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ITERATIVE ALGEBRAS

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Contents

1. Introduction

The problem of finding suitable environments, in which so-called 'iterative equations' have solutions, occurs frequently in connection with the theory of computing and programming languages. The equations, or rather sets of equations, which are to be solved, are of the form $y_j = t_j(x_1, \ldots, x_k, y_1, \ldots, y_n)$ $(1 \leq j \leq n)$, for polynomials t_j and 'fixed' x_i. Such equations occur, for example, when modelling loops in flow charts or when studying formal grammars.

Elgot [4] has proposed iterative algebraic theories as models in which such equations have unique solutions, and the existence of free iterative theories was established in Bloom and Elgot [2]. Subsequently, Ginali [5] gave a construction

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of the free iterative theory using regular trees, analogous to the familiar construction of ordinary free algebras which uses finite labelled trees.

Now, algebraic theories were put forward by Lawvere [6]; they provide a homogeneous approach to (not necessarily finitary) universal algebra via category theory. However, iterative theories are not entirely homogeneous, and Bloom and Elgot [2, abstract] suggest that this is the obstacle to proving the existence of free iterative theories from 'general algebraic considerations'.

This paper outlines a new approach to modelling unique solutions of polynomial equations, namely that of iterative algebras. These are algebras in which some, or all, iterative equations are uniquely solvable. This approach has the advantage that it is simpler than Elgot's, dealing only with the familiar notion of polynomial, has Elgot's results as corollaries, and is applicable to some situations that do arise in computer science which do not fit into the framework of iterative theories.

Iterative algebras are introduced in Section 2.2, and general algebraic considerations are seen to indeed directly yield the existence of free iterative algebras (Section 2.3). This fact, together with some general properties of iterative systems of equations (Section 2.4) then has as a corollary the existence of free iterative theories (Section 4.1).

This approach is more general than that developed by Elgot et al., not only because their results are consequences of the ones presented here, but also because this approach lends itself to a consideration of algebras where there are unique solutions, not to all iterative systems of polynomial equations, but to specified subcollections. The existence of free algebras of this sort also follow from general considerations (Section 2.3), the point always being that the required solutions can be viewed as additional operations, and the algebras in question are then realized as an implicationally defined class of algebras of an expanded type, in much the same way that torsion-free divisible abelian groups may be viewed as vector spaces over the field of rationals. This approach does not work in case the solutions are not required to be unique, and an example to show that free algebras need not exist in this situation is given at the end of Section 2.3.

The free iterative algebra is explicitly constructed in Section 3.1 as the collection of all regular trees, and free iterative algebras with respect to certain subcollections of equations (including the one related to context-free grammars) are realized as subalgebras of the algebra of regular trees in Sections 3.2 and 3.3. In Section 3.4 we see that, in contrast to the situation for theories (Bloom, Ginali and Rutledge [3]), an algebra may have unique solutions to all single polynomial equations but still may not be iterative.

All of the material on algebraic theories is postponed until Section 4, where the general existence results claimed above are proved and some examples from Elgot [4] are discussed. Although this chapter is independent of the explicit constructions given in Section 3, when combined with 3.1 it yields Ginali's explicit description of the free iterative theory.

2. Definitions and basic results

In this section, the definitions of an algebra and a free algebra of a given type are recalled, and the notion of an iterative algebra is introduced, analogous to the iterative theories of Elgot [4], Bloom, Ginali and Rutledge [3] and Ginali [5]. In addition, basic results from universal algebra are used to establish the existence of free iterative algebras.

