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Duality Modules and Morita Duality

J. M. ZELMANOWITZ

University of California, Santa Barbara, California 93106

AND

W. JANSEN

University of Toledo, Toledo, Ohio 43606

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Every duality between a full subcategory $\mathscr{C}$ of topological R-modules and a full subcategory $\mathscr{D}$ of Mod-S which contains S and is closed under isomorphisms is determined in a natural way by a bimodule $_RU_S$. This article examines the situation when $\mathscr{D} = \text{Mod-}S$, in which case one gets a duality that may be considered as being one-half of a Morita duality. © 1989 Academic Press, Inc.

INTRODUCTION

By a *duality* we mean a contravariant category equivalence. Covariant category equivalences will be referred to as *equivalences.*

A bimodule $_RU_S$ will be called a *Morita-duality module* if it is a balanced injective cogenerator for both R-Mod and Mod-S. A left R-module M is here called *U°-reflexive* if the natural R-homomorphism $M \to \text{Hom}_S(\text{Hom}_R(M, U), U)$ is an isomorphism. When $_RU_S$ is a Morita-duality module, $\text{Hom}_R(\ , U)$ defines a duality between the U°-reflexive left R-modules and the U°-reflexive right S-modules and these subcategories are closed under submodules and epimorphic images. Conversely, if $\mathscr{C} \subseteq R$-Mod and $\mathscr{D} \subseteq$ Mod-S are full subcategories closed under submodules and epimorphic images and containing $_RR$ and S_S, respectively, then any duality between $\mathscr{C}$ and $\mathscr{D}$ is naturally equivalent to $\text{Hom}(\ , U)$ for some Morita-duality module $_RU_S$. This is a theorem due to Azumaya [2] and Morita [9], and we will refer to such a duality, when it exists, as a *Morita duality* between R and S.

In contrast to equivalences, which always exist for an arbitrary module category (e.g., between R-Mod and R_n-Mod, where R_n is the $n \times n$ matrix

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ring over R), category dualities between entire module categories R-Mod and Mod-S never exist [1; Lemma 24.7]. The reason for this is essentially an elaboration of the fact that the dimension of the dual of a countable dimensional vector space is not countable.

When a Morita duality exists between R and S then R and S must be semiperfect rings [14] and the U-reflexive modules are precisely the modules which are linearly compact in the discrete topology. This was shown by Müller in [10]. In fact, a Morita duality exists for a ring R precisely when ${}_RR$ and its minimal injective cogenerator are linearly compact in the discrete topology [10]. (The question of for which rings Morita dualities actually exist was already asked for commutative rings by Zelinsky [17]. It remains open at this time.)

Topology having now entered the picture, however slightly, one should recall the classical Lefschetz duality [7] which states that when R is a field the standard Morita duality between finite dimensional vector spaces can be extended to a duality between all vector spaces which are linearly compact in a Hausdorff linear topology and all discrete vector spaces.

The goals of this paper are twofold. First, we wish to extend Lefschetz duality to module categories over arbitrary rings, insofar as possible. A parallel objective is to provide an asymmetric generalization of Morita duality, which is itself of course symmetric with respect to R and S. The vehicle for doing this is the study of *duality* R-modules; these are discrete bimodules ${}_RU_S$ such that ${}_RU$ is a finitely cogenerated, linearly compact, quasi-injective self-cogenerator and S is naturally isomorphic to $\mathrm{Hom}_R(U, U)$. Duality modules should be regarded as "half-Morita-duality modules" because a discrete bimodule ${}_RU_S$ is a Morita duality module if and only if it is a duality R-module and a (right) duality S-module.

Duality R-modules are a particular instance of the *strongly quasi-injective* R-modules defined in [8]. It was shown there that a strongly quasi-injective R-module U yields a duality between the class of topological left R-modules which are isomorphic to closed submodules of a product of copies of ${}_RU$ and the class of right S-modules which are isomorphic to submodules of products of U_S.

In our particular situation, it is easy to deduce from [8] and [13] that duality R-modules are indeed associated with an asymmetrical form of Morita duality, and that their existence is necessary and sufficient for a Lefschetz-type duality between a certain subcategory of linearly compact left R-modules and the category Mod-S of right S-modules. This kind of duality may exist even without an underlying Morita duality being present. If a Morita duality does exist between R and S, defined by the (discrete) Morita-duality module ${}_RU_S$, then it extends to a duality between the category of linearly compact left R-modules which are (topologically) cogenerated by ${}_RU$ and the module category Mod-S.

We have organized our presentation so that the topological dualities appear first, and then the algebraic dualities occur as subdualities. To describe them we first establish some conventions and definitions.

${}_RU_S$ will always denote an R-S-bimodule endowed with a Hausdorff topology such that the elements of S induce continuous R-homomorphisms. R-Top will stand for the category of Hausdorff topological left R-modules with the ring of operators R having the discrete topology. For $M, N \in R$-Top, $\mathrm{Cont}_R(M, N)$ will denote the group of continuous R-homomorphisms from M to N. We call $L \in \text{Mod-}S$ *U-reflexive* if the natural S-homomorphism $L \to \mathrm{Cont}_R(\mathrm{Hom}_S(L, U), U)$ is an S-isomorphism; here $\mathrm{Hom}_S(L, U)$ is endowed with the subspace topology induced by the product topology on U^L. Similarly, $M \in R$-Top is *U-reflexive* if the natural R-homomorphism $M \to \mathrm{Hom}_S(\mathrm{Cont}_R(M, U), U)$ is a homeomorphism. Finally, it is convenient to say that there is a *U-duality* between full subcategories $\mathscr{C} \subseteq R$-Top and $\mathscr{D} \subseteq \text{Mod-}S$ when $\mathrm{Cont}_R(\ , U)\colon \mathscr{C} \to \mathscr{D}$, $\mathrm{Hom}_S(\ , U)\colon \mathscr{D} \to \mathscr{C}$, and every module in $\mathscr{C}$ and $\mathscr{D}$ is U-reflexive.

Section 1 is devoted to basic facts about U-dualities. There is always a U-duality between the subcategories of U-reflexive modules in R-Top and in Mod-S (Theorem 1.6). In Theorem 1.3 we show that the dualities which will be of interest to us are always U-dualities. For if $\mathscr{C}$ is a full subcategory of R-Top which is closed under isomorphisms, direct products, and closed submodules and $\mathscr{D}$ is a full subcategory of Mod-S which contains S_S and is closed under isomorphisms, and if there is a duality between $\mathscr{C}$ and $\mathscr{D}$, then this duality is naturally isomorphic to a U-duality with $S \cong \mathrm{Cont}_R(U, U)$.

We consider when the U-reflexives are closed under the formation of quotients in Section 2. A key idea is that quotient modules in R-Top should not be endowed with the quotient topology; rather we assign quotients the *U-topology* which is defined as follows. For N a closed submodule of $M \in R$-Top, the sub-basic open neighborhoods of 0 in the U-topology are of the form $\mathscr{O}f^{-1}/N$ for $\mathscr{O}$ an open neighborhood of 0 in U and $f \in \mathrm{Cont}_R(M, U)$ with $Nf = 0$. With the U-topology, M/N will be called a *U-quotient* of M. In Theorem 2.3, we show that if every U-quotient of ${}_RU$ and every quotient module of S_S is U-reflexive then ${}_RU_S$ is a duality R-module.

Using results from [8] and [13], we present a converse in Section 3. To describe it, let $\mathrm{Cogen}_R U$ (respectively, $\overline{\mathrm{Cogen}}_R U$) denote the family of modules which are topologically isomorphic to a (closed) submodule of a direct product of copies of ${}_RU$. We prove, as part of Theorem 3.1, that if ${}_RU_S$ is a duality R-module then there is a U-duality between $\overline{\mathrm{Cogen}}_R U$ and Mod-S, $\overline{\mathrm{Cogen}}_R U = \{M \in \mathrm{Cogen}_R U \mid M$ is linearly compact$\}$, and $\overline{\mathrm{Cogen}}_R U$ is closed under U-quotients. In particular, this is true for any Morita-duality module ${}_RU_S$.

