The classification of inherited hyperconics in Hall planes of even order

William Cherowitzo
Department of Mathematical and Statistical Sciences, University of Colorado Denver, Campus Box 170, P.O. Box 173364, Denver, CO 80217-3364, USA

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A B S T R A C T

In this note we complete the classification of inherited hyperconics in Hall planes of even order that was started by O’Keefe and Pascasio by proving that in the cases left open in [C.M. O’Keefe, A.A. Pascasio, Images of conics under derivation, Discrete Math. 151 (1996) 189–199] there are no inherited hyperconics.

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1. Introduction

A $k$-arc in a projective plane is a set of $k$ points no three of which are collinear. In a projective plane of order $n$ the maximum size of a $k$-arc depends on the parity of $n$; if $n$ is odd then the largest $k$-arcs are $(n + 1)$-arcs and are called ovals, while if $n$ is even the largest arcs are $(n + 2)$-arcs and are called hyperovals. In Desarguesian planes, examples of ovals are given by conics and in the even characteristic case, a conic can be extended by a uniquely determined point (the intersection of all the tangent lines of the conic) called the nucleus to form a hyperconic, which is a hyperoval (a.k.a. regular hyperoval, complete conic, etc.) In 1955, Segre [8] proved that the only ovals in Desarguesian planes of odd characteristic are the conics. The situation for even characteristic Desarguesian planes is far from determined (see Cherowitzo [2]). Ovals and hyperovals are known to exist in many non-Desarguesian planes, but Penttila, Royle and Simpson [7] have found some non-Desarguesian planes of order 16 which contain no hyperovals.

It is current practice to define Hall planes as the planes obtained by a single derivation of the corresponding Desarguesian plane of the same order, but they were originally introduced algebraically via Hall quasifields. We shall just sketch this well-known construction. Let $\ell_\infty$ be the infinite line of the affine plane $A = AG(2, q^2)$ (thus $A \cup \ell_\infty$ is the projective plane $P = PG(2, q^2)$). A set of $q + 1$ points...
on \( \ell_\infty \), denoted by \( \mathcal{D} \), is called a derivation set if there exists a set \( \mathcal{B} \) of Baer subplanes (subplanes of order \( q \)) of \( \mathcal{P} \) such that:

1. If \( b \in \mathcal{B} \), then \( b \cap \ell_\infty = \mathcal{D} \), and
2. If \( P, Q \in A \) with \( P \neq Q \) and \( PQ \cap \ell_\infty \in \mathcal{D} \), then there exists \( b \in \mathcal{B} \) with \( P, Q \in b \).

The affine Hall plane is constructed, with respect to a given derivation set \( \mathcal{D} \), by taking as points, the points of \( A \), and as lines, the lines of \( A \) whose projective extensions meet \( \ell_\infty \) outside of \( \mathcal{D} \) and the restrictions to \( A \) of the Baer subplanes in \( \mathcal{B} \). In the situation we have described, the affine planes obtained from different derivation sets are all isomorphic, so we may refer to the affine Hall plane of order \( q^2 \), without reference to the particular choice of derivation set. We shall use the term Hall plane to indicate the standard projective extension of an affine Hall plane and denote it by \( \text{HALL}(q^2) \), or if specific planes are under consideration by the slightly abbreviated \( \text{HALL}9, \text{HALL}16 \), etc. Desarguesian planes will be denoted in a similar manner by \( \text{DES}(q^2) \), \( \text{DES}9 \), etc. It is an easy exercise to show that \( \text{HALL}4 \) is isomorphic to \( \text{DES}4 \), but all other Hall planes are non-Desarguesian. It is clear from the construction that any collineation of \( \text{DES}(q^2) \) which stabilizes the derivation set will be a collineation of \( \text{HALL}(q^2) \), and it can be shown that if \( q > 3 \) these are the only collineations of \( \text{HALL}(q^2) \). In particular, since all translations of \( A \) fix \( \ell_\infty \) pointwise, they are also collineations of \( \text{HALL}(q^2) \) and so, the Hall planes are translation planes.

If one takes the viewpoint that non-Desarguesian projective planes are just Desarguesian planes (taken as point sets) in which some (possibly all) of the lines have been redefined, then it is natural to define the concept of an inherited arc, i.e. a \( k \)-arc in the Desarguesian plane which as a point set remains a \( k \)-arc after the lines have been redefined to form the non-Desarguesian plane. The examples of ovals and hyperovals in Hall planes to be found in the literature are all inherited arcs, with only one, computer obtained, exception in \( \text{HALL}16 \) ([11]).