2.1. *Algebras and free algebras*

As usual, a *type* of algebras is a set Σ (of 'operation symbols') together with a function which assigns to each $\sigma \in \Sigma$ a natural number $|\sigma|$ called the *arity* of σ. An *algebra of type* Σ is a set A together with, for each $\sigma \in \Sigma$, a $|\sigma|$-ary operation on A, i.e. a function $\sigma_A : A^{|\sigma|} \to A$. A *homomorphism* $f: A \to B$ between algebras A and B is a function from A to B such that, for all $\sigma \in \Sigma$ and all $a_1, \ldots, a_{|\sigma|} \in A$, $f(\sigma_A(a_1, \ldots, a_{|\sigma|})) = \sigma_B(f(a_1), \ldots, f(a_{|\sigma|}))$. For a set V (of 'variables') the free algebra F (word algebra) of type Σ over the set V is characterized (up to isomorphism over V) by the property that $V \subseteq F$, and every function from V into an algebra A of type Σ extends uniquely to a homomorphism from F into A. The existence of this free algebra is well known, as is the following explicit description as a collection of finite trees with labels from $\Sigma \cup V$: let ω^* be the set of finite sequences of natural numbers ≥ 1. A *V-labelled Σ-tree* is a partial function $t: \omega^* \rightharpoonup \Sigma \cup V$ with non-empty domain, $\operatorname{dom}(t)$, such that

(1) for all $u \in \omega^*$, $k \in \omega$, if $uk \in \operatorname{dom}(t)$ then $u \in \operatorname{dom}(t)$ *and* $t(u) = \sigma$ for some $\sigma \in \Sigma$ with $|\sigma| \geq k$;

(2) if $t(u) = \sigma$ then $uk \in \operatorname{dom}(t)$ for all $k \leq |\sigma|$.

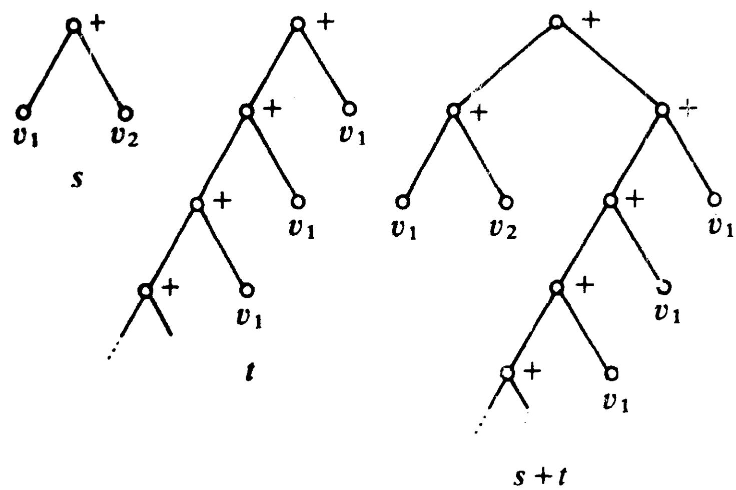
The set of all such trees forms an algebra $T_\Sigma V$, where the operators are defined by

$$\sigma_{T_\Sigma V}(t_1, \ldots, t_{|\sigma|})(\emptyset) = \sigma,$$

$$\sigma_{T_\Sigma V}(t_1, \ldots, t_{|\sigma|})(ku) = t_k(u) \quad \text{for all } u \in \omega^*, \text{ all } 1 \leq k \leq |\sigma|.$$

Also, the function which assigns to each $v \in V$ the partial function $\bar{v}$ with domain $\{\emptyset\}$, such that $\bar{v}(\emptyset) = v$, gives an embedding of V into $T_\Sigma V$. The *complexity* (or degree) of an element $t \in T_\Sigma V$ is the number of operation symbols appearing in t, i.e. the number of sequences $u \in \omega^*$ with $t(u) \in \Sigma$; the complexity can of course be infinite.

Examples. Σ has a binary operation $+$,



domain of $s = \{\emptyset, 1, 2\}$,

domain of $t = \{\emptyset, 1, 2, 11, 12, 111, 112, 1111, 1112, \ldots\}$

= all sequences containing at most one 2, which must occur at the end,

domain of $s+t = \{\emptyset, 1, 2, 11, 12, 21, 22, 211, 212, 2111, 2112, \ldots\}$

$= \{\emptyset\} \cup \{1u \mid u \in \mathrm{dom}(s)\} \cup \{2u \mid u \in \mathrm{dom}(t)\}$.

