Homoclinic solutions for a class of the second order Hamiltonian systems

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Abstract

We study the existence of homoclinic orbits for the second order Hamiltonian system \( \ddot{q} + V_q(t, q) = f(t) \), where \( q \in \mathbb{R}^n \) and \( V \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \), \( V(t, q) = -K(t, q) + W(t, q) \) is \( T \)-periodic in \( t \). A map \( K \) satisfies the “pinching” condition \( b_1|q|^2 \leq K(t, q) \leq b_2|q|^2 \), \( W \) is superlinear at the infinity and \( f \) is sufficiently small in \( L^2(\mathbb{R}, \mathbb{R}^n) \). A homoclinic orbit is obtained as a limit of \( 2kT \)-periodic solutions of a certain sequence of the second order differential equations.

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1. Introduction

In this paper, we shall be concerned with the existence of homoclinic orbits for the second order Hamiltonian system:

\[ \ddot{q} + V_q(t, q) = f(t), \]  

(HS)
where $t \in \mathbb{R}$, $q \in \mathbb{R}^n$ and functions $V: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $f: \mathbb{R} \to \mathbb{R}^n$ satisfy:

(H1) $V(t, q) = -K(t, q) + W(t, q)$, where $K, W: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are $C^1$-maps, $T$-periodic with respect to $t$, $T > 0$,

(H2) there are constants $b_1, b_2 > 0$ such that for all $(t, q) \in \mathbb{R} \times \mathbb{R}^n$

\[ b_1 |q|^2 \leq K(t, q) \leq b_2 |q|^2, \]

(H3) for all $(t, q) \in \mathbb{R} \times \mathbb{R}^n$, $K(t, q) \leq (q, K_q(t, q)) \leq 2K(t, q)$,

(H4) $W_q(t, q) = o(|q|)$, as $|q| \to 0$ uniformly with respect to $t$,

(H5) there is a constant $\mu > 2$ such that for every $t \in \mathbb{R}$ and $q \in \mathbb{R}^n \setminus \{0\}$

\[ 0 < \mu W(t, q) \leq (q, W_q(t, q)), \]

(H6) $f: \mathbb{R} \to \mathbb{R}^n$ is a continuous and bounded function.

Here and subsequently, $(\cdot, \cdot): \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the standard inner product in $\mathbb{R}^n$ and $|\cdot|$ is the induced norm.

We will say that a solution $q$ of (HS) is homoclinic (to 0) if $q(t) \to 0$ as $t \to \pm \infty$. In addition, if $q \not\equiv 0$ then $q$ is called a nontrivial homoclinic solution.

For each $k \in \mathbb{N}$, let $E_k := W^{1,2}_{2kT}(\mathbb{R}, \mathbb{R}^n)$, the Hilbert space of $2kT$-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norm

\[ \|q\|_{E_k} := \left( \int_{-kT}^{kT} (|\dot{q}(t)|^2 + |q(t)|^2) \, dt \right)^{1/2}. \]

Furthermore, let $L^{\infty}_{2kT}(\mathbb{R}, \mathbb{R}^n)$ denote a space of $2kT$-periodic essentially bounded (measurable) functions from $\mathbb{R}$ into $\mathbb{R}^n$ equipped with the norm

\[ \|q\|_{L^{\infty}_{2kT}} := \text{ess sup}\{|q(t)|: t \in [-kT, kT]\}. \]

We begin with a result which is a direct consequence of estimations made by Rabinowitz in [12].

**Proposition 1.1.** There is a positive constant $C$ such that for each $k \in \mathbb{N}$ and $q \in E_k$ the following inequality holds:

\[ \|q\|_{L^{\infty}_{2kT}} \leq C \|q\|_{E_k}. \] (1)

One can easily show that the inequality (1) holds true with constant $C = \sqrt{2}$ if $T \geq \frac{1}{2}$ (see Fact 2.8).
Set \( M := \sup\{W(t, q) : t \in [0, T], |q| = 1\} \), \( \bar{b}_1 := \min\{1, 2b_1\} \), \( \bar{b}_2 := \max\{1, 2b_2\} \) and suppose that:

\[
(H_7) \ 2M < \bar{b}_1 \quad \text{and} \quad (\int_{\mathbb{R}} |f(t)|^2 \, dt)^{1/2} \leq \frac{\beta}{2\sqrt{r}}, \quad \text{where} \quad 0 < \beta < \bar{b}_1 - 2M \quad \text{and} \quad C \quad \text{is a constant from Proposition 1.1.}
\]

We will prove the following theorem:

**Theorem 1.2.** If the conditions (H1)–(H7) are satisfied then the system (HS) possesses a nontrivial homoclinic solution \( q \in W^{1, 2}(\mathbb{R}, \mathbb{R}^n) \) such that \( \dot{q}(t) \to 0 \) as \( t \to \pm \infty \).

