a-c} \psi(x+y)\, d\mu(x)\, d\nu(y) \right| \\
&\leqslant \int_c^{a-b} \int_b^{a-c} |\psi(x+y)|\, d|\mu|\,(x)\, d|\nu|\,(y) \\
&\leqslant \int_c^{a-b} \{\sup\{|\psi(t)| : b+y \leqslant t \leqslant a-c+y\} \cdot |\mu|([b, a-c])\}\, d\nu(y) \\
&\leqslant C \cdot \sup\{|\psi(t)| : b+c \leqslant t \leqslant 2a-b-c\} \\
&\leqslant C' \cdot \sup\{|\psi(t)| : b+c \leqslant t \leqslant a\}. \qquad (4.8)
\end{aligned}$$

Therefore, $\mu * \nu$ is a continuous linear form on each $C_r(-\infty, a]$, so it belongs to $M_l(\mathbf{R})$. ∎

COROLLARY 4.9. *If* $\mu, \nu \in M_c(\mathbf{R}^-)$, *then* $\mu * \nu \in M_c(\mathbf{R}^-)$.

Proof. It suffices to prove that supp $(\mu * \nu)$ is contained in $(-\infty, 0]$. For this, pick any continuous ψ of compact support contained in $(0, \infty)$. Then, by definition,

$$\langle \mu * \nu, \psi \rangle = \iint \psi(x+y)\, d\mu(x)\, d\nu(y) = \int_{-\infty}^0 \int_{-\infty}^0 \psi(x+y)\, d\mu(x)\, d\nu(y) = 0$$

because $\psi(x+y) = 0$ if $x \leqslant 0$ and $y \leqslant 0$. Hence, supp$(\mu * \nu)$ is contained in $(-\infty, 0]$. ∎

Now consider the following question: Given $\mu \in M_l(\mathbf{R})$ and $f \in L^2_+(\mathbf{R})$, we have $\mu * f \in M_l(\mathbf{R})$ since we may regard $L^2_+(\mathbf{R}) \subset M_l(\mathbf{R})$, naturally. Now does it also belong to $L^2_+(\mathbf{R})$? The following proposition answers this question affirmatively. (This result will be used in Section 5.)

PROPOSITION 4.10. *Let* $\mu \in M_l(\mathbf{R})$ *and* $f \in L^2_+(\mathbf{R})$. *Then* $\mu * f$ *belongs to* $L^2_+(\mathbf{R})$.

Proof. In view of the fact $\mu * f \in M_l(\mathbf{R})$, it suffices only to show that there exists a function $g \in L^2_+(\mathbf{R})$ such that $\mu * f = g$ as a measure.

For each n, let α_n be a continuous function with values in $[0, 1]$ such that $\alpha_n(t) = 1$ for $t \leqslant n$ and $\alpha_n(t) = 0$ for $t \geqslant n+1$. Then $\alpha_n \mu \in M_c(\mathbf{R})$, and $\alpha_n f$ belongs to $L^2(\mathbf{R})$ and has compact support. Now define $g_n := (\alpha_n \mu) * (\alpha_n f)$. Then g_n belongs to $L^2(\mathbf{R})$ (Dieudonné [6, 14.9.2]). Furthermore, if supp $\mu \subset (a, \infty)$, supp $f \subset (a, \infty)$, and $m > n$, then it is easy to check, via direct calculation, $g_m|_{(-\infty, n+a)} = g_n|_{(-\infty, n+a)}$. Hence, as $n \to \infty$, g_n defines a locally L^2-function g whose support is bounded on the left.

Now take any $\psi \in C_r(\mathbf{R})$. Then $\operatorname{supp} \psi \subset (-\infty, b)$ for some b. Then by (4.8) we have

$$\begin{aligned}\langle \mu * f, \psi \rangle &= \int_a^{b-a} \int_a^{b-a} \psi(x+y)\, d\mu(x) f(y)\, dy \\ &= \int_a^{b-a} \int_a^{b-a} \psi(x+y)\, \alpha_n(x)\, d\mu(x)\, \alpha_n(y) f(y)\, dy \qquad (n > b-a) \\ &= \langle (\alpha_n \mu) * (\alpha_n f), \psi \rangle \\ &= \langle g_n, \psi \rangle \\ &= \langle g, \psi \rangle \qquad (\text{because } g_n = g \quad \text{on} \quad (-\infty, b)).\end{aligned}$$

Hence, $\mu * f = g$ in the sense of measures. ∎

Now given $\alpha \in M_l(\mathbf{R})$, we define $l(\alpha)$ by

$$l(\alpha) := \inf\{t \colon t \in \operatorname{supp} \alpha\}.$$

The following lemma is adopted from Kamen [12] (modified to the present context) and will be used in the next section.