In [5], O’Keefe and Pascaio consider images of conics in \( \text{DES}(q^2) \) under the operation of derivation. They examine both the odd and even characteristic cases. The scheme that is used involves a first level classification into 3 cases according to whether the line \( \ell_\infty \) is:

A: a secant line with points \( P \) and \( Q \) of the conic (the hyperbolic case),
B: a tangent line with point \( P \) of the conic (the parabolic case), or
C: an exterior line (the elliptic case).

We shall restrict ourselves to the even characteristic case, and thus, in the parabolic case, the nucleus \( N \) also lies on \( \ell_\infty \). There is a refinement of the scheme in the hyperbolic and parabolic cases depending on the relationship of these points with the derivation set \( \mathcal{D} \). With \( X = Q \) in the hyperbolic case or \( X = N \) in the parabolic case, we have the subcases:

1. \( P, X \in \mathcal{D} \),
2. exactly one of \( P \) or \( X \) is in \( \mathcal{D} \), or
3. \( P, X \not\in \mathcal{D} \).

(Note that this does not correspond precisely to the subcase numbering of [5].) O’Keefe and Pascaio then show that there are no inherited conics (and hence hyperconics) in cases A(1) and B(1). Every hyperconic of case B(2) inherits [6], and the two situations give non-isomorphic hyperovals in \( \text{HALL}(q^2) \). Hyperconics in case B(3) which inherit are classified by Glynn and Steinke in [3].

In this paper we will finish the classification by proving:

**Theorem (A).** There are no inherited hyperconics in Hall planes of even order in the elliptic case (C) or case A(3).

in Section 3 and

**Theorem (B).** There are no inherited hyperconics in Hall planes of even order in the hyperbolic case A(2).

in Section 4.

2. Algebraic preliminaries

In \( \text{DES}(q^2) \) let \( \ell_\infty \) be the line \( z = 0 \) containing the derivation set \( \mathcal{D} = \{(1, \eta, 0) \mid \eta \in GF(q^2), \eta^{q+1} = 1\} \). Let \( \mathcal{D} = \{\eta \mid \eta^{q+1} = 1, \eta \in GF(q^2)\} \) and note that \( \mathcal{D} \), the set of norm 1 elements of
If any hyperoval of DES(q^2), is a multiplicative subgroup of GF(q^2) = GF(q^2) \ {0}. If a Baer subplane contains a derivation set, then we say that the subplane belongs to that derivation set. The affine points of a Baer subplane that belongs to D satisfy an equation of the form y = ηx^q + k for some η ∈ D and k ∈ GF(q^2) (see Hughes and Piper [4]).

We first collect some simple but useful properties of the subgroup D.

**Lemma 1.** Let D = {η | η^{q+1} = 1, η ∈ GF(q^2)}, then:
(a) d ∈ D ⇒ d^q ∈ D ∀ integers a,
(b) d ∈ D ⇒ d^q = \frac{1}{d},
(c) m^{q-1} ∈ D ∀ m ∈ GF(q^2)*.
(d) D ∩ GF(q) = {1} (valid only when q is even). □

**Lemma 2.** Each non-zero element m of GF(q^2), q even, can be written uniquely as m = ηα where η ∈ D and α ∈ GF(q)^*.

**Proof.** Since | D | = q + 1 there are (q + 1)(q − 1) non-zero products of the form ηα where η ∈ D and α ∈ GF(q)^*.

Since q is even, (q + 1, q − 1) = 1 and there are elements a, b ∈ Z_{q^2} such that 1 = a(q + 1) + b(q − 1). Thus, for any n ∈ Z_{q^2}, n = an(q + 1) + bn(q − 1) = c(q + 1) + d(q − 1) (mod q^2 − 1) where c = an (mod q − 1) and d = bn (mod q + 1). Let ω be a primitive element of GF(q^2).

Then ω^q = ω^{a(q+1)+d(q-1)} = (ω^q)^{a(q+1)}(ω^q)^{d(q-1)} = αη where η ∈ D and α ∈ GF(q)^*.

In what follows, we shall also need certain collineations with special properties.

**Lemma 3.** Any hyperoval of DES(q^2), q even, containing two points on ℓ∞, one in the derivation set D and the other not, can be mapped to a hyperoval containing (1, 1, 0) and (0, 1, 0) by a homomorphism which preserves D.