In recent years several authors studied homoclinic orbits for Hamiltonian systems via critical point theory. In particular, the second order systems were considered in [1,3,5–7,11–13,16], and those of the first order in [4,8–10,14,15]. Our study is motivated by a paper of Rabinowitz [12] in which the existence of a nontrivial homoclinic solution for the second order Hamiltonian system

\[
\ddot{q} + V_q(t, q) = 0
\]

was proved. The function \( V \) considered by the author is of the form

\[
V(t, q) = -\frac{1}{2}(L(t)q, q) + W(t, q), \quad (2)
\]

where \( L \) is a continuous \( T \)-periodic matrix valued function such that \( L(t) \) is positive definite and symmetric for all \( t \in [0, T] \), \( W \) satisfies (H4) and (H5). Let us note that conditions (H2) and (H3) are satisfied if \( K(t, q) = \frac{1}{2}(L(t)q, q) \). On the other hand, one can easily check that if

\[
K(t, x) = \begin{cases} 
(1 + \frac{1}{1+x^2})x^2 & \text{for } x \geq 0, \\
(1 + \frac{2}{1+x^2})x^2 & \text{for } x < 0
\end{cases}
\]

and \( W(t, x) = x^4 \), where \( t, x \in \mathbb{R} \), then \( V(t, x) = -K(t, x) + W(t, x) \) cannot be represented in the form (2) with \( W \) satisfying (H4), (H5) while \( V \) satisfies conditions (H1)–(H5). Hence, our theorem extends the result from [12] even if \( f(t) = 0 \). It follows from our assumptions that \( q(t) = 0 \) is a solution of (HS) only if \( f(t) = 0 \). Therefore, if \( f \) is a nonzero function the existence of a homoclinic solution of (HS) implies its nontriviality.

Similarly to [12] a homoclinic solution of (HS) is obtained as a limit, as \( k \to +\infty \), of a certain sequence of functions \( q_k \in E_k \). However, in our approach, we consider a sequence of systems of differential equations:

\[
\ddot{q} + V_q(t, q) = f_k(t), \quad (HS_k)
\]
where for each \( k \in \mathbb{N} \), \( f_k : \mathbb{R} \to \mathbb{R}^n \) is a \( 2kT \)-periodic extension of the restriction of \( f \) to the interval \([-kT, kT]\) and \( q_k \) is a \( 2kT \)-periodic solution of \((\text{HS}_k)\) obtained via the Mountain Pass Theorem.

Part of the difficulty in treating \((\text{HS})\) is caused by the fact that in order to get appropriate convergence of the sequence of approximative functions \( \{ q_k \}_{k \in \mathbb{N}} \) we need the constants \( \rho \) and \( \zeta \) appearing in the condition (iii) of the Mountain Pass Theorem (see Theorem 2.5) to be independent of \( k \).

2. Proof of Theorem 1.2

At first let us recall some properties of the function \( W(t, q) \) from [12]. They all are necessary to the proof of Theorem 1.2.

**Fact 2.1.** For every \( t \in [0, T] \) the following inequalities hold:

\[
W(t, q) \leq W \left( t, \frac{q}{|q|} \right) |q|^\mu \quad \text{if } 0 < |q| \leq 1, \tag{3}
\]

\[
W(t, q) \geq W \left( t, \frac{q}{|q|} \right) |q|^\mu \quad \text{if } |q| \geq 1. \tag{4}
\]

To prove this fact it suffices to show that for every \( q \neq 0 \) and \( t \in [0, T] \) the function \((0, +\infty) \ni \zeta \mapsto W(t, \zeta^{-1} q) \zeta^\mu \) is nonincreasing. It is an immediate consequence of \((\text{H}_5)\).

**Fact 2.2.** Set \( m := \inf \{ W(t, q) : t \in [0, T], |q| = 1 \} \). Then for every \( \zeta \in \mathbb{R} \setminus \{ 0 \} \) and \( q \in E_k \setminus \{ 0 \} \) we have

\[
\int_{-kT}^{kT} W(t, \zeta q(t)) \, dt \geq m |\zeta|^\mu \int_{-kT}^{kT} |q(t)|^\mu \, dt - 2kTm. \tag{5}
\]

**Proof.** Fix \( \zeta \in \mathbb{R} \setminus \{ 0 \} \) and \( q \in E_k \setminus \{ 0 \} \). Set \( A_k = \{ t \in [-kT, kT] : |\zeta q(t)| \leq 1 \} \), and \( B_k = \{ t \in [-kT, kT] : |\zeta q(t)| \geq 1 \} \). From (4) we obtain

\[
\int_{-kT}^{kT} W(t, \zeta q(t)) \, dt \geq \int_{B_k} W(t, \zeta q(t)) \, dt \geq \int_{B_k} W \left( t, \frac{\zeta q(t)}{|\zeta q(t)|} \right) |\zeta q(t)|^\mu \, dt
\]

\[
\geq m \int_{B_k} |\zeta q(t)|^\mu \, dt
\]

\[
\geq m \int_{-kT}^{kT} |\zeta q(t)|^\mu \, dt - m \int_{A_k} |\zeta q(t)|^\mu \, dt
\]

\[
\geq m |\zeta|^\mu \int_{-kT}^{kT} |q(t)|^\mu \, dt - 2kTm. \quad \square
\]
Fact 2.3. Let $Y: [0, +\infty) \to [0, +\infty)$ be given as follows: $Y(0) = 0$ and

$$Y(s) = \max_{t \in [0, T]} \left(\frac{q, W_q(t, q)}{|q|^2}\right)$$

for $s > 0$. Then $Y$ is continuous, nondecreasing, $Y(s) > 0$ for $s > 0$ and $Y(s) \to +\infty$ as $s \to +\infty$.