LEMMA 4.11. *Every element of the form* $\delta_\tau + \alpha \in M_c(\mathbf{R}^-)$, $l(\alpha) > \tau$, $\alpha \in \Omega$, *has an inverse, with respect to convolution, in* $M_l(\mathbf{R})$.

Proof. Write $\mu\nu$ and α^n instead of $\mu * \nu$ and $\alpha * \cdots * \alpha$ (n times), respectively. Since $\delta_{-\tau}(\delta_\tau + \alpha) = \delta_0 + \delta_{-\tau}\alpha$, it clearly suffices to prove that $\delta_0 + \delta_{-\tau}\alpha$ is invertible in $M_l(\mathbf{R})$. Consider the following formal power series expansion of $(\delta_0 + \delta_{-\tau}\alpha)^{-1}$:

$$(\delta_0 + \delta_{-\tau}\alpha)^{-1} = \sum_{n=0}^{\infty} (-\delta_{-\tau}\alpha)^n. \tag{4.12}$$

We claim that the right-hand side of (4.12) converges in $M_l(\mathbf{R})$. In view of the fact that $M_l(\mathbf{R})$ is the dual of $C_r(\mathbf{R})$, it is enough to show that (4.12) converges pointwise. (For, then, we can apply the Banach–Steinhaus theorem since $C_r(\mathbf{R})$ is a barrelled space as the inductive limit of Frechét spaces.)

Take any $\psi \in C_r(\mathbf{R})$. Then $\operatorname{supp} \psi \subset (-\infty, b]$ for some b. Further, $\operatorname{supp}(\delta_{-\tau}\alpha) \subset [c, \infty)$ for some $c > 0$ since $l(\alpha) > \tau$. It follows that

$$\inf\{\operatorname{supp}(\delta_{-\tau}\alpha)^n\} \geqslant nc$$

for all $n > 0$ because $\operatorname{supp}(\mu * \nu) \subset \operatorname{supp} \mu + \operatorname{supp} \nu$. Let n_0 be the first integer such that $n_0 c > b$. We then obtain

$$\left\langle \sum_{n=0}^{\infty} (-\delta_{-\tau}\alpha)^n, \psi \right\rangle = \lim_{m \to \infty} \left\langle \sum_{n=0}^{m} (-\delta_{-\tau}\alpha)^n, \psi \right\rangle$$
$$= \sum_{n=0}^{n_0} \langle (-\delta_{-\tau}\alpha)^n, \psi \rangle$$

because $\langle (-\delta_{-\tau}\alpha)^n, \psi \rangle = 0$ for all $n > n_0$. Thus, (4.12) converges pointwise.

Now consider the following equality:

$$\left(\sum_{n=0}^{m} (-\delta_{-\tau}\alpha)^n \right) * (\delta_0 + \delta_{-\tau}\alpha) = \delta_0 - (-\delta_{-\tau}\alpha)^{m+1},$$

the right-hand side of which obviously tends to δ_0 as $m \to \infty$. Hence, (4.12) gives the inverse of $\delta_0 + \delta_{-\tau}\alpha$. ∎

5. Applications to Reachability and Controllability

We have seen that every constant linear system can be endowed with an $M_c(\mathbf{R}^-)$-module structure. As an application of this, we give a sufficient condition for a system to have the property that every reachable state is controllable to zero. Kamen [12] and Denham and Yamashita [5] already gave some solutions to this problem, but since their input space contains distributions, it is in general necessary to apply a distribution input to steer a state to zero. (For some class of systems, e.g., delay-differential systems, Kamen [12] shows that control can be done with smooth inputs; see Kamen [12, Theorem 6.4].) We shall, on the contrary, seek the condition under which reachable states can be controlled to zero by L^2-inputs. This type of question is not throughly investigated in the existing literature.

