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Exceptional Values of the Dedekind Symbol

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Five new exceptional values of the Dedekind symbol are presented, and a conjecture is proposed on the necessary and sufficient conditions for integers to be exceptional values.

1. INTRODUCTION

For coprime integers h and k with $k>0$, the Dedekind symbol (h, k) is defined as

$$(h, k) = 6ks(h, k) = 6k \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left[\frac{hr}{k}\right] - \frac{1}{2}\right),$$

where $s(h, k)$ is the Dedekind sum (cf. Hirzebruch and Zagier [2], Rademacher and Grosswald [4]). This symbol is integer-valued. The following properties are well known:

$$2h(h, k) + 2k(k, h) = h^2 + k^2 - 3hk + 1, \tag{1}$$

$$(h', k) = (h, k) \qquad \text{if} \quad h' \equiv h \pmod{k}, \tag{2}$$

$$(-h, k) = -(h, k), \tag{3}$$

$$(h, k) \equiv 0, \pm 1, \pm 3 \pmod{9}.$$

For $w \equiv 0, \pm 1, \pm 3 \pmod{9}$ we define the *order* of w by

$$\text{ord } w = \inf\{k \mid (h, k) = w \text{ for some } h\};$$

if w does not occur as a value (h, k), we set ord $w = \infty$ and call w an *exceptional value*. Since (3) shows that w and $-w$ are of the same order, we may consider only the non-negative exceptional values. Salié who investigated

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exceptional values for the first time stated in [5] that 12, 17, 44, and 107 would be exceptional. Recently, Asai [1] has proved this and further shown that 152, 172, 197, 530 are exceptional. Very recently, Norimune Saito has found exceptional values 962, 1025, 1682, 2402, 4625, 8837, 10610, 12545, and 12770. It is of interest that the above numbers except for 12 and 172 are congruent to -1 modulo 9. This fact motivated us to search for further exceptional values. In fact, we have determined the exceptional values between 12770 and 56645. They are the following five numbers:

24965, 27890, 33857, 37250, and 37637.

Our computational results strongly suggest:

CONJECTURE 1. *Suppose* $w \in \mathbb{N}$, $w^2 \equiv 0$ *or* 1 (mod 9). *Then* w *is exceptional if and only if*

(i) $w-1$ *is a square not divisible by any prime* $\equiv \pm 3 \pmod 8$, *or*

(ii) $w = 12, 44, 107, 152,$ *or* 172.

2. Computation and Result

In what follows we always assume $w \in \mathbb{N}$, $w^2 \equiv 0$ or 1 (mod 9). First we recall Asai's criterion.

THEOREM A[1]. *If* $w > 1$ *is of finite order, then* ord $w < 2w$.

Remark. Salié's result [5, Satz 5′] is equivalent to the lower bound ord $w \geqslant \frac{3}{2} + \sqrt{2w + (1/4)}$.

Let N be a given positive integer. With the aid of Theorem A, (2) and (3), we can determine all exceptional $w \leqslant N$ by evaluating (h, k) for all fractions h/k in the first half of the Farey series of order $2N-1$, namely for all non-negative reduced fractions $h/k \leqslant \frac{1}{2}$ with $0 < k \leqslant 2N-1$. More precisely, if $w \leqslant N$ does not appear in the absolute values of such (h, k), then w is exceptional. For the evaluation, as Rademacher and Asai stated, we can employ the relation:

$$(h_1 + h_2, k_1 + k_2) = (h_1, k_1) + (h_2, k_2) - k_1 + k_2 \tag{4}$$

for adjacent Farey fractions $h_1/k_1 < h_2/k_2$ [1, Lemma 3] (the symbol version of [3, Satz 5]). But, Saito used (1).

Actually, utilizing (4) with the initial values $(0, 1) = (1, 2) = 0$, we have computed the values of the Dedekind symbol for all 877, 848, 679 fractions in the first half of the Farey series of order 76,000. Simultaneously sifting