It is easy to verify this fact applying $(H_4)$, $(H_5)$ and (4).

Assumptions $(H_4)$ and $(H_5)$ imply that $W(t, q) = o(|q|^2)$ as $q \to 0$ uniformly for $t \in [0, T]$ and $W(t, 0) = 0, W_q(t, 0) = 0$. Moreover, from $(H_2)$ we conclude that $K(t, 0) = 0, K_q(t, 0) = 0$.

Before we will prove Theorem 1.2, we have to introduce more notation and some necessary definitions. For each $k \in \mathbb{N}$, let $L^2_{2kT}(\mathbb{R}, \mathbb{R}^n)$ denote the Hilbert space of $2kT$-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norm $\|q\|_{L^2_{2kT}} = \left(\int_{-kT}^{kT} |q(t)|^2 dt\right)^{1/2}$.

Let $f_k: \mathbb{R} \to \mathbb{R}^n$ be a $2kT$-periodic extension of $f|_{[-kT,kT]}$ onto $\mathbb{R}$. From $(H_7)$ it follows that $\|f_k\|_{L^2_{2kT}} \leq \beta/2C$. Consider the second order Hamiltonian system:

$$\ddot{q} + V_q(t, q) = f_k(t).$$

(HS$_k$)

Let $\eta_k: E_k \to [0, +\infty)$ be given by

$$\eta_k(q) = \left(\int_{-kT}^{kT} \left[\frac{1}{2}|\dot{q}(t)|^2 + 2K(t, q(t))\right] dt\right)^{1/2}.$$

By $(H_2)$,

$$\bar{b}_1 \|q\|_{E_k}^2 \leq \eta_k^2(q) \leq \bar{b}_2 \|q\|_{E_k}^2.$$

(7)

(8)

It is worth pointing out that if the function $K(t, q)$ is of the form $\frac{1}{2}(L(t)q, q)$ with a matrix valued function $L$ satisfying the same conditions as in [12] then $\eta_k$ determined by (7) is a norm in $E_k$ equivalent to the norm $\|\cdot\|_{E_k}$. Let $I_k: E_k \to \mathbb{R}$ be defined by

$$I_k(q) = \int_{-kT}^{kT} \left[\frac{1}{2}|\dot{q}(t)|^2 - V(t, q(t))\right] dt + \int_{-kT}^{kT} (f_k(t), q(t)) dt$$

$$= \frac{1}{2} \eta_k^2(q) - \int_{-kT}^{kT} W(t, q(t)) dt + \int_{-kT}^{kT} (f_k(t), q(t)) dt.$$

(9)

Then $I_k \in C^1(E_k, \mathbb{R})$ and it is easy to check that

$$I_k'(q)v = \int_{-kT}^{kT} \left[(\dot{q}(t), \dot{v}(t)) - (V_q(t, q(t)), v(t))\right] dt + \int_{-kT}^{kT} (f_k(t), v(t)) dt,$$

(10)
and
\[
I_k'(q)q \leq \eta_k^2(q) - \int_{-kT}^{kT} (W_q(t, q(t)), q(t)) \, dt + \int_{-kT}^{kT} (f_k(t), q(t)) \, dt.
\] (11)

Moreover, it is clear that critical points of \( I_k \) are classical \( 2kT \)-periodic solutions of \( (HS_k) \).

We have divided the proof of Theorem 1.2 into a sequence of lemmas.

**Lemma 2.4.** If \( V \) and \( f \) satisfy \((H_1)-(H_7)\) then for every \( k \in \mathbb{N} \) the system \( (HS_k) \) possesses a \( 2kT \)-periodic solution.

We will obtain a critical point of \( I_k \) by the use of a standard version of the Mountain Pass Theorem (see [2]). It provides the minimax characterisation which is important for what follows. Therefore, we state this theorem precisely.

**Theorem 2.5** (see Ambrosetti and Rabinowitz [2]). Let \( E \) be a real Banach space and \( I: E \to \mathbb{R} \) be a \( C^1 \)-smooth functional. If \( I \) satisfies the following conditions:

(i) \( I(0) = 0 \),

(ii) every sequence \( \{u_j\}_{j \in \mathbb{N}} \) in \( E \) such that \( \{I(u_j)\}_{j \in \mathbb{N}} \) is bounded in \( \mathbb{R} \) and \( I'(u_j) \to 0 \) in \( E^* \), as \( j \to +\infty \), contains a convergent subsequence (the Palais-Smale condition),

(iii) there exist constants \( \varrho, \alpha > 0 \) such that \( I|_{\varrho B_\varrho(0)} \geq \alpha \),