Definition 5.1. A constant linear system $\Sigma = (X, \varphi)$ is *weakly smooth* (or *accepts M_c-inputs*) if

(i) the associated semigroup $\{\Phi(t)\}_{t \geqslant 0}$ is locally equicontinuous;

(ii) there exists a (unique) continuous linear extension $\tilde{g}: M_c(\mathbf{R}^-) \to X$ of the reachability map g of Σ.

In the sequel, $\tilde{g}$ will also be denoted by g. Note that smooth systems as defined by Yamamoto [23b] (roughly speaking, systems whose state transition is governed by a functional differential equation) are weakly smooth. So this class is not overly special.

DEFINITION 5.2. Let $\Sigma = (X, \varphi)$ be a constant linear system. A state $x \in X$ is *controllable* (*to zero*) if there exist $T > 0$ and $u \in L^2[0, T)$ such that $\varphi(T, x, u) = 0$. Also, x is *controllable to zero in the sense of Kamen* (or *K-controllable to zero*) if there exist $T, \varepsilon > 0$ ($T > \varepsilon$) and $u \in L^2[0, T)$ such that $u(t) = 0$ on $[0, \varepsilon)$ and $\varphi(T, x, u) = 0$.

Since $\varphi(T, x, u) = \Phi(T)x + g(\sigma_T^l u)$, x is controllable (K-controllable) to zero if and only if $\Phi(T)x + g(\omega) = 0$ for some $\omega \in \Omega$ such that $l(\omega) \geqslant -T$ ($l(\omega) > -T$, respectively). In other words, $\delta_\tau \cdot x + g(\omega) = 0$ for some $\tau < 0$ and $\omega \in \Omega$ with $l(\omega) \geqslant \tau$ ($l(\omega) > \tau$) using the module action on Σ defined in Section 3.

Remark 5.3. Kamen [12] introduced the above notion of controllability. These two notions of controllability obviously coincide when Σ is finite-dimensional. Whether or not controllability implies K-controllability for infinite-dimensional systems seems to be unknown.

Now our basic question is the following: When is the reachable set X_r $(= g(\Omega))$ controllable to zero (which means every element of X_r is controllable)? In order to build a decent controller at least every element of X_r must be controllable to zero. Further, if this is the case, then every element of X_r can be steered to any other element of X_r by the application of a suitable input. This implies, in particular, that if the system Σ is quasi-reachable (i.e., X_r is dense in the whole state space) and possesses the above property, then every state can be steered to any other state arbitrarily closely *because our module action is continuous.*

We can rewrite the reachable set X_r, by virtue of Proposition 3.30, as follows:

$$X_r = g(\Omega) = \{g(\omega * \delta_0) \colon \omega \in \Omega\} = \{\omega \cdot g(\delta_0) \colon \omega \in \Omega\} = \Omega \cdot G, \quad (5.4)$$

where $G := g(\delta_0)$. Similarly, $g(M_c(\mathbf{R}^-)) = M_c(\mathbf{R}^-) \cdot G$.

Let M be a subset of the state space X of a system Σ, and let Ann M denote the annihilator (ideal) of M, i.e.,

$$\text{Ann } M := \{\mu \in M_c(\mathbf{R}^-) \colon \mu \cdot x = 0 \quad \text{for all} \quad x \in M\}. \quad (5.5)$$

We start with the following lemma:

LEMMA 5.6. *Let $\Sigma = (X, \varphi)$ be a weakly smooth constant linear system. Then* Ann $G =$ Ann $X_r =$ Ann $\bar{X}_r$.

Proof. The second equality is an immediate consequence of the separate continuity of our module action (Theorem 3.24).

It, therefore, suffices to prove Ann $G \supset$ Ann $\bar{X}_r$ and Ann $G \subset$ Ann X_r. Since Ω is easily seen to be dense in $M_c(\mathbf{R}^-)$ (for a proof, see, e.g., Yamamoto [23b]), δ_0 can be approximated by elements of Ω. Hence $G = g(\delta_0)$ belongs to $\bar{X}_r$ due to the continuity of g. It, thus, follows that Ann $G \supset$ Ann $\bar{X}_r$.