**Proof.** If (1, a, 0) and (1, b, 0) are the points of a hyperoval on the line ℓ∞ with a ∉ D and b ∈ D then any homomorphism in PGL(3, q^2) of the form

\[(x, y, z) \mapsto (x, y, z) \begin{pmatrix} a & 1 + a^q b & 1 \\ a + b & 1 + a^q b & a^q \\ a + b & a^q & 1 \end{pmatrix}, \] (1)

maps (1, a, 0) → (0, 1, 0), (1, b, 0) → (1, 1, 0) and preserves the set D. If, on the other hand, the point of this hyperoval which is not in D is (0, 1, 0) then a map of the form

\[(x, y, z) \mapsto (x, y, z) \begin{pmatrix} b & 0 & 0 \\ 0 & 1 & 0 \\ * & * & 1 \end{pmatrix}, \] (2)

fixes (0, 1, 0), maps (1, b, 0) → (1, 1, 0) and preserves the set D. □

3. The elliptic case and case A(3)

The open cases C and A(3) can be dealt with simultaneously. Let Ω be a conic of DES(q^2) whose nucleus does not lie on ℓ∞. Using a translation, we may assume that the nucleus N of Ω is the origin, i.e. the point with affine coordinates (0, 0). Hence, Ω = {(x, y) | Ax^2 + xy + By^2 + F = 0} where F ≠ 0 since the nucleus is not a point of the conic and if Tr(AB) = 1, ℓ∞ is an exterior line and we are in case C, while if Tr(AB) = 0, ℓ∞ is a secant line and we shall assume that we are in case A(3) (Tr is the absolute trace function of GF(q^2)).

If Ω is an inherited oval in HALL(q^2) then (0, 0) is still the nucleus of this oval in this plane. Thus, in DES(q^2), each Baer subplane belonging to D which contains (0, 0) must contain exactly one point.
of $\Omega$ (in the A(3) case, this follows since the two infinite points are not in $\mathcal{D}$). We now proceed to investigate these intersections.

In the A(3) case, we may use a collineation of Lemma 3 to ensure that $\Omega$ contains the point $(0, 1, 0)$. This implies that $B = 0$. If $A \neq 0$ then in either case $A \neq B^q$ (note that $A = 0$ can only occur in the A(3) case) since if $A = B^q$ then $AB = B^{q+1} \in GF(q)$ and $Tr(AB) = 0$, which does not occur in case C.

The affine points in Baer subplanes of DES($q^2$) containing $(0, 0)$ and belonging to $\mathcal{D}$ satisfy $y = \eta x^q$ for $\eta \in \mathbf{D}$. The $x$ coordinates of the points of intersection of these subplanes with $\Omega$ satisfy

$$x^{q+1} + \frac{A}{\eta} x^2 + B \eta x^{2q} + \frac{F}{\eta} = 0. \quad (3)$$

Adding this to its $q$th power gives the equation

$$\eta(A^q + B)x^{2q} + \frac{1}{\eta}(A + B^q)x^2 + \left(\frac{F}{\eta} + \eta F^q\right) = 0. \quad (4)$$

With $z = \eta^q (A + B^q)x^2$ this becomes

$$z^q + z + \left(\frac{F}{\eta} + \eta F^q\right) = 0,$$

whose $q$ solutions are $z = \alpha + \eta^q F$ as $\alpha$ runs through $GF(q)$. As each solution of (3) is also a solution of (4), we obtain the solutions of (3) by finding the appropriate values of $\alpha$. The $\alpha$’s that give solutions of (3) satisfy

$$\alpha S + \sqrt{\alpha T} + M = 0,$$

where

$$S = \frac{A}{A + B^q} + \frac{B}{A^q + B} + \frac{1}{\sqrt{(A + B^q)^{q+1}}},$$

and

$$T = \sqrt{\eta F^q + \eta^{q+1} F^{q+1}}.$$

unless $A = 0$ and $B = 0$. This quadratic equation in $\sqrt{\alpha}$ has a unique solution only in the cases where exactly one of $S$ or $T$ is zero. Notice that $T = 0$ if and only if $\frac{F}{\eta} \in GF(q)$. By Lemma 2, $F = \eta_1 \alpha_1$ with $\eta_1 \in \mathbf{D}$ and $\alpha_1 \in GF(q)^*$. Hence, if $S = 0$, by choosing $\eta = \eta_1$, Eq. (3) has either 0 or $q$ solutions and if $S \neq 0$, for each $\eta \neq \eta_1$, Eq. (3) has either 0 or 2 solutions. Finally, in the special case that $A = 0$ and $B = 0$, Eq. (3) has either 0 or $q + 1$ solutions for any value of $\eta$. Thus, in all cases, there will exist Baer subplanes belonging to $\mathcal{D}$ through the point $(0, 0)$ which do not contain exactly one point of $\Omega$, and so, $\Omega$ cannot be an inherited arc in HALL($q^2$).