(iv) there exists \( e \in E \setminus \overline{B_\varrho(0)} \) such that \( I(e) \leq 0 \),

where \( B_\varrho(0) \) is an open ball in \( E \) of radius \( \varrho \) centred at 0, then \( I \) possesses a critical value \( c \geq \alpha \) given by

\[
c = \inf_{g \in \Gamma} \max_{s \in [0,1]} I(g(s)),
\]

where

\[
\Gamma = \{ g \in C([0,1], E) : g(0) = 0, \ g(1) = e \}.
\]

**Proof of Lemma 2.4.** In our case it is clear that \( I_k(0) = 0 \). We show that \( I_k \) satisfies the Palais-Smale condition. Assume that \( \{u_j\}_{j \in \mathbb{N}} \) in \( E_k \) is a sequence such that \( \{I_k(u_j)\}_{j \in \mathbb{N}} \) is bounded and \( I_k'(u_j) \to 0 \) as \( j \to +\infty \). Then there exists a constant \( C_k > 0 \) such that

\[
|I_k(u_j)| \leq C_k, \quad \|I'_k(u_j)\|_{E_k^*} \leq C_k
\] (12)
for every \( j \in \mathbb{N} \). We first prove that \( \{u_j\}_{j \in \mathbb{N}} \) is bounded. By (9) and (H5),

\[
\eta_k^2(u_j) \leq 2I_k(u_j) + \frac{2}{\mu} \int_{-kT}^{kT} (W_q(t, u_j(t)), u_j(t)) \, dt
-2 \int_{-kT}^{kT} (f_k(t), u_j(t)) \, dt.
\]  

(13)

From (13) and (11) we obtain

\[
\left(1 - \frac{2}{\mu}\right) \eta_k^2(u_j) \leq 2I_k(u_j) - \frac{2}{\mu} I'_k(u_j) u_j
- \left(2 - \frac{2}{\mu}\right) \int_{-kT}^{kT} (f_k(t), u_j(t)) \, dt.
\]  

(14)

From (14) and (8) it follows that

\[
\left(1 - \frac{2}{\mu}\right) \tilde{b}_1 \|u_j\|_{E_k}^2 \leq 2I_k(u_j)
+ \left(\frac{2}{\mu} \|I'_k(u_j)\|_{E_k^*} + \left(2 - \frac{2}{\mu}\right) \|f_k\|_{L_{2kT}^2}\right) \|u_j\|_{E_k}.
\]  

(15)

Combining (15) with (H7) and (12) we get

\[
\left(1 - \frac{2}{\mu}\right) \tilde{b}_1 \|u_j\|_{E_k}^2 - \left(\frac{2C_k}{\mu} + \left(2 - \frac{2}{\mu}\right) \frac{\beta}{2C}\right) \|u_j\|_{E_k} - 2C_k \leq 0.
\]  

(16)

Since \( \mu > 2 \), (16) shows that \( \{u_j\}_{j \in \mathbb{N}} \) is bounded in \( E_k \). Going if necessary to a subsequence, we can assume that there exists \( u \in E_k \) such that \( u_j \rightharpoonup u \), as \( j \to +\infty \), in \( E_k \), which implies \( u_j \to u \) uniformly on \([-kT, kT]\). Hence \( (I'_k(u_j) - I'_k(u))(u_j - u) \to 0 \), \( \|u_j - u\|_{L_{2kT}^2} \to 0 \) and

\[
\int_{-kT}^{kT} (V_q(t, u_j(t)) - V_q(t, u(t)), u_j(t) - u(t)) \, dt \to 0,
\]

as \( j \to +\infty \). Moreover, an easy computation shows that

\[
(I'_k(u_j) - I'_k(u))(u_j - u) = \|\dot{u}_j - \dot{u}\|_{L_{2kT}^2}^2
- \int_{-kT}^{kT} (V_q(t, u_j(t)) - V_q(t, u(t)), u_j(t) - u(t)) \, dt,
\]

and so \( \|\dot{u}_j - \dot{u}\|_{L_{2kT}^2}^2 \to 0 \). Consequently, \( \|u_j - u\|_{E_k} \to 0 \).
We now show that there exist constants \( a, \alpha > 0 \) independent of \( k \) such that every \( I_k \) satisfies the assumption (iii) of Theorem 2.5 with these constants. Assume that \( 0 < \| q \|_{L^\infty_{2kT}} \leq 1 \). By (3) we have

\[
\int_{-kT}^{kT} W(t, q(t)) \, dt \leq \int_{-kT}^{kT} W\left(t, \frac{q(t)}{|q(t)|}\right) |q(t)|^\mu \, dt \\
\leq M \int_{-kT}^{kT} |q(t)|^2 \, dt \leq M \| q \|_{E_k}^2,
\]

and, in consequence, combining this with (8) and \((H_7)\) we obtain

\[
I_k(q) \geq \frac{1}{2} \tilde{b}_1 \| q \|_{E_k}^2 - M \| q \|_{E_k}^2 - \| f_k \|_{L^2_{2kT}} \| q \|_{L^2_{2kT}} \\
\geq \frac{1}{2} \tilde{b}_1 \| q \|_{E_k}^2 - M \| q \|_{E_k}^2 - \frac{\beta}{2C} \| q \|_{E_k} \\
= \frac{1}{2}(\tilde{b}_1 - \beta - 2M)\| q \|_{E_k}^2 + \frac{\beta}{2} \| q \|_{E_k}^2 - \frac{\beta}{2C} \| q \|_{E_k}.
\]

(17)

Note that \((H_7)\) implies \( \tilde{b}_1 - \beta - 2M > 0 \). Set

\[
q = \frac{1}{C}, \quad \alpha = \frac{\tilde{b}_1 - \beta - 2M}{2C^2}.
\]

By (1), if \( \| q \|_{E_k} = q \) then \( 0 < \| q \|_{L^\infty_{2kT}} \leq 1 \) and (17) gives \( I_k(q) \geq \alpha \).