The subset $F_\Sigma V$ of $T_\Sigma V$ consisting of all finite trees (i.e. all trees with finite domains) is clearly closed under the Σ-operations, and hence is an algebra; it is well known that this algebra is the free Σ-algebra over V. Moreover, if $U \subseteq V$ then the subalgebra of $F_\Sigma V$ generated by U is $F_\Sigma U$.

If $U = \{u_1, \ldots, u_m\}$ is an m-element set then the elements of $F_\Sigma U$ are called *m-ary polynomials*. For each algebra A of type Σ, each $t \in F_\Sigma U$ induces an m-ary operation $t_A : A^m \to A$; for $a_1, \ldots, a_m \in A$, $t_A(a_1, \ldots, a_m)$ is the image of t under the unique homomorphism $F_\Sigma U \to A$ mapping u_i to a_i for each $i \leq m$. Note that if $t = u_i$ then $t_A : A^m \to A$ is just the ith projection, i.e. $t_A(a_1, \ldots, a_m) = a_i$, and if $t = \sigma(u_1, \ldots, u_{|\sigma|})$ for $\sigma \in \Sigma$ then $t_A = \sigma_A$.

Each homomorphism $f: A \to B$ between algebras A and B preserves all these polynomials, i.e. for each $t \in F_\Sigma\{u, \ldots, u_m\}$ and for all $a_1, \ldots a_m \in A$

$$f(t_A(a_1, \ldots, a_m)) = t_B(f(a_1), \ldots, f(a_m)).$$

This fact is well known, and in any case not difficult to prove; one merely shows that the set of all $t \in F_\Sigma\{u_1, \ldots, u_m\}$ which are preserved by f contains $u_1, \ldots, u_m$ and is closed under all the Σ-operations, and uses the fact that $F_\Sigma\{u_1, \ldots, u_m\}$ is generated by $\{u_1, \ldots, u_m\}$.

Actually, since $F_\Sigma\{u_1, \ldots, u_m\} \subseteq F_\Sigma\{u_1, \ldots, u_p\}$ whenever $m \leq p$, each $t \in F_\Sigma\{u_1, \ldots, u_m\}$ induces a p-ary operation $A^p \to A$ for each $p \geq m$, which, of course, does not depend on its last $p-m$ arguments. Thus, strictly speaking, our notation 't_A' is ambiguous and should also include the 'm'; however this will normally be clear from context.

2.2. Iterative systems and their solutions

The main concern of this paper is algebras in which certain systems of polynomial equations of the form

$$\begin{aligned} y_1 &= t_1(x_1,\ldots,x_k,y_1,\ldots,y_n)\\ y_2 &= t_2(x_1,\ldots,x_k,y_1,\ldots,y_n)\\ &\;\vdots\\ y_n &= t_n(x_1,\ldots,x_k,y_1,\ldots,y_n) \end{aligned}$$

have, for each choice of the x_i, unique solutions $y_1,\ldots,y_n$. Since the x_i and y_j really play different roles, it will be convenient to introduce two disjoint sets of variables for them, and so we make the following definition.

For the rest of this paper, let $X=\{x_i \mid 1\leq i\in\omega\}\cup\{y_i \mid 1\leq i\in\omega\}$ be a countable set, where $x_i\neq y_i$, and for $i\neq j$, $x_i\neq x_j\neq y_i\neq y_j$, and let $F_\Sigma X$ be, as described above, the free algebra of type Σ over the set X. For each pair $n\geq 1$ and k of natural numbers, let $F_\Sigma(k,n)$ be the subalgebra of $F_\Sigma X$ generated by $\{x_1,\ldots,x_k,y_1,\ldots,y_n\}$. Further, for each $t\in F_\Sigma(k,n)$ and each Σ-algebra A, let t_A be the $(k+n)$-ary polynomial on A induced by t such that $t_A(a_1,\ldots,a_k,b_1,\ldots,b_n)$ is the image of t under the homomorphism from $F_\Sigma(k,n)$ to A mapping x_i to a_i for $1\leq i\leq k$ and y_j to b_j for $1\leq j\leq n$. (Note that k may be zero.)