This proves

**Theorem** (A). *There are no inherited hyperconics in Hall planes of even order in the elliptic case (C) or the hyperbolic case A(3).* \(\square\)

**Remark 4.** By being a bit more careful with the counting of solutions, a proof of the non-existence of inherited hyperconics in the A(1) case can also be obtained from this argument. Case A(2), however, cannot be done this way since all the appropriate Baer subplanes do contain exactly one point of $\Omega$, with a single exception (needed in this case) that contains no such point.

### 4. The hyperbolic case A(2)

To deal with the remaining hyperbolic case we shall start with a special instance whose proof clearly indicates the general method.
Lemma 5. The hyperconic with equation \( x^2 + xy + z^2 = 0 \) is not inherited in a Hall plane of even order \( q^2 \) \((q \geq 4)\).

Proof. The points of this hyperconic on \( \ell_\infty \) are \((0, 1, 0)\) and \((1, 1, 0)\). We will carry out all computations using affine coordinates. The point \((\alpha, 1)\) where \(\alpha^2 + \alpha + 1 = 0\) lies on the conic. Each line through \((\alpha, 1)\) and \((1, \eta, 0)\), where \(\eta \in D \setminus \{1\}\), is a secant line of the conic and contains the additional conic point with coordinates

\[
\left( \frac{\alpha + 1}{\eta + 1}, \frac{\eta^2 \alpha + 1}{\eta + 1} \right). \tag{5}
\]

If \(q = 2^e\) then \(\alpha \in GF(q)\) and we can calculate that the Baer subplane belonging to \(\mathcal{D}\) and containing \((\alpha, 1)\) and the point of (5) has equation

\[
y = \left( \frac{\eta(\alpha \eta + 1)}{\eta + \alpha} \right) x^q + 1 + \frac{\alpha \eta(\alpha \eta + 1)}{\eta + \alpha}. \tag{6}
\]

On the other hand, if \(q = 2^{e+1}\) then \(\alpha^q = \alpha^2 = \alpha + 1\) and the corresponding Baer subplane has equation

\[
y = \left( \frac{\eta(\alpha \eta + 1)}{\eta + \alpha^2} \right) x^q + \frac{\alpha \eta + \eta^2 + \alpha^2}{\eta + \alpha^2}. \tag{7}
\]

Now, we determine the condition that implies that the points

\[
(\alpha, 1), \left( \frac{\alpha + 1}{\eta_1 + 1}, \frac{\eta_1^2 \alpha + 1}{\eta_1 + 1} \right), \quad \text{and} \quad \left( \frac{\alpha + 1}{\eta_2 + 1}, \frac{\eta_2^2 \alpha + 1}{\eta_2 + 1} \right) \tag{8}
\]

lie in the same Baer subplane belonging to \(\mathcal{D}\). When \(q = 2^e\) this condition is

\[
\eta_2 = \frac{\alpha \eta_1 + 1}{\alpha + \eta_1}, \tag{9}
\]

while for \(q = 2^{e+1}\) the condition becomes

\[
\eta_2 = \frac{\alpha^2 + \eta_1}{1 + \alpha \eta_1}. \tag{10}
\]

Finally, we verify that the three points of (8) are defined and distinct. Conditions (9) and (10) imply that when defined, \(\eta_2 \in D\). When \(\eta_2\) is defined, in order to get three distinct points we must have \(\eta_1 \neq 1, \eta_2 \neq 1\) and \(\eta_1 \neq \eta_2\). When \(q = 2^e\) all these conditions are met provided \(\eta_1 \neq 1\). When \(q = 2^{e+1}\), the conditions are all met provided \(\eta_1 \neq 1\) and \(\eta_1 \neq \alpha^2\). Thus in all cases, if \(q \geq 4\) there will be a triple of conic points contained in a Baer subplane belonging to \(\mathcal{D}\) and so, this hyperconic is not an inherited arc in the corresponding Hall plane. \(\square\)