It remains to prove that for every \( k \in \mathbb{N} \) there exists \( e_k \in E_k \) such that \( \| e_k \|_{E_k} > q \) and \( I_k(e_k) \leq 0 \). By the use of (5), (9) and (8) we have that for every \( \zeta \in \mathbb{R} \setminus \{0\} \) and \( q \in E_k \setminus \{0\} \) the following inequality holds:

\[
I_k(\zeta q) \leq \tilde{b}_2 \zeta^2 \| q \|_{E_k}^2 - m |\zeta|^{\mu} \int_{-kT}^{kT} |q(t)|^\mu \, dt \\
+ |\zeta| \cdot \| f_k \|_{L^2_{2kT}} \| q \|_{L^2_{2kT}} + 2kTm.
\]

(18)

Take \( Q \in E_1 \) such that \( Q(\pm T) = 0 \). Since \( \mu > 2 \) and \( m > 0 \), (18) implies that there exists \( \tilde{\zeta} \in \mathbb{R} \setminus \{0\} \) such that \( \| \tilde{\zeta} Q \|_{E_1} > q \) and \( I_1(\tilde{\zeta} Q) < 0 \). Set \( e_1(t) = \tilde{\zeta} Q(t) \) and

\[
e_k(t) = \begin{cases} 
 e_1(t) & \text{for } |t| \leq T, \\
 0 & \text{for } T < |t| \leq kT
\end{cases}
\]

(19)

for \( k > 0 \). Then \( e_k \in E_k \), \( \| e_k \|_{E_k} = \| e_1 \|_{E_1} > q \) and \( I_k(e_k) = I_1(e_1) < 0 \) for every \( k \in \mathbb{N} \). By Theorem 2.5, \( I_k \) possesses a critical value \( c_k \geq \alpha \) given by

\[
c_k = \inf_{g \in \Gamma_k} \max_{s \in [0,1]} I_k(g(s)),
\]

(20)
where

\[ \Gamma_k = \{ g \in C([0, 1], E_k) : g(0) = 0, \ g(1) = e_k \}. \]

Hence, for every \( k \in \mathbb{N} \), there is \( q_k \in E_k \) such that

\[ I_k(q_k) = c_k, \quad I_k'(q_k) = 0. \quad (21) \]

The function \( q_k \) is a desired classical \( 2kT \)-periodic solution of \((HS_k)\). Since \( c_k > 0 \), \( q_k \) is a nontrivial solution even if \( f_k(t) = 0 \). □

Let \( C_{\text{loc}}^p(\mathbb{R}, \mathbb{R}^m) \), where \( p \in \mathbb{N} \cup \{0\} \), denote the space of \( C^p \) functions on \( \mathbb{R} \) with values in \( \mathbb{R}^m \) under the topology of almost uniformly convergence of functions and all derivatives up to the order \( p \). Using the Arzelà-Ascoli theorem we prove what follows.

**Lemma 2.6.** Let \( \{q_k\}_{k \in \mathbb{N}} \) be the sequence given by (21). There exist an increasing function \( \phi : \mathbb{N} \to \mathbb{N} \) and a \( C^1 \) function \( q_0 : \mathbb{R} \to \mathbb{R}^m \) such that \( q_{\phi(k)} \to q_0 \), as \( k \to +\infty \), in \( C^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^m) \).

**Proof.** The first step in the proof is to show that the sequences \( \{c_k\}_{k \in \mathbb{N}} \) and \( \{\|q_k\|_{E_k}\}_{k \in \mathbb{N}} \) are bounded. For every \( k \in \mathbb{N} \), let \( g_k : [0, 1] \to E_k \) be a curve given by \( g_k(s) = seq_k \), where \( e_k \) is determined by (19). Then \( g_k \in \Gamma_k \) and \( I_k(g_k(s)) = I_1(g_1(s)) \) for all \( k \in \mathbb{N} \) and \( s \in [0, 1] \). Therefore, by (20),

\[ c_k \leq \max_{s \in [0, 1]} I_1(g_1(s)) \equiv M_0 \quad (22) \]

independently of \( k \in \mathbb{N} \). As \( I_k'(q_k) = 0 \), we receive from (9), (11) and \((H_5)\) that

\[
\begin{align*}
    c_k &= I_k(q_k) - \frac{1}{2} I_k'(q_k)q_k \\
    &\geq \left( \frac{\mu}{2} - 1 \right) \int_{-kT}^{kT} W(t, q_k(t)) \, dt + \frac{1}{2} \int_{-kT}^{kT} (f_k(t), q_k(t)) \, dt,
\end{align*}
\]

and hence

\[
\int_{-kT}^{kT} W(t, q_k(t)) \, dt \leq \frac{1}{\mu - 2} \left( 2c_k - \int_{-kT}^{kT} (f_k(t), q_k(t)) \, dt \right).
\]