Now let $\lambda \in \operatorname{Ann} G$ and $x \in X_r$. Then $x = g(\omega)$ for some $\omega \in \Omega$. Hence, $\lambda \cdot x = \lambda \cdot g(\omega) = \lambda \cdot g(\omega * \delta_0) = \lambda \cdot (\omega \cdot g(\delta_0)) = (\lambda * \omega) \cdot G = (\omega * \lambda) \cdot G = \omega \cdot (\lambda \cdot G) = 0$ by Theorem 3.28 and Proposition 3.30. Thus $\lambda \in \operatorname{Ann} X_r$, which completes the proof. ∎

PROPOSITION 5.7. *Let $\Sigma = (X, \varphi)$ be a weakly smooth constant linear system. Suppose $G = g(\delta_0)$ is K-controllable to zero. Then there exist $\tau < 0$ and $\alpha \in \Omega$ with $l(\alpha) > \tau$ such that $\delta_\tau + \alpha \in \operatorname{Ann} X_r$.*

Proof. Since G is K-controllable to zero, there exist $\tau < 0$ and $\alpha \in \Omega$ such that $l(\alpha) > \tau$ and $\Phi(-\tau)G + g(\alpha) = 0$. But this means that $(\delta_\tau + \alpha) \cdot G = \Phi(-\tau)G + \alpha \cdot g(\delta_0) = \Phi(-\tau)G + g(\alpha * \delta_0) = \Phi(-\tau)G + g(\alpha) = 0$, i.e., $\delta_\tau + \alpha \in \operatorname{Ann} G$. Since $\operatorname{Ann} G = \operatorname{Ann} X_r$ by the preceding lemma, the proof is complete. ∎

The following lemma is essentially due to Kamen [12] and modified to the present context.

LEMMA 5.8. *Let $\Sigma = (X, \varphi)$ be a weakly smooth constant linear system, and g its reachability map. Suppose that there exists $\beta \in \operatorname{Ann} X_r$ having an inverse β^{-1} in $M_l(\mathbf{R})$. Let $\tau < l(\beta)$. Then for each $x = g(\omega) \in X_r$ $(\omega \in \Omega)$, there exists $\alpha \in \Omega$ such that $x = g(\alpha)$ and $l(\alpha) > \tau$.*

Proof. If $l(\omega) > \tau$, there is nothing to prove. So assume that $l(\omega) \leqslant \tau$. We have $\beta * (\beta^{-1} * \omega) = \omega$ in $M_l(\mathbf{R})$. Hence, $l(\omega) \geqslant l(\beta) + l(\beta^{-1} * \omega)$ because $\operatorname{supp} \mu * \nu \subset \operatorname{supp} \mu + \operatorname{supp} \nu$. Note that $\beta^{-1} * \omega \in L^2_+(\mathbf{R})$ by Proposition 4.10. Since $l(\omega) \leqslant \tau$ and $\tau < l(\beta)$,

$$l(\beta^{-1} * \omega) \leqslant l(\omega) - l(\beta) \leqslant \tau - l(\beta) < 0.$$

Now choose $a_1, a_2, b_1, b_2 \in \mathbf{R}$ such that $a_2 < a_1 < l(\beta^{-1} * \omega)$ and $\tau - l(\beta) < b_1 < b_2 < 0$. And let ψ be a continuous function with compact support such that $0 \leqslant \psi(t) \leqslant 1$ for all t and

$$\begin{aligned} \psi(t) &= 1 \qquad \text{if} \quad a_1 \leqslant t \leqslant b_1 \\ &= 0 \qquad \text{if} \quad t \leqslant a_2 \quad \text{or} \quad t \geqslant b_2. \end{aligned}$$

Let $(\beta^{-1} * \omega)\psi$ denote the multiplication of $\beta^{-1} * \omega$ with ψ. Then $\operatorname{supp} (\beta^{-1} * \omega)\psi \subset [l(\beta^{-1} * \omega), \; b_2]$, which implies that $(\beta^{-1} * \omega)\psi \in M_c(\mathbf{R}^-)$. Furthermore, since $(\beta^{-1} * \omega) \in L^2_+(\mathbf{R})$, $(\beta^{-1} * \omega)\psi \in \Omega$ follows.