Contained in such a duality between $\overline{\mathrm{Cogen}}_R U$ and Mod-S lies a purely algebraic subduality between subcategories of R-Mod and Mod-S. The elucidation of this subduality occupies our attention in Section 4. The key step is to consider R-modules in the *U-adic-topology* (here called the *$U°$-topology* for brevity). This is the weakest topology with respect to which the natural R-homomorphism $M \to U^{\mathrm{Hom}_R(M,U)}$ is continuous. $M \in R$-Mod will be $U°$-reflexive if and only if M, endowed with the $U°$-topology, is U-reflexive. This enables us to describe the algebraic subduality associated with a duality R-module in Theorem 4.1. For instance, it turns out that the $U°$-reflexive R-modules are precisely those which are cogenerated by ${}_R U$ and are linearly compact in the $U°$-topology. Combining Theorems 4.1 and 4.5, one learns that a necessary and sufficient condition for the discrete bimodule ${}_R U_S$ to be a duality R-module is that all quotient modules of ${}_R U$ and S_S be $U°$-reflexive. This shows in a very precise way that a duality R-module provides one-half of a Morita duality. We also indicate in this section how the theorems of Morita and Müller on Morita duality are consequences of our approach. We conclude with a partial summary of the main results of this paper in Theorem 4.6.

Section 5 is devoted to providing examples, particularly of duality R-modules which are not Morita-duality modules.

Much of the machinery used in this article already existed in the literature, some of it for some time. We could not have conceived of this project without Müller's linkage of Morita duality with linear compactness in [10]; and we acknowledge our debt to Sandomierski's pioneering work on duality modules [15]. We rely heavily, particularly in Section 3, on the machinery developed in Menini and Orsatti's comprehensive study [8] of the duality associated with a discrete (or compact) strongly quasi-injective module. They were apparently the first to recognize that a Morita duality extends to a Lefschetz-type duality.

Another part of our inspiration for this project came from the work of Fuller for the case of equivalence between a completely additive subcategory of R-mod and a module category S-Mod. Fuller noted in [4] that such an equivalence is determined by a finitely generated, quasi-projective self-generator ${}_R U$ with $S \cong \mathrm{Hom}_R(U, U)$. Our results include a formal dual of this theorem. We are also indebted to J. Lambek for helpful seminar talks and private conversations. In addition, Lambek's observation in [6] that a discrete quasi-injective R-module U is U^A-injective in R-Top plays a key role in Section 3 of this paper, as it does also in [8]; and it was Lambek who suggested the use of U-copresented modules in Theorem 3.1.

Finally, we note how our results compare with those of Oberst. In an ambitious and important work [12], Oberst produced a general duality theorem for Grothendieck categories. When specialized to module categories [12, p. 528], it produces a subcategory $STC(\mathrm{Hom}_S(U, U))$ of linearly compact (in the sense of [12]) $\mathrm{Hom}_S(U, U)$-modules and a Lefschetz-type duality $STC(\mathrm{Hom}_S(U, U)) \rightleftarrows$ Mod-S, under the assumption that every right ideal of S is the annihilator of a finite subset of U. When restricted further to a duality R-module ${}_RU_S$, this differs from our work in at least two respects. First, we avoid the hypotheses on the right ideals of S (which in view of Theorem 2.4 would restrict ${}_RU$ to be Noetherian); and, second, we get a duality directly between Mod-S and linearly compact R-modules, without having to pass to the completion $\mathrm{Hom}_S(U, U)$.

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Throughout this paper, unless otherwise indicated, *R will denote an arbitrary ring, S a ring with identity element, and U an R-S-bimodule endowed with a structure of a topological group such that the elements of S induce continuous R-homomorphisms.* Homomorphisms will be written on the side of a module opposite to that of the scalars. R-Mod will stand for the category of left R-modules, Mod-S for the category of right S-modules, and $\mathrm{Hom}_S(K, L)$ will often be abbreviated $H_S(K, L)$.

For M and N topological R-modules, $\mathrm{Cont}_R(M, N)$ or $C_R(M, N)$ will denote the group of continuous R-homomorphisms from M to N. Whenever we write $S \cong \mathrm{Cont}_R(U, U)$ we will mean that S is naturally isomorphic to $\mathrm{Cont}_R(U, U)$ via its right action on U. By a *topological R-monomorphism* $f\colon M \to N$ we shall mean an R-monomorphism which is continuous and open onto Mf endowed with the subspace topology; and a *topological R-isomorphism* will indicate an R-isomorphism which is also a homeomorphism. R-Top will denote the category of Hausdorff topological R-modules with the ring of operators R having the discrete topology. Note that to say that the topology is Hausdorff is equivalent to assuming that the intersection of the (basic) open neighborhoods of 0 equals 0.

A topology on a left R-module M is called *linear* (and M is said to be *linearly topologized*) if it is invariant under translations and there is a basis of neighborhoods of 0 which consists of submodules of M. M thus becomes a topological module when R is endowed with the discrete topology. We will let R-Lin denote the category of Hausdorff linearly topologized left R-modules with continuous homomorphisms. Whereas all topologies in our main results will be linear and Hausdorff, this will not necessarily be required for the intermediate propositions.

For any right S-module L, *the group* $\mathrm{Hom}_S(L, U)$ *will always be assumed to carry the topology induced by considering* $\mathrm{Hom}_S(L, U)$ *as an R-submodule of the product module U^L.* That is, the basic open neighborhoods of 0 are of the form $\mathcal{O}(x_1, ..., x_t) = \{f \in \mathrm{Hom}_S(L, U) \mid fx_1, ..., fx_t \in \mathcal{O}\}$ for some

$x_1, ..., x_t \in L$ and open neighborhood $\mathcal{O}$ of 0 in U. When U is discrete this is usually called the "finite topology" on $\mathrm{Hom}_S(L, U)$.

The following elementary observations are well known.

PROPOSITION 1.1. *For any Hausdorff topology on* U, $\mathrm{Hom}_S(L, U)$ *is a closed submodule of* U^L.

PROPOSITION 1.2. *For* $M \in R$-Top *and* $L \in$ Mod-S, *there is a natural isomorphism of groups* $\mathrm{Hom}_S(L, \mathrm{Cont}_R(M, U)) \cong \mathrm{Cont}_R(M, \mathrm{Hom}_S(L, U))$.

Proof. The mutually inverse isomorphisms ϕ and ψ are defined by $(mf^{\phi})x = m(fx)$ and $m(g^{\psi}x) = (mg)x$ for any $m \in M$, $x \in L$, $f \in H_S(L, C_R(M, U))$, $g \in C_R(M, H_S(L, U))$. It is easy to verify that $f^{\phi} \in C_R(M, H_S(L, U))$, that $g^{\psi}x \in C_R(M, U)$, and that ϕ and ψ are mutually inverse group homomorphisms with g^{ψ} an S-homomorphism and f^{ϕ} an R-homomorphism. ▌

For $M \in R$-Top and $L \in$ Mod-S we define natural homomorphisms $\mu_M: M \to \mathrm{Hom}_S(\mathrm{Cont}_R(M, U), U)$ and $\nu_L: L \to \mathrm{Cont}_R(\mathrm{Hom}_S(L, U), U)$ in the usual way: $(m\mu_M)f = mf$ and $g(\nu_L x) = gx$ for $m \in M$, $x \in L$, $f \in \mathrm{Cont}_R(M, U)$, and $g \in \mathrm{Hom}_S(L, U)$. $\nu_L x$ is indeed a continuous map because for any open neighborhood $\mathcal{O}$ of 0 in U, $\mathcal{O}(\nu_L x)^{-1} = \mathcal{O}(x)$ is an open set in $\mathrm{Hom}_S(L, U)$. Similarly, μ_M is a continuous map.