Combining the above with (8), (9) and (22) we have

\[
\beta_1 \|q_k\|_{E_k}^2 \leq \frac{2\mu M_0}{\mu - 2} + \frac{2\mu - 2}{\mu - 2} \|f_k\|_{L^2_{2kT}} \|q_k\|_{L^2_{2kT}},
\]
and, in consequence, by \((H_7)\)
\[
\bar{b}_1 \|q_k\|_{E_k}^2 - \frac{\beta(\mu - 1)}{C(\mu - 2)} \|q_k\|_{E_k} - \frac{2\mu M_0}{\mu - 2} \leq 0.
\]
(23)

Since \(\bar{b}_1 > 0\) and all coefficients of (23) are independent of \(k\), we see that there is \(M_1 > 0\) independent of \(k\) such that
\[
\|q_k\|_{E_k} \leq M_1.
\]
(24)

We now observe that the sequences \(\{q_k\}_{k \in \mathbb{N}}, \{\dot{q}_k\}_{k \in \mathbb{N}}\) and \(\{\ddot{q}_k\}_{k \in \mathbb{N}}\) are uniformly bounded. By (1),
\[
\|q_k\|_{L^\infty_{2kT}} \leq CM_1 \equiv M_2
\]
(25)
for every \(k \in \mathbb{N}\). Since \(q_k\) satisfies \((HS_k)\), if \(t \in [-kT,kT]\) we have
\[
|\ddot{q}_k(t)| \leq |f_k(t)| + |V_q(t,q_k(t))| = |f(t)| + |V_q(t,q_k(t))|.
\]

Therefore (25), \((H_1)\) and \((H_6)\) imply that there is \(M_3 > 0\) independent of \(k\) such that
\[
\|\ddot{q}_k\|_{L^\infty_{2kT}} \leq M_3.
\]
(26)

From the Mean Value Theorem it follows that for every \(k \in \mathbb{N}\) and \(t \in \mathbb{R}\) there exists \(\tau_k \in [t-1,t]\) such that
\[
\dot{q}_k(\tau_k) = \int_{t-1}^t \ddot{q}_k(s) \, ds = q_k(t) - q_k(t-1).
\]

In consequence, combining the above with (25) and (26)
\[
|\ddot{q}_k(t)| = \left| \int_{\tau_k}^t \dddot{q}_k(s) \, ds + \dot{q}_k(\tau_k) \right|
\leq \int_{t-1}^t |\dddot{q}_k(s)| \, ds + |q_k(t) - q_k(t-1)| \leq M_3 + 2M_2 \equiv M_4,
\]
and hence for every \(k \in \mathbb{N}\)
\[
\|\ddot{q}_k\|_{L^\infty_{2kT}} \leq M_4.
\]
(27)
The task is now to show that \( \{q_k\}_{k \in \mathbb{N}} \) and \( \{\dot{q}_k\}_{k \in \mathbb{N}} \) are equicontinuous. Of course, it suffices to prove that both sequences satisfy the Lipschitz condition with some constants independent of \( k \). Let \( k \in \mathbb{N} \) and \( t, t_0 \in \mathbb{R} \). Then
\[
|q_k(t) - q_k(t_0)| = \left| \int_{t_0}^{t} \dot{q}_k(s) \, ds \right| \leq \left| \int_{t_0}^{t} |\dot{q}_k(s)| \, ds \right| \leq M_4 |t - t_0|,
\]
by (27), and analogously,
\[
|\dot{q}_k(t) - \dot{q}_k(t_0)| \leq M_3 |t - t_0|,
\]
by (26). Since \( \{q_k\}_{k \in \mathbb{N}} \) and \( \{\dot{q}_k\}_{k \in \mathbb{N}} \) are bounded in \( L^\infty_{2kT}(\mathbb{R}, \mathbb{R}^n) \) and equicontinuous, we obtain the existence of a subsequence \( \{q_{\theta(k)}\}_{k \in \mathbb{N}} \) convergent to a certain \( q_0 \) in \( C^{1}_{\text{loc}}(\mathbb{R}, \mathbb{R}^n) \) by using the Arzelà-Ascoli theorem. □

Our next goal is to show that \( q_0 \) is the desired homoclinic solution of (HS). For this purpose, we need the following observations.