TABLE I

w	$\sqrt{w-1}$	w	$\sqrt{w-1}$	w	$\sqrt{w-1}$
12	—	198917	$446 = 2 \cdot 223$	954530	977
17	$4 = 2^2$	214370	463	988037	$994 = 2 \cdot 7 \cdot 71$
44	—	232325	$482 = 2 \cdot 241$	1008017[a]	$1004 = 2^2 \cdot 251$
107	—	258065	$508 = 2^2 \cdot 127$	1026170[a]	1013
152	—	276677	$526 = 2 \cdot 263$	1042442[a]	1021
172	—	277730	$527 = 17 \cdot 31$	1044485	$1022 = 2 \cdot 7 \cdot 73$
197	$14 = 2 \cdot 7$	295937	$544 = 2^5 \cdot 17$	1062962	1031
530	23	305810	$553 = 7 \cdot 79$	1079522	1039
962	31	315845	$562 = 2 \cdot 281$	1100402	1049
1025	$32 = 2^5$	358802	599	1117250	$1057 = 7 \cdot 151$
1682	41	368450	607	1119365	$1058 = 2 \cdot 23^2$
2402	$49 = 7^2$	380690	617	1157777[a]	$1076 = 2^2 \cdot 269$
4625	$68 = 2^2 \cdot 17$	391877	$626 = 2 \cdot 313$	1175057	$1084 = 2^2 \cdot 271$
8837	$94 = 2 \cdot 47$	414737	$644 = 2^2 \cdot 7 \cdot 23$	1194650[a]	1093
10610	103	461042	$679 = 7 \cdot 97$	1196837[a]	$1094 = 2 \cdot 547$
12545	$112 = 2^4 \cdot 7$	485810	$697 = 17 \cdot 41$	1216610	1103
12770	113	498437	$706 = 2 \cdot 353$	1274642	1129
24965	$158 = 2 \cdot 79$	537290[a]	733	1295045	$1138 = 2 \cdot 569$
27890	167	538757	$734 = 2 \cdot 367$	1317905	$1148 = 2^2 \cdot 7 \cdot 41$
33857	$184 = 2^3 \cdot 23$	552050	743	1336337	$1156 = 2^2 \cdot 17^2$
37250	193	564002	751	1378277[a]	$1174 = 2 \cdot 587$
37637	$194 = 2 \cdot 97$	565505	$752 = 2^4 \cdot 47$	1420865[a]	$1192 = 2^3 \cdot 149$
56645	$238 = 2 \cdot 7 \cdot 17$	579122	761	1423250	1193
57122	239	591362	769	1442402	1201
61505	$248 = 2^3 \cdot 31$	619370[a]	787	1444805	$1202 = 2 \cdot 601$
65537	$256 = 2^8$	633617	$796 = 2^2 \cdot 199$	1466522[a]	$1211 = 7 \cdot 173$
66050	257	677330	823	1485962[a]	$1219 = 23 \cdot 53$
75077	$274 = 2 \cdot 137$	678977	$824 = 2^3 \cdot 103$	1507985[a]	$1228 = 2^2 \cdot 307$
80657	$284 = 2^2 \cdot 71$	693890	$833 = 7^2 \cdot 17$	1510442[a]	1229
85265	$292 = 2^2 \cdot 73$	737882[a]	859	1530170[a]	1237
91205	$302 = 2 \cdot 151$	753425	$868 = 2^2 \cdot 7 \cdot 31$	1532645[a]	$1238 = 2 \cdot 619$
96722	311	770885	$878 = 2 \cdot 439$	1552517	$1246 = 2 \cdot 7 \cdot 89$
107585	$328 = 2^3 \cdot 41$	784997[a]	$886 = 2 \cdot 443$	1577537[a]	$1256 = 2^3 \cdot 157$
108242	$329 = 7 \cdot 47$	786770	887	1597697	$1264 = 2^4 \cdot 79$
113570	337	802817	$896 = 2^7 \cdot 7$	1643525	$1282 = 2 \cdot 641$
126737	$356 = 2^2 \cdot 89$	817217	$904 = 2^3 \cdot 113$	1646090[a]	1283
145925	$382 = 2 \cdot 191$	835397	$914 = 2 \cdot 457$	1666682[a]	1291
146690	383	868625	$932 = 2^2 \cdot 233$	1692602[a]	1301
152882	$391 = 17 \cdot 23$	885482[a]	941	1737125[a]	$1318 = 2 \cdot 659$
153665	$392 = 2^3 \cdot 7^2$	917765	$958 = 2 \cdot 479$	1739762	1319
160802	401	919682	$959 = 7 \cdot 137$	1760930	1327
167282	409	935090	967	1784897	$1336 = 2^3 \cdot 167$