We will call $M \in R$-Top *U-reflexive* if μ_M is a topological R-isomorphism; similarly, $L \in$ Mod-S is called *U-reflexive* if ν_L is an S-isomorphism. It will be useful to say that there is a *U-duality* between full subcategories $\mathscr{C} \subseteq R$-Top and $\mathscr{D} \subseteq$ Mod-S when $\mathrm{Cont}_R(\ , U): \mathscr{C} \to \mathscr{D}$, $\mathrm{Hom}_S(\ , U): \mathscr{D} \to \mathscr{C}$, and every module in $\mathscr{C}$ and $\mathscr{D}$ is U-reflexive.

The main result of this section is an analogue of Theorem 23.5 in [1]. It shows that the dualities which will be of interest to us are naturally isomorphic to U-dualities. (The proof given below demonstrates how to use Proposition 1.2 to simplify the proof given in [1].)

THEOREM 1.3. *Suppose that* $\mathscr{C}$ *is a full subcategory of* R-Top *and that* $\mathscr{D}$ *is a full subcategory of* Mod-S *which contains* S_S *and is closed under isomorphisms. Let* $\mathscr{C} \rightleftarrows^{F}_{G} \mathscr{D}$ *be a pair of mutually inverse dualities and let* $U = G(S)$ *endowed with its canonical structure of R-S-bimodule. Then*

(i) *S is naturally isomorphic to* $\mathrm{Cont}_R(U, U)$, *and*

(ii) *there is a natural isomorphism* $F(\ ) \cong \mathrm{Cont}_R(\ , U)$.

If, additionally, $\mathscr{C}$ *is closed under topological R-isomorphisms, direct products and closed submodules, then*

(iii) *there is a natural isomorphism* $G(\ )\cong \mathrm{Hom}_S(\ , U)$, *and*

(iv) *every* $M\in\mathscr{C}$ *and every* $L\in\mathscr{D}$ *is U-reflexive.*

Proof. The S-module structure on $U=G(S)$ is defined by $u\cdot s=u(G(s_\lambda))$, where $s_\lambda\in \mathrm{Hom}_S(S, S)$ is left multiplication by $s\in S$.

(i) $S\cong \mathrm{Hom}_S(S, S)\cong \mathrm{Cont}_R(G(S), G(S))$ as rings, with the second isomorphism given by G.

(ii) Let $\zeta: I_{\mathscr{C}}\to G\circ F$ and $\eta: I_{\mathscr{D}}\to F\circ G$ denote the assumed natural isomorphisms. Since F and G are full and faithful we have natural isomorphisms of groups

$$C_R(M, G(L)) \xrightarrow{C_R(\zeta_M^{-1}, 1)} C_R(GF(M), G(L)) \xrightarrow{G^{-1}} H_S(L, F(M))$$

for any $M\in\mathscr{C}$, $L\in\mathscr{D}$. Hence we have natural S-isomorphisms $F(M)\cong \mathrm{Hom}_S(S, F(M))\cong \mathrm{Cont}_R(M, G(S))=\mathrm{Cont}_R(M, U)$, so $\mathrm{Cont}_R(M, U)\in\mathscr{D}$ and it follows that $F(\ )\cong \mathrm{Cont}_R(\ , U)$.

(iii) Next, suppose also that $\mathscr{C}$ is closed under topological R-isomorphisms, direct products, and closed submodules. We show that $G(L)\cong \mathrm{Hom}_S(L, U)$ for any $L\in\mathscr{D}$ by showing that $\mathrm{Hom}_S(\ , U)$ is adjoint to F. We have that $\mathrm{Hom}_S(L, U)$ is closed in U^L by Proposition 1.1. Hence $\mathrm{Hom}_S(L, U)\in\mathscr{C}$ and therefore $\mathrm{Hom}_S(\ , U)$ is a functor $\mathscr{D}\to\mathscr{C}$. For any $L\in\mathscr{D}$ and $M\in\mathscr{C}$ we have by Proposition 1.2 that $\mathrm{Hom}_S(L, F(M))\cong \mathrm{Hom}_S(L, \mathrm{Cont}_R(M, U))\cong \mathrm{Cont}_R(M, \mathrm{Hom}_S(L, U))$. Hence $\mathrm{Hom}_S(\ , U)$ is adjoint to F, and the proof of (iii) is complete.

(iv) To show that each $M\in\mathscr{C}$ is U-reflexive, observe first that μ_M is a monomorphism because $M\cong GF(M)$ in R-Top. We may assume that $F(\ )=\mathrm{Cont}_R(\ , U)$ and $G(\ )=\mathrm{Hom}_S(\ , U)$, and it will suffice to prove that $\zeta_M=h\mu_M$ for some $h\in \mathrm{Cont}_R(M, M)$. For it then follows, sequentially, that μ_M is an R-epimorphism (because ζ_M is); that h is an R-isomorphism (because ζ_M and μ_M are); that h is a topological R-isomorphism (because ζ_M is and μ_M is a continuous R-isomorphism); and hence, finally, that $\mu_M=h^{-1}\zeta_M$ is a topological R-isomorphism.

Let ψ be the natural isomorphism defined in the proof of Proposition 1.2. $\zeta_M\in \mathrm{Cont}_R(M, GF(M))$, so $\zeta_M^\psi\in \mathrm{Hom}_S(F(M), F(M))$. Also, F induces an isomorphism $F(\ )$: $\mathrm{Cont}_R(M, M)\to \mathrm{Hom}_S(F(M), F(M))$, so there exists $h\in \mathrm{Cont}_R(M, M)$ with $\zeta_M^\psi=F(h)$. Hence, for every $f\in F(M)$ and every $m\in M$, $m(\zeta_M^\psi f)=m(F(h)f)$ or $(m\zeta_M)f=m(hf)=(mh)f=(mh\mu_M)f$. Thus $m\zeta_M=mh\mu_M$ for all $m\in M$, whence $\zeta_M=h\mu_M$.

By a similar argument, one shows that ν_L is an S-isomorphism for each $L\in\mathscr{D}$. ∎

One should recognize that there is always a U-duality between the subcategories of U-reflexive modules in R-Top and Mod-S, just as is the case for module categories [1, Proposition 23.1]. We show this in Theorem 1.6 as a consequence of the following pair of fairly well-known results.

PROPOSITION 1.4. *For $L, L' \in$ Mod-S, each $g \in \mathrm{Hom}_S(L, L')$ induces an element $g^* = \mathrm{Hom}_S(g, 1) \in \mathrm{Cont}_R(\mathrm{Hom}_S(L', U), \mathrm{Hom}_S(L, U))$ defined by $(f)g^* = f \circ g$ for $f \in \mathrm{Hom}_S(L', U)$. If g is an epimorphism then g^* is a topological R-monomorphism.*

Proof. Let $\mathcal{O}(x_1, ..., x_t)$ be a basic open neighborhood of 0 in $\mathrm{Hom}_S(L, U)$ with $x_1, ..., x_t \in L$ and $\mathcal{O}$ an open neighborhood of 0 in U. Then $(\mathcal{O}(x_1, ..., x_t))(g^*)^{-1} = \mathcal{O}(gx_1, ..., gx_t)$ which is a basic open neighborhood of 0 in $\mathrm{Hom}_S(L', U)$. If g is an epimorphism then every basic open neighborhood of 0 in $\mathrm{Hom}_S(L', U)$ is of this form, and the fact that g^* is an R-monomorphism is well-known. ∎

Similarly, if $f \in \mathrm{Cont}_R(M, N)$ then we will write f^* for the induced map $\mathrm{Cont}_R(f, 1) \in \mathrm{Hom}_S(\mathrm{Cont}_R(N, U), \mathrm{Cont}_R(M, U))$.