**Fact 2.7.** Let \( q: \mathbb{R} \to \mathbb{R}^n \) be a continuous mapping. If a weak derivative \( \dot{q}: \mathbb{R} \to \mathbb{R}^n \) is continuous at \( t_0 \), then \( q \) is differentiable at \( t_0 \) and
\[
\lim_{t \to t_0} \frac{q(t) - q(t_0)}{t - t_0} = \dot{q}(t_0).
\]

**Proof.** Fix \( \varepsilon > 0 \). By the assumption, there exists \( \delta > 0 \) such that for every \( t \in \mathbb{R} \), if \( |t - t_0| < \delta \) then \( |\dot{q}(t) - \dot{q}(t_0)| < \varepsilon \). Hence
\[
\left| \frac{q(t) - q(t_0)}{t - t_0} - \dot{q}(t_0) \right| = \left| \frac{\int_{t_0}^{t} (\dot{q}(s) - \dot{q}(t_0)) \, ds}{t - t_0} \right| \leq \frac{\int_{t_0}^{t} |\dot{q}(s) - \dot{q}(t_0)| \, ds}{|t - t_0|} \leq \varepsilon
\]
provided that \( 0 < |t - t_0| < \delta \). □

Let \( L^2_{\text{loc}}(\mathbb{R}, \mathbb{R}^n) \) denote the space of functions on \( \mathbb{R} \) with values in \( \mathbb{R}^n \) locally square integrable.

**Fact 2.8.** Let \( q: \mathbb{R} \to \mathbb{R}^n \) be a continuous mapping such that \( \dot{q} \in L^2_{\text{loc}}(\mathbb{R}, \mathbb{R}^n) \). For every \( t \in \mathbb{R} \) the following inequality holds:
\[
|q(t)| \leq \sqrt{2} \left( \int_{t-1/2}^{t+1/2} \left( |q(s)|^2 + |\dot{q}(s)|^2 \right) \, ds \right)^{1/2}. \tag{28}
\]
Proof. Fix \( t \in \mathbb{R} \). For every \( \tau \in \mathbb{R} \),
\[
|q(t)| \leq |q(\tau)| + \left| \int_{\tau}^{t} \dot{q}(s) \, ds \right|.
\tag{29}
\]
Integrating (29) over \([t - \frac{1}{2}, t + \frac{1}{2}]\) and using the Hölder inequality we obtain
\[
|q(t)| \leq \int_{t-1/2}^{t+1/2} \left( |q(\tau)| + \left| \int_{\tau}^{t} \dot{q}(s) \, ds \right| \right) \, d\tau
\]
\[
\leq \left( \int_{t-1/2}^{t+1/2} \left( |q(\tau)| + \left| \int_{\tau}^{t} \dot{q}(s) \, ds \right| \right)^2 \, d\tau \right)^{1/2}
\]
\[
\leq \left( 2 \int_{t-1/2}^{t+1/2} |q(\tau)|^2 + \left| \int_{\tau}^{t} \dot{q}(s) \, ds \right|^2 \, d\tau \right)^{1/2}
\]
\[
\leq \sqrt{2} \left( \int_{t-1/2}^{t+1/2} |q(\tau)|^2 \, d\tau + \int_{t-1/2}^{t+1/2} |\dot{q}(s)|^2 \, ds \right)^{1/2}.
\]
\(\square\)

Lemma 2.9. The function \(q_0\) determined by Lemma 2.6 is the desired homoclinic solution of (HS).

Proof. The proof will be divided into four steps.

Step 1: We show that \(q_0\) is a solution of (HS). For every \( k \in \mathbb{N} \) and \( t \in \mathbb{R} \) we have
\[
\ddot{q}_{\varphi(k)}(t) = f_{\varphi(k)}(t) - V_{q}(t, q_{\varphi(k)}(t)).
\tag{30}
\]
Since \(q_{\varphi(k)} \to q_0\) and \(f_{\varphi(k)} \to f\) almost uniformly on \(\mathbb{R}\), we obtain that \(\ddot{q}_{\varphi(k)} \to w\) almost uniformly on \(\mathbb{R}\), where \(w(t) = f(t) - V_{q}(t, q_0(t))\). Fix \(a, b \in \mathbb{R}\) such that \(a < b\). There is \(k_0 \in \mathbb{N}\) such that for every \(k \geq k_0\) and \(t \in [a, b]\), (30) becomes
\[
\ddot{q}_{\varphi(k)}(t) = f(t) - V_{q}(t, q_{\varphi(k)}(t)).
\]
Hence, if \(k \geq k_0\) then the restriction of \(\ddot{q}_{\varphi(k)}\) onto \([a, b]\) is continuous. From Fact 2.7 it follows that \(\ddot{q}_{\varphi(k)}\) is a derivative of \(q_{\varphi(k)}\) in \((a, b)\) for every \(k \geq k_0\). Since \(\ddot{q}_{\varphi(k)} \to w\) and \(\dot{q}_{\varphi(k)} \to \dot{q}_0\) almost uniformly on \(\mathbb{R}\), we have \(w = \ddot{q}_0\) in \((a, b)\). By the above, we conclude that \(w = \ddot{q}_0\) in \(\mathbb{R}\) and \(q_0\) satisfies (HS). Moreover, note that we have actually proved that \(\{q_{\varphi(k)}\}_{k \in \mathbb{N}}\) converges to \(q_0\) in the topology of \(C^2_{\text{loc}}(\mathbb{R}, \mathbb{R}^n)\).
Step 2: We prove that \( q_0(t) \to 0 \), as \( t \to \pm \infty \). We have
\[
\int_{-\infty}^{+\infty} \left( |q_0(t)|^2 + |\dot{q}_0(t)|^2 \right) \, dt = \lim_{i \to +\infty} \int_{-iT}^{iT} \left( |q_0(t)|^2 + |\dot{q}_0(t)|^2 \right) \, dt \\
= \lim_{i \to +\infty} \lim_{k \to +\infty} \int_{-iT}^{iT} \left( |q_{\varphi(k)}(t)|^2 + |\dot{q}_{\varphi(k)}(t)|^2 \right) \, dt.
\]
Clearly, for every \( i \in \mathbb{N} \) there exists \( k_i \in \mathbb{N} \) such that for all \( k \geq k_i \) we have
\[
\int_{-iT}^{iT} \left( |q_{\varphi(k)}(t)|^2 + |\dot{q}_{\varphi(k)}(t)|^2 \right) \, dt \leq \| q_{\varphi(k)} \|_{E_{\varphi(k)}}^2 \leq M_1^2,
\]
by (24). Letting \( k \to +\infty \), we get
\[
\int_{-iT}^{iT} \left( |q_0(t)|^2 + |\dot{q}_0(t)|^2 \right) \, dt \leq M_1^2,
\]
and now, letting \( i \to +\infty \), we have
\[
\int_{-\infty}^{+\infty} \left( |q_0(t)|^2 + |\dot{q}_0(t)|^2 \right) \, dt \leq M_1^2,
\]
and so
\[
\int_{|t| \geq r} \left( |q_0(t)|^2 + |\dot{q}_0(t)|^2 \right) \, dt \to 0, \tag{31}
\]
as \( r \to +\infty \). Combining (31) with (28) we receive our claim.