Now define $\alpha := -\beta * [(\beta^{-1} * \omega)\psi - \beta^{-1} * \omega]$. Then $\alpha = -\beta * [(\beta^{-1} * \omega)\psi] + \omega$. Since $\beta \in M_c(\mathbf{R}^-)$, $\beta * [(\beta^{-1} * \omega)\psi]$ belongs to Ω by Theorem 3.28 and the remark after the proof of this theorem. Hence, $\alpha \in \Omega$. Also, $g(\beta * [(\beta^{-1} * \omega)\psi]) = \beta \cdot g([(\beta^{-1} * \omega)\psi]) = 0$ because $\beta \in \operatorname{Ann} X_r$. Hence, $g(\alpha) = g(\omega)$ follows.

It remains only to prove $l(\alpha) > \tau$. By construction, $(\beta^{-1} * \omega)\,\psi = \beta^{-1} * \omega$ on $(-\infty, b_1)$ and hence

$$\operatorname{supp}\,[-(\beta^{-1} * \omega)\,\psi + \beta^{-1} * \omega] \subset [b_1, \infty).$$

Then by the definition of α, $\operatorname{supp} \alpha \subset [b_1 + l(\beta), 0]$ because $\alpha \in \Omega$. Therefore, $l(\alpha) \geqslant b_1 + l(\beta) > \tau$. ∎

We are now ready to prove the main theorem of this section.

THEOREM 5.9. *Let $\Sigma = (X, \varphi)$ be a weakly smooth constant linear system, and g its reachability map. Suppose that $G = g(\delta_0)$ is K-controllable to zero in time T. Then the reachable set X_r is reachable and controllable in bounded time $T + \varepsilon$, where ε is an arbitrary positive number.*

Proof. Let α be an input which steers G to 0 in time T. Then by Proposition 5.7, $\delta_{-T} + \alpha \in \operatorname{Ann} X_r$. Since $\delta_{-T} + \alpha$ has an inverse in $M_l(\mathbf{R})$ according to Lemma 4.11, every element of X_r is reachable in bounded time $T + \varepsilon$ by Lemma 5.8; in other words, $g(\Omega) = g(L^2[-T - \varepsilon, 0])$.

Now take any $x \in X_r$. Then $x = g(\omega)$ for some $\omega \in L^2[-T - \varepsilon, 0]$. Since $\sigma_{T+\varepsilon}\omega \in \Omega$, $\Phi(T + \varepsilon)\,x = g(\sigma_{T+\varepsilon}\omega)$ is reachable, so it must belong to $g(L^2[-T - \varepsilon, 0])$. Hence, there exists $\hat{\omega} \in L^2[-T - \varepsilon, 0]$ such that $-\Phi(T + \varepsilon)\,x = g(\hat{\omega})$. It now follows that

$$\varphi(T + \varepsilon, x, \sigma^r_{T+\varepsilon}\hat{\omega}) = \Phi(T + \varepsilon)x + g(\hat{\omega}) = 0,$$

i.e., x is controllable in time $T + \varepsilon$. Since $T + \varepsilon$ is independent of each x, X_r is controllable in bounded time $T + \varepsilon$. ∎

Remark 5.10. Generalization of this result to the multi-input case can be done in exactly the same way as in the work of Kamen [12].

Remark 5.11. Suppose that a constant linear system Σ is smooth, i.e., its state transition is described by the functional differential equation:

$$\dot{x} = Fx + Gu, \qquad x, G \in X,$$

such that the semigroup generated by F is locally equicontinuous. It is easy to verify that $g(\delta_0) = G$. This means that the bounded-time reachability (controllability) of Σ can be determined by testing K-controllability of G.

6. AN EXAMPLE

Consider the delay-differential system specified by the functional differential equation:

$$\frac{d}{dt}\begin{pmatrix} x \\ z(\theta) \end{pmatrix} = \begin{pmatrix} x + z(0) \\ dz(\theta)/d\theta \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u =: F\begin{pmatrix} x \\ z(\theta) \end{pmatrix} + Gu, \qquad -1 \leqslant \theta \leqslant 0, \tag{6.1}$$

where $(x, z(\theta))'$ denotes the state of this system which belongs to $M_2 = \mathbf{R} \times L^2[-1, 0]$, and u the input. (For more detail, see Manitius and Triggiani [14].) The domain of the operator F is given by

$$D(F) := \left\{ \begin{pmatrix} x \\ z(\theta) \end{pmatrix} \in M_2 : z \in W_2^1[-1,0], z(-1) = x \right\}, \tag{6.2}$$

where $W_2^1[-1, 0]$ is the first-order Sobolev space on $[-1, 0]$. For a more intuitive interpretation of this system, consult Fig. 1. Although we do not give details here, it is easy to see that the above given G is precisely the image of the delta function under the reachability map. Hence, according to Theorem 5.9, *if G is K-controllable to zero, then every reachable state is reachable and controllable in bounded time.*