We also need the following fact, whose proof is routine and is therefore omitted.

PROPOSITION 1.5. *Suppose that $M' \to^f M \to^g M'' \to 0$ is an exact sequence in R-Top. If g is an open map then the induced sequence $0 \to \mathrm{Cont}_R(M'', U) \to^{g^*} \mathrm{Cont}_R(M, U) \to^{f^*} \mathrm{Cont}_R(M', U)$ is exact.*

THEOREM 1.6. *There is a U-duality between the subcategories $\mathscr{C} = \{M \in R\text{-Top} \mid M \text{ is } U\text{-reflexive}\}$ and $\mathscr{D} = \{L \in \text{Mod-}S \mid L \text{ is } U\text{-reflexive}\}$.*

Proof. We need only show that for each $M \in \mathscr{C}$ and $L \in \mathscr{D}$, $M^* = \mathrm{Cont}_R(M, U)$ and $L^* = \mathrm{Hom}_S(L, U)$ are U-reflexive.

We first establish by a routine calculation that $\mu_M^* \circ \nu_{M^*} = 1_{M^*}$ and $\mu_{L^*} \circ \nu_L^* = 1_{L^*}$ (see [1, Proposition 20.14]). Next, for $M \in \mathscr{C}$, μ_M is a topological R-isomorphism. By Proposition 1.5, μ_M^* is then an S-isomorphism. Hence ν_{M^*} is an S-isomorphism. Similarly, for $L \in \mathscr{D}$, ν_L is an S-isomorphism. It then follows from Proposition 1.4 that ν_L^* is a topological R-isomorphism. Hence, so is μ_{L^*}. ∎

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Henceforth, *U is assumed to be Hausdorff.* Our objective in this section is to describe the properties that are necessary for ${}_RU_S$ in order that quotients of certain U-reflexive modules be U-reflexive. We begin by defining these properties.

Recall that a module is *finitely cogenerated* (or *finitely embedded*) if whenever an intersection of submodules is 0 then already a finite intersection is 0; this is equivalent to having a finitely generated essential socle [1, Proposition 10.7]. Observe that a finitely cogenerated Hausdorff linearly topologized module is necessarily discrete.

$M \in R$-Top will be called *quasi-injective* if every element of $\mathrm{Cont}_R(N, M)$ with N a submodule of M can be extended to an element of $\mathrm{Cont}_R(M, M)$; for M discrete this agrees with the usual definition in R-Mod. ${}_RU$ will be called a *self-cogenerator* if for every closed submodule K of U and every $x \in U \backslash K$ there exists $f \in \mathrm{Cont}_R(U, U)$ with $Kf = 0$ and $xf \neq 0$. For ${}_RU$ discrete, this says precisely that every quotient module of ${}_RU$ can be embedded in a direct product of copies of ${}_RU$. (This deviates from the most common definition; see [8] and [15]. For discrete quasi-injective modules the two concepts will agree; see the proof of Theorem 3.1. Our preference for this definition lies in the fact that it is dual to the usual definition of self-generator; see [4, p. 537].)

For $K \subseteq U$ and $L \subseteq S$, we set $l_U(L) = \{u \in U \mid uL = 0\}$, $r_S(K) = \{s \in S \mid Ks = 0\}$. *Observe that when $S \cong \mathrm{Cont}_R(U, U)$, U is a self-cogenerator if and only if $K = l_U r_S(K)$ for every closed submodule K of ${}_RU$.*

A linearly topologized module M is *linearly compact* if every family of cosets $\{m_i + M_i\}_{i \in I}$ with each M_i a closed submodule of M satisfies the finite intersection property; that is, if $\bigcap_{i \in I} m_i + M_i = \varnothing$ then $\bigcap_{i \in F} m_i + M_i = \varnothing$ for some finite set $F \subseteq I$.

The following facts are well known [3].

(1) A linearly compact module is linearly compact in any weaker topology.

(2) A direct product of linearly compact modules is linearly compact.

(3) A linearly compact submodule of a Hausdorff linearly topologized module is closed.

(4) A closed submodule N of a Hausdorff linearly compact module M is linearly compact and M/N is linearly compact in the quotient topology.

(5) A continuous homomorphic image of a linearly compact module in a Hausdorff linearly topologized module is closed. In particular, a continuous homomorphism of a Hausdorff linearly compact module into a Hausdorff linearly topologized module is a closed mapping on submodules.

We will frequently use the following basic fact. Its proof is straightforward and so will be omitted.

PROPOSITION 2.1. *The canonical R-homomorphism* $\mathrm{Hom}_S(S^{(X)}, U) \to^{\theta} U^X$ *is a topological R-isomorphism. In particular,* $\mathrm{Hom}_S(S, U) \cong U$ *is a topological R-isomorphism.*

We now recall the connection between linear compactness of ${}_RU$ and injectivity of U_S; see [15, Corollary 2, p. 342] or [8, Theorem 9.4]. A straightforward adaptation of the arguments used in the discrete case, which we omit, allows one to extend the same result to linearly topologized modules.

PROPOSITION 2.2. *Assume that* $U \in R$-Lin *and that* $S \cong \mathrm{Cont}_R(U, U)$.

(i) *If* ${}_RU$ *is a self-cogenerator, then* ${}_RU$ *is linearly compact when* U_S *is injective.*

(ii) *If* $U^{(n)}$ *is a self-cogenerator for every positive integer n, then* U_S *is injective when* ${}_RU$ *is linearly compact.*

We are almost ready for the main result of this section. It turns out that a key step is to use a topology on quotient modules which is not, in general, the quotient topology.

For N a closed submodule of $M \in R$-Top, we define the *U-topology* on M/N to be the topology whose sub-basic open neighborhoods of 0 are of the form $\mathcal{O}f^{-1}/N$, where $\mathcal{O}$ is an open nieghborhood of 0 in U and $f \in \mathrm{Cont}_R(M, U)$ with $Nf = 0$. By the U-topology on M we mean the U-topology on $M/0$. (Really, we should speak of the $U(M, N, \sigma)$-topology since this topology depends on the topology σ on M as well as on the choice of the submodule N of M, and not just on the isomorphism type of M/N. But no confusion should arise in practice.) The U-topology is, in general, weaker than the quotient topology (see Example 5.1) and is the weakest topology on M/N with respect to which every $f \in \mathrm{Cont}_R(M, U)$ with $Nf = 0$ induces a continuous homomorphism of M/N into U. It is a linear topology whenever U is linearly topologized. It will be convenient to call a quotient module M/N with N a closed submodule of M a *U-quotient* of M to indicate that it is endowed with the U-topology.

THEOREM 2.3. *Let* $U \in R$-Lin *and suppose that every U-quotient of* ${}_RU$ *and every quotient module of* S_S *is U-reflexive. Then* ${}_RU$ *is a discrete, finitely cogenerated, quasi-injective self-cogenerator which is linearly compact, and* $S \cong \mathrm{Hom}_R(U, U)$.

Furthermore, $r_S(\ )$ *and* $l_U(\ )$ *determine mutually inverse lattice anti-isomorphisms between the R-submodules of U and the right ideals of S.*

Proof. From Proposition 2.1 and since S_S is U-reflexive, we have natural isomorphisms $S \cong \mathrm{Cont}_R(\mathrm{Hom}_S(S, U), U) \cong \mathrm{Cont}_R(U, U)$.