Step 3: We now show that \( \dot{q}_0(t) \to 0 \), as \( t \to \pm \infty \). To do this, observe that
\[
|\dot{q}_0(t)|^2 \leq 2 \int_{t-1/2}^{t+1/2} \left( |q_0(s)|^2 + |\dot{q}_0(s)|^2 \right) \, ds + 2 \int_{t-1/2}^{t+1/2} |\ddot{q}_0(s)|^2 \, ds, \tag{32}
\]
by (28). Since we have (31) and (32) it suffices to prove that
\[
\int_r^{r+1} |\ddot{q}_0(s)|^2 \, ds \to 0, \tag{33}
\]
as \( r \to \pm \infty \). By (HS) we obtain
\[
\int_r^{r+1} |\ddot{q}_0(s)|^2 \, ds = \int_r^{r+1} \left( |V_q(s, q_0(s))|^2 + |f(s)|^2 \right) \, ds \\
- 2 \int_r^{r+1} (V_q(s, q_0(s)), f(s)) \, ds.
\]
Since $V_q(t, 0) = 0$ for all $t \in \mathbb{R}$, $q_0(t) \to 0$, as $t \to \pm \infty$ and $\int_r^{r+1} |f(s)|^2 \, ds \to 0$, as $r \to \pm \infty$, (33) follows.

Step 4: In the end, we have to show that if $f \equiv 0$ then $q_0 \equiv 0$. For this purpose, as Rabinowitz we use the properties of $Y$ given by (6). The definition of $Y$ implies

$$\int_{-kT}^{kT} (W_q(t, q_k(t)), q_k(t)) \, dt \leq Y(\|q_k\|_{L^\infty_{2kT}})\|q_k\|_{E_k}^2$$

(34)

for every $k \in \mathbb{N}$. Since $I'_k(q_k)q_k = 0$, (10) gives

$$\int_{-kT}^{kT} (W_q(t, q_k(t)), q_k(t)) \, dt = \int_{-kT}^{kT} |\dot{q}_k(t)|^2 \, dt + \int_{-kT}^{kT} (K_q(t, q_k(t)), q_k(t)) \, dt.$$  

(35)

Substituting (35) into (34), and next applying (H3) and (H2) we obtain

$$Y(\|q_k\|_{L^\infty_{2kT}})\|q_k\|_{E_k}^2 \geq \min\{1, b_1\}\|q_k\|_{E_k}^2,$$

and hence

$$Y(\|q_k\|_{L^\infty_{2kT}}) \geq \min\{1, b_1\} > 0.$$  

(36)

The remainder of the proof is the same as in [12]. If $\|q_k\|_{L^\infty_{2kT}} \to 0$, as $k \to +\infty$, we would have $Y(0) \geq \min\{1, b_1\} > 0$, a contradiction. Thus there is $\gamma > 0$ such that

$$\|q_k\|_{L^\infty_{2kT}} \geq \gamma$$

(37)

for every $k \in \mathbb{N}$. Clearly, $q_k(t + jT)$ is a $2kT$-periodic solution of (HS$_k$) for every $j \in \mathbb{Z}$. By replacing earlier, if necessary, $q_k$ by $q_k(t + jT)$ for some $j \in [-k, k] \cap \mathbb{Z}$, one can assume that the maximum of $q_k$ occurs in $[-T, T]$. Suppose, contrary to our claim, that $q_0 \equiv 0$. Then, by Lemma 2.6,

$$\|q_{\varphi(k)}\|_{L^\infty_{2\varphi(k)T}} = \max_{t \in [-T, T]} |q_{\varphi(k)}(t)| \to 0,$$

which contradicts (37). □

References