In order to see that this is indeed the case, consider a function u which satisfies

(i) $u(t) = 0$ on $[0, \varepsilon)$, $[2\varepsilon, 1)$ and $[1 + 2\varepsilon, \infty)$;

(ii) $1 + \int_{\varepsilon}^{2\varepsilon} u(s)\, ds = 0$;

(iii) $u(t) = -1 - \int_0^{t-1} u(s)\, ds$ for $1 \leqslant t < 1 + 2\varepsilon$.

We claim that this input drives G to 0. To see this, consider the diagram in Fig. 1. For the first ε seconds, nothing is done to the system. Then for the next ε seconds, the input u is fed into the system to steer the integrator part of G to 0 at time 2ε. Of course, the integrator produces an output in the meantime which is $1 + \int_0^t u(s)\, ds$. This output is stored in the delay, and at time 1 it starts to be fed back to the integrator through the feedback path. But then the newcoming input through the input channel cancels this feedback according to (iii) of (6.4). Hence, at time $1 + 2\varepsilon$, the state of the system becomes 0 and remains 0 afterward.

We may thus conclude, from Theorem 5.9, that in this system every reachable state can be controlled to zero in bounded time $1 + \varepsilon'$, where ε' is an arbitrary positive number.

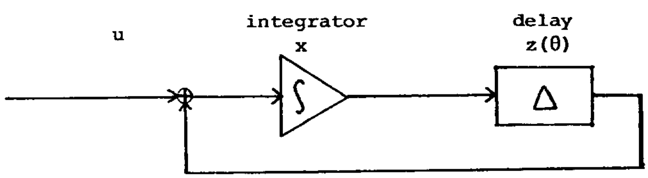


FIGURE 1

7. Concluding Remarks

In this work we have introduced a new module structure to a large class of constant linear systems. In view of the recent growing interest in module theoretic approaches to structural properties of constant linear systems (see, for example, Fuhrmann [8], Kamen [12], and Denham and Yamashita [5]), it appears important to have a firm theoretical link between this type of approach and those via functional–analytic methods. It is hoped that our approach will serve as a common language in this type of investigations.

Though the machinery employed here is rather heavy, its implication is quite straightforward and easy to understand: *Every (weakly smooth) constant linear system admits an* $M_c(\mathbf{R}^-)$*-module structure.* This is a natural extension of Kalman's $k[z]$-module framework for finite-dimensional constant linear systems. Further, there is no need to know deep results on topological vector spaces, measures, etc., once this result is accepted as such. Afterward, one can proceed with essentially algebraic operations.

Our approach has an added advantage: this module action is separately continuous. Hence, one can be quite free from possible dangers encountered in handling infinite-dimensional systems.

Also, unlike in Fuhrmann [7], the infinitesimal generator of the associated semigroup need not have a spectral representation. Of course, when the semigroup has such a representation, our result essentially coincides with that of Fuhrmann [7].

There is already a vast amount of literature on systems over rings, or ring-theoretic approaches to constant linear systems (see, e.g., Kamen [11], Morse [15], Williams and Zakian [22], and Sontag [18] for an excellent survey). It is not yet entirely clear how our present approach is related to this type of research, but it is certainly an interesting theme for further investigations.

Though we have given the results for scalar (single-input) systems mainly for simplicity of notations, there is no difficulty in generalizing the results in Sections 3 and 4 to the multi-input case. Also, as mentioned in Remark 5.10, Theorem 5.11 can be generalized to the multi-input systems in the same way as in the work of Kamen [12].

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Note added in proof. We later found that the condition $l(\alpha) > \tau$ is not necessary in Lemma 4.11, but $l(\alpha) \geqslant \tau$ is enough (see, e.g., Schwartz [24, III.2, Theorem 16]). Hence, one does not have to require K-controllability in Theorem 5.9, but simply controllability of G is enough.

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