For any right ideal L of S, $l_U(L) = \bigcap_{s \in L} 0s^{-1}$ is closed in U and the kernel of $\nu_{S/L}: S/L \to C_R(H_S(S/L, U), U)$ is $r_S l_U(L)/L$. Since $\nu_{S/L}$ is an isomorphism, $r_S l_U(L) = L$ for each right ideal L of S.

On the other hand, for K a closed submodule of ${}_RU$, the U-quotient U/K is assumed to be U-reflexive. Since the kernel of $\mu_{U/K}$ is $l_U r_S(K)/K$, $l_U r_S(K) = K$. Hence, ${}_RU$ is a self-cogenerator and $r_S(\ )$ and $l_U(\ )$ are mutually inverse lattice anti-isomorphisms between the closed submodules of ${}_RU$ and the right ideals of S.

Since ${}_RU$ is Hausdorff, $\bigcap_{i \in I} U_i = 0$, where the U_i, $i \in I$, are the open submodule neighborhoods of 0 in U. Each U_i is also closed, and the lattice anti-isomorphism above together with the fact that S_S is finitely generated guarantees that already $\bigcap_{i \in F} U_i = 0$ for some finite subset $F \subseteq I$. Hence $\{0\}$ is an open set which implies that ${}_RU$ has the discrete topology and is finitely cogenerated. In particular, $S \cong \mathrm{Cont}_R(U, U) = \mathrm{Hom}_R(U, U)$.

In view of Proposition 2.2(i), it remains only to show that ${}_RU$ is quasi-injective and U_S is injective. To see this, we proceed as follows. Since ${}_RU$ is discrete, one has that:

(i) For every ${}_RK \subseteq {}_RU$, the U-topology on U/K is the weak topology of $\mathrm{Hom}_R(U/K, U)$ and hence $\mathrm{Hom}_S(r_S(K), U) \cong U/K$; and

(ii) For every $L_S \subseteq S_S$, $\mathrm{Hom}_R(l_U(L), U) \cong S/L$.

From (i) and the fact that every right ideal of S is of the form $r_S(K)$, U_S is injective. Moreover, since every submodule of ${}_RU$ is of the form $l_U(L)$, we conclude from (ii) that ${}_RU$ is quasi-injective. ∎

In view of the preceding theorem, it is of interest to know when the quotient topology and the U-topology coincide. This is answered by the following elementary observation.

PROPOSITION 2.4. *For $M \in R$-*Top *and N a closed submodule of ${}_RM$, the following conditions are equivalent for M/N with the quotient topology.*

(i) *The quotient topology on M/N coincides with the U-topology on M/N.*

(ii) *$\mu_{M/N}$ is a topological R-monomorphism.*

(iii) *The canonical mapping $j: M/N \to U^{\mathrm{Cont}_R(M/N, U)}$ is a topological R-monomorphism.*

(iv) *M/N is topologically isomorphic to an R-submodule of U^X for some set X.*

Proof. (i) $\Rightarrow$ (ii) The quotient topology on M/N is Hausdorff because N

is closed in M, so $\bigcap (\mathcal{O}f^{-1}/N)=0$ where the intersection runs over all the sub-basic U-open neighborhoods of 0 in M/N. In particular, $\bigcap \ker f/N=0$ where the intersection is over all $f\in \mathrm{Cont}_R(M, U)$ with $Nf=0$. Since each such f induces an element $\bar{f}\in \mathrm{Cont}_R(M/N, U)$, we have that $\bigcap_{g\in C_R(M/N, U)} \ker g=0$. Therefore, $\mu_{M/N}$ is a continuous R-monomorphism; and it is open because the quotient topology coincides with the U-topology on M/N.

(ii) $\Rightarrow$ (iii) j: $M/N\to U^{\mathrm{Cont}_R(M/N, U)}$ is defined by $\bar{m}j=\{\bar{m}g\}_{g\in C_R(M/N, U)}$ for $\bar{m}\in M/N$. Observe that $j=\mu_{M/N}\circ i$, where i is the inclusion mapping of $\mathrm{Hom}_S(\mathrm{Cont}_R(M/N, U), U)$ into $U^{\mathrm{Cont}_R(M/N, U)}$. Hence j is a topological R-monomorphism when $\mu_{M/N}$ is.

(iii) $\Rightarrow$ (iv) is trivially true.

(iv) $\Rightarrow$ (i) The U-topology on M/N is, in general, weaker than the quotient topology. On the other hand, let i: $M/N\to U^X$ be the assumed topological R-monomorphism, π_x: $U^X\to U$ the projection map of U^X onto the x-coordinate, and π: $M\to M/N$ the quotient map. Then a family of basic open neighborhoods of 0 in M/N comprises those of the form $\bigcap_{k=1}^{t} \mathcal{O}(i\pi_{x_k})^{-1}=\bigcap_{k=1}^{t} \mathcal{O}(\pi i\pi_{x_k})^{-1}/N$ for $\mathcal{O}$ an open neighborhood of 0 in U and $x_1, ..., x_k\in X$. Since each $\pi i\pi_{x_k}\in \mathrm{Cont}_R(M, U)$ and $N(\pi i\pi_{x_k})=0$, these neighborhoods are U-open. Thus the two topologies coincide. ∎

3

In this section we produce a converse to Theorem 2.3. That is, we give a description of the duality associated with a finitely cogenerated, quasi-injective self-cogenerator which is linearly compact.

We will let $\mathrm{Cogen}_R U$ (respectively, $\overline{\mathrm{Cogen}}_R U$) denote the family of modules which are topologically isomorphic to a (closed) submodule of a direct product of copies of ${}_RU$. $M\in R$-Top is said to be *copresented* by ${}_RU$ if there exists an exact sequence $0\to M\to^{i} U^X\to U^Y$ in R-Top with i a topological R-monomorphism.

The main result of this section can now be stated.

THEOREM 3.1. *Suppose that ${}_RU$ is a discrete, finitely cogenerated, quasi-injective self-cogenerator which is linearly compact and that $S\cong \mathrm{Hom}_R(U, U)$. Then the following hold.*

(i) *There is a U-duality between* $\overline{\mathrm{Cogen}}_R U$ *and* Mod-S.

(ii) *U_S is an injective cogenerator.*

(iii) $\overline{\mathrm{Cogen}}_R U=\{$*linearly compact modules in* $\mathrm{Cogen}_R U\}=\{$*modules copresented by* ${}_RU\}=\{$*U-reflexive R-modules*$\}$.

(iv) $\overline{\mathrm{Cogen}}_R U$ *is closed under isomorphisms, direct products, closed submodules, and U-quotients.*

(v) *For each* $L \in \text{Mod-}S$, $r_L(\ )$ *and* $l_{H_S(L,U)}(\ )$ *are mutually inverse lattice anti-isomorphisms between the closed R-submodules of* $\mathrm{Hom}_S(L, U)$ *and the S-submodules of L. Similarly, for each* $M \in \overline{\mathrm{Cogen}}_R U$, $l_M(\ )$ *and* $r_{C_R(M,U)}(\ )$ *are mutually inverse lattice anti-isomorphisms between the S-submodules of* $\mathrm{Cont}_R(M, U)$ *and the closed R-submodules of M.*

The new information here is the deescription of $\overline{\mathrm{Cogen}}_R U$, and particularly its closure property under U-quotients. However, we emphasize that this result is essentially available in the existing literature. In particular, the second conclusion can be found in [10, Lemma 4], and the first conclusion is an immediate consequence of [8, Theorem 4.7] and the second conclusion. The proof presented here relies on interesting density argument [8, Theorem 2.4] as well as other machinery from [8] and [13].