TABLE I—*Continued*

w	$\sqrt{w-1}$	w	$\sqrt{w-1}$	w	$\sqrt{w-1}$
1787570	$1337 = 7 \cdot 191$	2758922[a]	$1661 = 11 \cdot 151$	3045698	—
1811717	$1346 = 2 \cdot 673$	2764277	—	3063772	—
1882385	$1372 = 2^2 \cdot 7^3$	2785562[a]	1669	3066002	$1751 = 17 \cdot 103$
1885130[a]	1373	2796029	—	3073967	—
1907162[a]	1381	2815685	$1678 = 2 \cdot 839$	3076442	—
1909925[a]	$1382 = 2 \cdot 691$	2817386	—	3077632	—
1957202	1399	2819042	$1679 = 23 \cdot 73$	3079313	—
1985282	1409	2838311	—	3084902	—
2010725[a]	$1418 = 2 \cdot 709$	2845970	$1687 = 7 \cdot 241$	3090187	—
2033477	$1426 = 2 \cdot 23 \cdot 31$	2849345[a]	$1688 = 2^3 \cdot 211$	3094082	1759
2036330[a]	1427	2857157	—	3112948	—
2062097	$1436 = 2^2 \cdot 359$	2876417[a]	$1696 = 2^5 \cdot 53$	3114116	—
2111210[a]	1453	2878507	—	3116897	—
2114117	$1454 = 2 \cdot 727$	2879308	—	3126529	—
2163842	1471	2879810	1697	3127157	—
2166785	$1472 = 2^6 \cdot 23$	2887127	—	3130397	—
2193362	1481	2887687	—	3134420	—
2217122	1489	2901167	—	3135743	—
2247002[a]	1499	2902706	—	3138112	—
2298257[a]	$1516 = 2^2 \cdot 379$	2905597	—	3157352	—
2328677[a]	$1526 = 2 \cdot 7 \cdot 109$	2909692	—	3157730	1777
2380850	1543	2910437[a]	$1706 = 2 \cdot 853$	3161285	$1778 = 2 \cdot 7 \cdot 127$
2383937	$1544 = 2^3 \cdot 193$	2926817	—	3165146	—
2391893	—	2930867	—	3167731	—
2408705	$1552 = 2^4 \cdot 97$	2937797	$1714 = 2 \cdot 857$	3182156	—
2411810	1553	2941073	—	3193370[a]	1787
2436722	$1561 = 7 \cdot 223$	2956067	—	3195152	—
2440412	—	2962367	—	3200197	—
2468042[a]	1571	2968730[a]	1723	3213523	—
2493242[a]	1579	2972177	$1724 = 2^2 \cdot 431$	3217877	—
2521745[a]	$1588 = 2^2 \cdot 397$	2982572	—	3219146	—
2524922[a]	$1589 = 7 \cdot 227$	2987162	—	3225518	—
2550410[a]	1597	2996497	—	3225617	$1796 = 2^2 \cdot 449$
2553605	$1598 = 2 \cdot 17 \cdot 47$	2999825	$1732 = 2^2 \cdot 433$	3239362	—
2582450	1607	3001607	—	3249332	—
2611457[a]	$1616 = 2^4 \cdot 101$	3003290[a]	1733	3251762	—
2637377[a]	$1624 = 2^3 \cdot 7 \cdot 29$	3003922	—	3255002	—
2666690	$1633 = 23 \cdot 71$	3018077	—	3255623	—
2670911	—	3025556	—	3258512	—
2696165[a]	$1642 = 2 \cdot 821$	3031082[a]	1741	3261158	—
2699450[a]	$1643 = 31 \cdot 53$	3031397	—	3263804	—
2729105[a]	$1652 = 2^2 \cdot 7 \cdot 59$	3043018	—		
2757851	—	3045212	—		

[a] Integers of the form $n^2 + 1$ with n divisible by some prime $\equiv \pm 3 \pmod 8$.

their absolute values from the searching range $w < 3{,}270{,}000$, we have obtained the 253 integers of order $> 76{,}000$ in the range, which are listed in Table I. The program which exploits the linked list data structure has been written in FORTAN 77 and run on FACOM M-382 at Computer Center, Kyushu University.

Table I and Theorem A give us the exceptional values exhibited in the Introduction. Furthermore Table I allows us to state our conjecture. Indeed, we verify that all 129 integers $< 3{,}270{,}000$ satisfying the conditions in Conjecture 1 are at least of order $> 76{,}000$ and that the first 17 of such numbers coincide with those of exceptional values except for the five irregulars.

3. Some Observations

We calculated the order of all $w \leqslant 200{,}000$ not listed in Table I. Our data indicate that $n^2 + 1$ with $n \equiv \pm 4 \pmod 9$, even if it is not exceptional, is of remarkably high order for some kind of n, e.g., a prime number n, and of course suggest that the result of Theorem A can be improved (see Fig. 1).