We can now consider the general case of a hyperconic with two infinite conic points, one and only one of which lies in the derivation set \(\mathcal{D}\). By Lemma 3, we may assume that these points are \((1, 1, 0)\) and \((0, 1, 0)\). Furthermore, by the use of an appropriate translation, we may assume that the nucleus of the conic (an affine point) is the point \((0, 0, 1)\). As all the collineations used here preserve the derivation set, any inherited hyperconic in the Hall plane of this type is projectively equivalent to one with these properties. In DES\((q^2)\), a conic passing through \((1, 1, 0), (0, 1, 0)\) with nucleus \((0, 0, 1)\) has an equation of the form \(x^2 + xy + Fz^2 = 0\), where \(F \neq 0\) since the nucleus is not a point of the conic and where we may assume, by Lemma 5, that \(F \neq 1\).

The point \((1, 1 + F, 1)\) lies on this conic and the lines determined by this point and the points \((1, \eta, 0)\), with \(\eta \in D\) are secant lines of the conic containing another conic point provided \(\eta \neq 1\) and \(\eta \neq 1 + F\) (in this latter case the line would be a tangent to the conic). Under these restrictions, the other conic points on these secant lines have, respectively, coordinates (we will from now on work with affine coordinates):
\[
\left( \frac{F}{1 + \eta}, \frac{1 + \eta^2 + F}{1 + \eta} \right). \tag{11}
\]

The Baer subplane belonging to \( D \) containing \((1, 1 + F)\) and a point of \( (11) \) has an equation of the form

\[
y = \frac{1}{(1 + \eta + F)^{q-1}} x^q + 1 + \eta + F + \frac{\eta F^q + F}{(1 + \eta + F)^q}. \tag{12}
\]

We now seek the condition that implies that the points

\[
(1, 1 + F), \quad \left( \frac{F}{1 + \eta_1}, \frac{1 + \eta_1^2 + F}{1 + \eta_1} \right) \quad \text{and} \quad \left( \frac{F}{1 + \eta_2}, \frac{1 + \eta_2^2 + F}{1 + \eta_2} \right) \tag{13}
\]

lie in the same Baer subplane belonging to \( D \). The necessary and sufficient condition is that \((1 + \eta_1 + F)^q = (1 + \eta_2 + F)^q\), and this implies that there exists a \( \beta \in GF(q) \) such that \( 1 + \eta_2 + F = \beta (1 + \eta_1 + F) \). Hence,

\[
\eta_2 = (1 + \beta)(1 + F) + \beta \eta_1. \tag{14}
\]

From this it follows that \( \beta = 1 \) if and only if \( \eta_1 = \eta_2 \) since \( \eta_1 \neq 1 + F \). Using \( (14) \) and the condition

\[
1 + \eta_1 + F + \frac{\eta_1 F^q + F}{(1 + \eta_1 + F)^q} = 1 + \eta_2 + F + \frac{\eta_2 F^q + F}{(1 + \eta_2 + F)^q},
\]

we can solve for \( \beta \) and obtain

\[
\eta_2 = \frac{1}{\eta_1 (1 + \eta_1 + F)^{q-1}}. \tag{15}
\]

Finally, we verify that the three points of \( (13) \) are defined and distinct. Condition \( (15) \) implies that \( \eta_2 \in D \) if \( \eta_1 \neq 1 + F \). Note that the possibility that \( \eta_2 = 1 + F \) leads, by \( (14) \), to the case \( \eta_1 = \eta_2 \).

In order to get three distinct points we must have \( \eta_1 \neq 1, \eta_2 \neq 1 \) and \( \eta_1 \neq \eta_2 \). If \( \eta_2 = 1 \) then \( (15) \) implies that \( \eta_1 = F^{1-q} \), and if \( \eta_2 = \eta_1 \) we obtain \( \eta_1^2 = (1 + F)^{1-q} \). Hence, if \( q + 1 > 4 \) there will exist an \( \eta_1 \) which provides a triple of conic points contained in a Baer subplane belonging to \( D \) and so, this hyperconic is not an inherited arc in the corresponding Hall plane. Note that there are at most 3 values of \( D \) that need to be avoided, since if \( 1 + F \in D \) the \( \eta_1 = \eta_2 \) condition becomes \( \eta_1 = 1 + F \).

This proves

**Theorem** (B). There are no inherited hyperconics in Hall planes of even order \( q^2 \) \((q \geq 4)\) in the hyperbolic case \( A(2) \). \( \Box \)

**References**