Proof. Since ${}_RU$ is a discrete quasi-injective self-cogenerator, it is strongly quasi-injective in the sense of [8, p. 200], and hence it is a self-cogenerator in the sense of [8, p. 194] and [15, p. 338]; see [8, Corollary 4.5]. From Proposition 2.2(ii), U_S is injective. Moreover, since the socle of ${}_RU$ is essential in ${}_RU$, U_S is a cogenerator in Mod-S by [13, Proposition 4.7]. Thus (ii) is proved and hence, by [8, Theorem 4.7], we get (i).

To see that $\overline{\mathrm{Cogen}}_R U$ is closed under the formation of U-quotients, let N be a closed submodule of $M \in \overline{\mathrm{Cogen}}_R U$. From [8, Theorem 4.7 and Proposition 2.6], we learn that M/N is in $\mathrm{Cogen}_R U$. Let $\varphi: M/N \to U^X$ be a topological R-monomorphism. ${}_RM$ is linearly compact and so M/N is linearly compact in the U-topology, and it follows that $(M/N)\varphi$ is closed in U^X. Hence the U-quotient M/N is in $\overline{\mathrm{Cogen}}_R U$. Thus (iv) is proved.

If $M \in \overline{\mathrm{Cogen}}_R U$, then one may apply [8, Proposition 2.6] in a straightforward manner to show that M is copresented by ${}_RU$. Conversely, if $M \in R$-Top is copresented by ${}_RU$, then M is topologically isomorphic to the kernel of a continuous homomorphism $\varphi: U^X \to U^Y$ for some sets X and Y. Hence $M \in \overline{\mathrm{Cogen}}_R U$. (iii) is now easily verified.

(v) is proved by the same arguments given in the second and third paragraphs of the proof of Theorem 2.3. ∎

4

The objective of this section is twofold: to elucidate the relationship between Morita dualities and the dualities described by Theorems 2.3 and

3.1 (the former are a special instance of the latter); and to provide purely algebraic restatements of these theorems.

For R a ring with identity element, we call the bimodule ${}_RU_S$ a *Morita-duality module* if it is a faithfully balanced bimodule (that is, there are natural isomorphisms $S \cong \mathrm{Hom}_R(U, U)$ and $R \cong \mathrm{Hom}_S(U, U)$) such that ${}_RU$ and U_S are injective cogenerators in R-Mod and Mod-S, respectively. The conditions are, of course, symmetric with respect to R and S. For convenience, we call a full subcategory of R-Mod which contains ${}_RR$ and ${}_RU$ and is closed under homomorphic images a *suitable subcategory* of R-Mod. One version of Morita's duality theorem for rings with identity element states that ${}_RU_S$ is a Morita-duality module if and only if $\mathrm{Hom}_R(\ , U)$ and $\mathrm{Hom}_S(\ , U)$ provide mutually inverse dualities between suitable subcategories of R-Mod and Mod-S, respectively [1, Theorems 23.5 and 24.1]. Müller showed that the natural domain for such a duality, when it exists, is the family of modules which is linearly compact in the discrete topology [10, Theorem 2].

The bimodules ${}_RU_S$ which were featured in Theorems 2.3 and 3.1 satisfied the requirement that ${}_RU$ be a discrete, finitely cogenerated, quasi-injective self-cogenerator which is linearly compact with $S \cong \mathrm{Hom}_R(U, U)$. We will call such a bimodule a *duality R-module* (or, simply, a *duality module*). From Proposition 2.2(i) and Theorem 3.1 it is readily deduced that a discrete bimodule ${}_RU_S$, over a ring R with identity element, is a Morita duality module if and only if it is a duality R-module and a (right) duality S-module. It is for this reason that the duality theory presented here may be regarded as providing an asymmetrical generalization of the Morita duality theory.

We begin by showing just how our results specialize when ${}_RU_S$ is a duality module which is also a cogenerator in R-Mod.

For $M \in R$-Mod we define the *U°-topology* on M (sometimes called the *U-adic topology* on M) to be the topology whose sub-basic open neighborhoods of 0 are of the form $\mathcal{O}f^{-1}$ for $\mathcal{O}$ an open neighborhood of 0 in U and $f \in \mathrm{Hom}_R(M, U)$. This is the weakest topology such that the canonical R-homomorphism $i: M \to U^{\mathrm{Hom}_R(M,U)}$ (defined for $m \in M$ by $mi = \{mf\}_{f \in \mathrm{Hom}_R(M,U)}$) is continuous. We will let M° denote the R-module M endowed with the U°-topology. Obviously, $\mathrm{Cont}_R(M^\circ, U) = \mathrm{Hom}_R(M, U)$.

Let $\mu_M^\circ: M \to \mathrm{Hom}_S(\mathrm{Hom}_R(M, U), U)$ be the canonical R-homomorphism defined for $m \in M, f \in \mathrm{Hom}_R(M, U)$ by $(m\mu_M^\circ)(f) = mf$. Observe that, as R-homomorphisms, $\mu_M^\circ = \mu_{M^\circ}$. An R-module M will be called *U°-reflexive* when μ_M° is an R-isomorphism; equivalently, when μ_{M° is a topological R-isomorphism. Thus, *M is U°-reflexive if and only if M° is U-reflexive.* Similarly, a right S-module L is *U°-reflexive* when the canonical S-homomorphism $\nu_L^\circ: L \to \mathrm{Hom}_R(\mathrm{Hom}_S(L, U), U)$ is an isomorphism.

We begin by stating a variant of Theorem 3.1 for R-Mod. Here we set $\mathscr{C}^\circ = \{M \in R\text{-Mod} \mid M$ is U°-reflexive$\}$ and $\mathscr{D}^\circ = \{L \in \text{Mod-}S \mid L$ is U°-reflexive$\}$. Compare [15, Theorem 3.8].

THEOREM 4.1. *Suppose that ${}_R U_S$ is a duality module. Then*

(i) $\text{Hom}_R(\ ,U)$ *and* $\text{Hom}_S(\ ,U)$ *are mutually inverse dualities between* $\mathscr{C}^\circ$ and $\mathscr{D}^\circ$.

(ii) $\mathscr{C}^\circ = \{M \in R\text{-Mod} \mid M^\circ$ *is linearly compact and U cogenerates M in* R-Mod$\} = \{M \in R\text{-Mod} \mid M^\circ \in \overline{\text{Cogen}}_R U\}$.

(iii) *If* $M \in \mathscr{C}^\circ$ *and N is a closed submodule of* M° *then* $M/N \in \mathscr{C}^\circ$. $\mathscr{C}^\circ$ *is closed under finite direct sums and* $\mathscr{C}^\circ \supseteq \{M \in R\text{-Mod} \mid M^\circ$ *is discrete*$\}$. *In particular,* $\mathscr{C}^\circ$ *contains every submodule and quotient module of* $U^{(n)}$ *for each positive integer n.*

(iv) $\mathscr{D}^\circ$ *is closed under submodules, extensions, and finite direct sums, and* $\mathscr{D}^\circ$ *contains every finitely generated right S-module.*

Proof. (i) is the algebraic analogue of Theorem 1.6 and is proved by a similar argument. It is true for any bimodule ${}_R U_S$; see [1, Proposition 23.1].

(ii) M is U°-reflexive if and only if M° is U-reflexive. Thus, by Theorem 3.1, $\mathscr{C}^\circ = \{M \in R\text{-Mod} \mid M^\circ \in \overline{\text{Cogen}}_R U\}$ and M° is linearly compact.

Conversely, assume that M° is linearly compact and that M is cogenerated by U in R-Mod. Then $i: M^\circ \to U^{\text{Hom}_R(M,U)}$ is a topological R-monomorphism. Since M° is linearly compact, $M^\circ i$ is closed in $U^{\text{Hom}_R(M,U)}$ and so $M^\circ \in \overline{\text{Cogen}}_R U$.