We also investigated solutions of the equation

$$(h, k) = n^2 + 1. \tag{5}$$

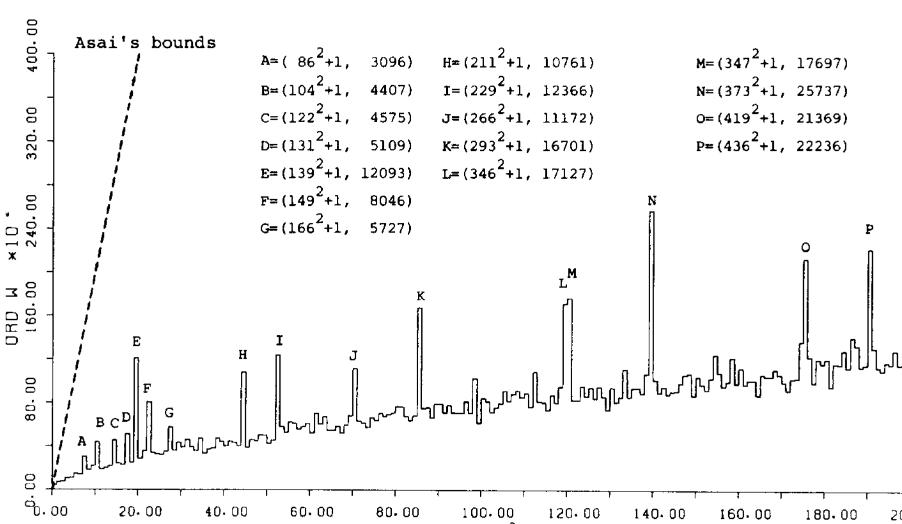

FIG. 1. $\max\{\text{ord } w \leqslant 76000 \mid 10^3(m-1) < w \leqslant 10^3 m\}$, $1 \leqslant m \leqslant 200$

CONJECTURE 2. *For each solution of* (5) *there exists a prime number* $p \equiv \pm 3 \pmod 8$ *such that*

$$k \equiv n \equiv 0 \pmod{p}. \tag{6}$$

Herein we note that (6) together with (1) yields $h \equiv 1 \pmod{p}$. When $3 \mid n$, this conjecture is obviously true because it is known [4, p. 27] that $3 \nmid (h, k)$ if and only if $3 \mid k$. There are no counterexamples in the 6,741 solutions of (5) with the restrictions $|h| \leqslant [k/2]$, $k \leqslant 26{,}000$, $n^2+1 <$ 800,000, and $3 \nmid n$ (see, e.g., Table II). Conjecture 2 enables us to find effectively solutions of (5) for fixed n.

Last, we easily verify by (1) that

$$(h, k, n) = (3m+1, 18m+3, 3m)$$

is a solution of (5) for any $m \geqslant 0$. In a future paper we shall give several families of solutions of (5) with $3 \nmid n$.

TABLE II

Solutions of (5) with $k = \mathrm{ord}(n^2+1)$, h Absolutely Least on Such k and n, and p a Common Prime Divisor $\equiv \pm 3 \pmod 8$ of k and n

n	h	k	p	n	h	k	p
5	11	30	5	149	−1489	8046	149
13	−25	78	13	157	158	3297	157
$22 = 2 \cdot 11$	23	132	11	$166 = 2 \cdot 83$	1163	5727	83
$40 = 2^3 \cdot 5$	11	195	5	$175 = 5^2 \cdot 7$	251	750	5
$50 = 2 \cdot 5^2$	26	375	5	$176 = 2^4 \cdot 11$	89	2607	11
$58 = 2 \cdot 29$	−115	696	29	$185 = 5 \cdot 37$	506	3945	5
59	−235	1062	59	$202 = 2 \cdot 101$	203	3636	101
67	68	1005	67	$203 = 7 \cdot 29$	−985	2349	29
$76 = 2^2 \cdot 19$	77	912	19	211	1478	10761	211
$77 = 7 \cdot 11$	89	1221	11	$212 = 2^2 \cdot 53$	107	3180	53
$85 = 5 \cdot 17$	−119	465	5	$220 = 2^2 \cdot 5 \cdot 11$	−1858	4587	11
$86 = 2 \cdot 43$	1205	3096	43	$221 = 13 \cdot 17$	14	1209	13
$95 = 5 \cdot 19$	296	1065	5	229	−1831	12366	229
$104 = 2^3 \cdot 13$	521	4407	13	$230 = 2 \cdot 5 \cdot 23$	221	3480	5
$121 = 11^2$	254	2013	11	$247 = 13 \cdot 19$	−1156	6981	13
$122 = 2 \cdot 61$	428	4575	61	$265 = 5 \cdot 53$	107	2862	53
$130 = 2 \cdot 5 \cdot 13$	236	1635	5	$266 = 2 \cdot 7 \cdot 19$	1331	11172	19
131	656	5109	131	$275 = 5^2 \cdot 11$	926	5085	5
139	−4447	12093	139	283	284	7641	283
$140 = 2^2 \cdot 5 \cdot 7$	71	1260	5	293	1466	16701	293
$148 = 2^2 \cdot 37$	−295	3552	37	$301 = 7 \cdot 43$	302	6321	43

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