(iii) Let N be a closed submodule of M° for $M \in \mathscr{C}^\circ$. Then $M^\circ \in \overline{\text{Cogen}}_R U$ and the U-topology on M°/N and the U°-topology on M/N coincide. Hence by Theorem 3.1, $(M/N)^\circ \in \overline{\text{Cogen}}_R U$ and so $M/N \in \mathscr{C}^\circ$.

Next, let $M_1, ..., M_t \in \mathscr{C}^\circ$ and set $M = \bigoplus_{i=1}^t M_i$. The U°-topology on M is weaker than the product topology obtained from the U°-topologies on the individual M_i. Hence by Theorem 3.1, M° is linearly compact and so $M \in \mathscr{C}^\circ$.

If ${}_R M$ is discrete in the U°-topology then $0 = \bigcap_{i=1}^n \ker f_i$ for some $f_1, ..., f_n \in \text{Hom}_R(M, U)$. Hence M is isomorphic to an R-submodule of $U^{(n)}$. Since $U^{(n)}$ is discrete, $M^\circ \in \overline{\text{Cogen}}_R U$ and so $M \in \mathscr{C}^\circ$.

(iv) Let $0 \to L' \to L \to L'' \to 0$ be an exact sequence in Mod-S. Since

U_S is an injective cogenerator (Theorem 3.1), we obtain a commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_R(H_S(L', U), U) & \longrightarrow & H_R(H_S(L, U), U) & \longrightarrow & H_R(H_S(L'', U), U) & & \\ & & \uparrow{\scriptstyle \nu^\circ_{L'}} & & \uparrow{\scriptstyle \nu^\circ_L} & & \uparrow{\scriptstyle \nu^\circ_{L''}} & & \\ 0 & \longrightarrow & L' & \longrightarrow & L & \longrightarrow & L'' & \longrightarrow & 0 \end{array}$$

with each vertical map a monomorphism.

If $L \in \mathscr{D}^\circ$ then ν°_L is an isomorphism. From a diagram chase, we conclude that $\nu^\circ_{L'}$ is an epimorphism. So $L' \in \mathscr{D}^\circ$ and $\mathscr{D}^\circ$ is closed under submodules.

If in the above diagram we choose $L = S^{(n)}$, then the upper row becomes right exact because $\mathrm{Hom}_S(S^{(n)}, U) \cong U^{(n)}$ and ${}_RU$ is quasi-injective. We conclude that $\nu^\circ_{L''}$ is an isomorphism in this case, and hence $\mathscr{D}^\circ$ contains every finitely generated right S-module.

A similar diagram chase establishes that $\mathscr{D}^\circ$ is closed under extensions.

Finally, suppose that $L_1, ..., L_t \in \mathscr{D}^\circ$ and set $L = \bigoplus_{i=1}^t L_i$. Then ν°_L equals the composition of natural isomorphisms

$$L = \bigoplus_{i=1}^{t} L_i \xrightarrow{\oplus \nu^\circ_{L_i}} \bigoplus_{i=1}^{t} H_R(H_S(L_i, U), U) \cong H_R(H_S(L, U), U).$$

Hence $L \in \mathscr{D}^\circ$ and $\mathscr{D}^\circ$ is closed under finite direct sums. ∎

COROLLARY 4.2. *Suppose that ${}_RU_S$ is a duality module which cogenerates each quotient module of $M \in R$-*Mod. *Then M is U°-reflexive if and only if ${}_RM$ is linearly compact in the discrete topology.*

Proof. If M is linearly compact in the discrete topology, then it is certainly linearly compact in the U°-topology, and hence M is U°-reflexive by Theorem 4.1.

Conversely, assume that M is U°-reflexive. It suffices by Theorem 4.1 to prove that every R-submodule N of M is closed in the U°-topology. This follows directly from Theorem 3.1(v) in view of the assumption that M/N is cogenerated by ${}_RU$. ∎

Observe that the previous corollary implies, in particular, that when ${}_RU_S$ is a Morita-duality module then the U°-reflexive modules are precisely the modules which are linearly compact in the discrete topology [10, Theorem 2]. It can also be used to derive Müller's important description of Morita dualities [10, Theorem 1].

PROPOSITION 4.3. *A ring R with identity element possesses a Morita-duality module ${}_RU_S$ if and only if ${}_RR$ and its minimal injective cogenerator are linearly compact in the discrete topology.*

Proof. Suppose that ${}_RU_S$ is a Morita-duality module and let ${}_RV$ be a minimal injective cogenerator for ${}_RR$ (that is, V is the injective hull of a direct sum of simple R-modules, one for each isomorphism type). Then ${}_RV$ is isomorphic to a direct summand of ${}_RU$ and hence is linearly compact in the discrete topology. Since ${}_RR$ is clearly U-reflexive, ${}_RR$ is linearly compact in the discrete topology by Corollary 4.2.

Conversely, suppose that ${}_RR$ and its minimal injective cogenerator ${}_RU$ are linearly compact in the discrete topology. Then, with $S = \mathrm{Hom}_R(U, U)$, ${}_RU_S$ is a Morita-duality module. For ${}_RU$ is finite dimensional by [15, Lemma 2.3], and hence is finitely cogenerated. U_S is therefore an injective cogenerator by Theorem 3.1. Finally, ${}_RR$ is U°-reflexive by Corollary 4.2, so we have the desired natural isomorphism $R \cong \mathrm{Hom}_S(\mathrm{Hom}_R(R, U), U) \cong \mathrm{Hom}_S(U, U)$. ▮

The next lemma prepares the way for an algebraic version of Theorem 2.3.

LEMMA 4.4. *Suppose that the canonical S-homomorphism* $v_L^\circ: L \to \mathrm{Hom}_R(\mathrm{Hom}_S(L, U), U)$ *is an epimorphism. Then the subspace topology on* $\mathrm{Hom}_S(L, U)$ *induced by* U^L *coincides with the* U°*-topology on* $\mathrm{Hom}_S(L, U)$.

Proof. The sub-basic U°-open neighborhoods of 0 in $\mathrm{Hom}_S(L, U)$ are of the form $\mathcal{O}\phi^{-1}$ for some $\phi \in \mathrm{Hom}_R(\mathrm{Hom}_S(L, U), U)$ and $\mathcal{O}$ an open neighborhood of 0 in U. Since v_L° is an epimorphism by hypothesis, we may write $\phi = v_L^\circ(x)$ for some $x \in L$. Hence $\mathcal{O}\phi^{-1} = \{f \in \mathrm{Hom}_S(L, U) \mid (f)\phi \in \mathcal{O}\} = \{f \in \mathrm{Hom}_S(L, U) \mid f(v_L^\circ(x)) \in \mathcal{O}\} = \{f \in \mathrm{Hom}_S(L, U) \mid f(x) \in \mathcal{O}\}$, which is a sub-basic open neighborhood of 0 in $\mathrm{Hom}_S(L, U)$. This argument is reversible and the topologies coincide. ▮

THEOREM 4.5. *If* ${}_RU_S$ *is a discrete bimodule such that all quotient modules of* ${}_RU$ *and* S_S *are* U°*-reflexive then* ${}_RU_S$ *is a duality R-module.*

Proof. By hypothesis, $(U/K)^\circ$ is U-reflexive for every submodule K of ${}_RU$. Since the U-topology on U/K and the U°-topology on U/K coincide, each U-quotient U/K of ${}_RU$ is U-reflexive.

Each cyclic right S-module S/L is U°-reflexive, so by the preceding lemma, $\mathrm{Hom}_S(S/L, U)$ has the U°-topology. It follows that each cyclic right S-module is U-reflexive. Theorem 2.3 may now be applied to give the desired conclusion. ▮

A Morita-duality module ${}_RU_S$ is characterized by the fact that all quotient modules of ${}_RR$, ${}_RU$, S_S, and U_S are U°-reflexive [1, Theorem 24.1]. (This and the characterization of a Morita-duality module given in the Introduction are straightforward applications of the results of this article. We will refrain from performing this exercise.) From Theorems 4.1 and 4.5 we know that the discrete bimodule ${}_RU_S$ is a duality module if and only if all quotient modules of ${}_RU$ and S_S are U°-reflexive. This shows in an explicit way that a duality module can be thought of as a half-Morita-duality module.

It is known that a ring S is right linearly compact in the discrete topology if and only if there exists a duality R-module ${}_RU_S$. Furthermore, in the "only if" part, U can be chosen to be a minimal injective cogenerator for Mod-S and $R = \text{Hom}_S(U, U)$; see [15, Theorem 3.10]. Müller asked in [10] whether a commutative linearly compact ring must have a Morita duality. For arbitrary rings the answer is negative; see Example 1 of [15]. Our results enable one to conclude that a right linearly compact ring S has a duality in the sense of Theorems 3.1 and 4.1.

The final item in this section summarizes our main results.

THEOREM 4.6. *For a bimodule* ${}_RU_S \in R$-Lin *the following conditions are equivalent.*

(1) *${}_RU_S$ is a duality R-module.*

(2) *Every U-quotient of ${}_RU$ and every quotient module of S_S is U-reflexive.*

(3) *U is discrete and every quotient module of ${}_RU$ and S_S is U°-reflexive.*

(4) *There is a U-duality between* $\overline{\text{Cogen}_R U}$ *and* Mod-S, *and* $\overline{\text{Cogen}_R U}$ *is closed under U-quotients.*

(5) *$U = G(S)$ for some duality $G: \mathscr{D} \to \mathscr{C}$ where $\mathscr{C}$ is a full subcategory of* R-Top *which is closed under isomorphisms, closed submodules, U-quotients, and direct products, and $\mathscr{D}$ is a full subcategory of* Mod-S *which is closed under isomorphisms and contains all cyclic right S-modules.*

Proof. (1) ⇒ (4) is just Theorem 3.1, and (4) ⇒ (5) is clear. (5) ⇒ (2) follows from Theorem 1.3, and (2) ⇒ (1) is in Theorem 2.3. Finally, as was noted above, the equivalence of (1) and (3) is contained in Theorems 4.1 and 4.5. ∎

5

This section is largely devoted to presenting examples of duality modules which are not already Morita-duality modules.

There are some obvious examples of duality modules. A simple left module U over an arbitrary ring R is a duality module and there is a duality between $\overline{\text{Cogen}_R U}$ and the right vector spaces over $S =$

$\mathrm{Hom}_R(U, U)$. (Since $S = \mathrm{Hom}_R(U, U)$ is a division ring in this case, every $M \in \overline{\mathrm{Cogen}_R U}$ is topologically isomorphic to U^X for some X.) This is but a slight generalization of Lefschetz duality for fields.

Next, let I be an ideal of R. If U is a duality module over R/I then U is also a duality module over R. Hence, if R/I has a Morita-duality module and U is a minimal injective cogenerator for R-Mod then $r_U(I)$ is a duality module over R.

From now on we assume that R is a ring with identity element.

In Section 2, we introduced the U-topology on a quotient module M/N with N closed in M. It was this topology which was instrumental in our description of dualities. The following example shows that the U-topology may be strictly weaker than the quotient topology.

EXAMPLE 5.1. Let ${}_R U_S$ be a Morita-duality module which is not left artinian (for example, U could be the minimal injective cogenerator of a commutative non-Noetherian maximal valuation ring R [16, Example 2.4]). By [1, Theorem 10.10] there is a submodule K of ${}_R U$ with U/K not finitely cogenerated.

Now ${}_R U$ has the discrete topology by assumption, so the quotient topology on U/K is also discrete. On the other hand, the U-topology on U/K has as basic open neighborhoods of 0 the finite intersections of kernels of elements of $\mathrm{Hom}_R(U/K, U)$; that is, the kernels of elements of $\mathrm{Hom}_R(U/K, U^{(n)})$ for various n. Since all R-submodules of $U^{(n)}$ are finitely cogenerated there can exist no R-monomorphisms from U/K to $U^{(n)}$. Hence $\{0\}$ cannot be open in the U-topology on U/K and the U-topology is not discrete. (However, every submodule of U/K is closed in the U-topology.)

We next show how a duality module gives rise to other duality modules.

EXAMPLE 5.2 (Subduality). If ${}_R U_S$ is a duality module and ${}_R V_S$ is a sub-bimodule of U which is essential in ${}_R U$ then ${}_R V$ is a duality module.

For, linear compactness in the discrete topology and being finitely cogenerated are inherited by submodules. That ${}_R V$ is quasi-injective follows from a standard argument. ${}_R V$ is quasi-injective if and only if $VE = V$, where $E = \mathrm{Hom}_R(\hat{U}, \hat{U}) = \mathrm{Hom}_R(\hat{V}, \hat{V})$ and $\hat{U}$ denotes the injective hull of ${}_R U$. Now $S = \mathrm{Hom}_R(U, U) \cong E/r_E(U)$ and $r_E(U) \subseteq r_E(V)$, so $VE = VS = V$ and ${}_R V$ is quasi-injective. (see [1, pp. 216–217].) Finally, a quasi-injective module is a self-cogenerator if and only if it contains all (isomorphism types of) simple submodules of factors of itself. Since ${}_R U$ is a self-cogenerator, $\mathrm{Soc}_R U$ contains all simple submodules of factors of ${}_R U$. But $\mathrm{Soc}_R U = \mathrm{Soc}_R V$, so $\mathrm{Soc}_R V$ must contain all simple submodules of factors of ${}_R V$, proving that ${}_R V$ is a self-cogenerator.

EXAMPLE 5.3 (Noetherian PI rings). Assume that R is a Noetherian PI ring and $\mathcal{M} = \{M_1, ..., M_n\}$ is a finite collection of maximal ideals of R, closed under outgoing links [11, p. 238]. Set $M = \bigcap_{i=1}^{n} M_i$ and let U be the R-injective hull of R/M. Then $_RU$ is a duality module.

For, by its definition, $_RU$ is finitely cogenerated. $_RU$ is in fact a left artinian injective module because it is a finite direct sum of injective hulls of simple modules, each of which is artinian by [5, Theorem 2]. A left artinian module is linearly compact in the discrete topology [17].

A left long link $P \rightsquigarrow \rightsquigarrow Q$ between maximal ideals of an FBN ring as defined in [12] is a nonzero homomorphism from E_P to E_Q, where E_P denotes the injective hull of the simple left module with annihilator P. Since $\mathcal{M}$ is closed under outgoing links there exists no nonzero homomorphism from $_RU$ to E_N for any maximal ideal $N \notin \mathcal{M}$. Every factor module of $_RU$ is finitely cogenerated [1, Proposition 10.10], and every simple submodule of a factor module of $_RU$ is isomorphic to a simple submodule of U (else there would be a nonzero homomorphism from $_RU$ to E_N for some $N \notin \mathcal{M}$). Hence we conclude that $_RU$ is a self-cogenerator, and $_RU$ is therefore a duality module.

The finite link-closed (that is, closed under both outgoing and incoming links) families of maximal ideals are precisely those whose intersections are classically localizable [11]. Hence, by [5, Theorem 3], the finite link-closed families of maximal ideals are exactly those for which localization and completion yields a ring with Morita duality. There exist, however, unpublished examples (due to Müller and the second author) of noetherian PI rings with finite families of maximal ideals closed under outgoing links but whose full link-closure is infinite. Thus, the embedding $R \to \mathrm{Biend}_R U$ for $_RU$ as above yields a localization and completion in a situation where a classical localization step is not available.

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