An *iterative system* is, for some $n\geq 1$ and k, an n-tuple $T=(t_1,\ldots,t_n)$ of elements of $F_\Sigma(k,n)$; T is called *uniquely solvable in* A (for a Σ-algebra A) iff

(*) for all $a_1,\ldots,a_k\in A$ there exists unique $b_1,\ldots,b_n\in A$ such that

$$b_j=t_{jA}(a_1,\ldots,a_k,b_1,\ldots,b_n)\quad\text{for } 1\leq j\leq n.$$

An iterative system T is called *degenerate* iff it is uniquely solvable in every Σ-algebra. $T=(t_1,\ldots,t_n)$ is called *trivial* iff there exist $i_1,\ldots,i_m\in\{1,2,\ldots,n\}$ such that $t_{i_1}=y_{i_2}$, $t_{i_2}=y_{i_3},\ldots,t_{i_m}=y_{i_1}$. Note that a solution in A for the system $T=(y_2,y_3,y_4,\ldots,y_n,y_1)$ consists of elements $b_1,\ldots,b_n\in A$, such that $b_1=b_2=b_3=\cdots=b_n$ and hence for any element $b\in A$, $b=b_1=b_2=\cdots=b_n$ provides such a solution. Thus, if this system is uniquely solvable in A, it follows that A is the trivial (= one-element) algebra.

Further, T is called *ideal* if $t_j\notin\{x_1,\ldots,x_k,y_1,\ldots,y_n\}$ for all $j\leq n$; i.e., if each t_j has complexity ≥ 1; the ideal iterative systems are the analogues for algebras of the ideal morphisms of Elgot [3]. Clearly every ideal system is non-trivial; we will see below that from the point of view of solvability these notions are equivalent. First some examples.

Example 1. Suppose Σ has one binary operation $+$ and let $T=(x_1+y_2, y_1)\in F_\Sigma(1,2)^2$; then T is non-trivial but not ideal. T is uniqely solvable in an algebra A iff for all $a\in A$ there exist unique b_1, $b_2\in A$ with $b_1=a+b_2$ and $b_2=b_1$. This is clearly equivalent to the existence of a unique $b\in A$ with $b=a+b$, and hence the solvability of T is equivalent with the solvability of the ideal system $\bar{T}=(x_1+y_1)\in F_\Sigma(1,1)^1$.

Example 2. Let $T=(x_1, y_1)\in F_\Sigma(1,2)^2$; then T is uniquely solvable in A iff for all $a\in A$ there are unique b_1 and b_2 in A with $b_1=a$, $b_2=b_1$. This system T is degenerate; for any A and $a\in A$, the unique solution for T is given by $b_1=b_2=a$.

Example 3. For any type Σ, let 2_Σ be the algebra whose elements are 0 and 1, with operations defined by

$$\sigma(a_1,\ldots,a_{|\sigma|})=\begin{cases}0 & \text{if } a_i=0 \text{ for some } i\leqslant|\sigma|,\\ 1 & \text{otherwise.}\end{cases}$$

We will see in Section 2.4 that an iterative system is uniquely solvable in 2_Σ iff it is degenerate, so that 2_Σ is a 'test object' for which systems are always uniquely solvable.

Example 4. For any type Σ and any set V, let $F_\Sigma^* V$ be the Σ-algebra obtained by adjoining one new element $*$ to $F_\Sigma V$ and defining the operations to act as usual in $F_\Sigma V$ and to take value $*$ otherwise, so that $\sigma(a_1,\ldots,a_{|\sigma|})=*$ whenever $a_i=*$ for some $i\leqslant|\sigma|$. Then $F_\Sigma^* V$ is a Σ-algebra in which every non-trivial iterative system is uniquely solvable; this is shown by the following discussion.

For any non-trivial and non-degerate iterative system $T=(t_1,\ldots,t_n)\in F_\Sigma(k,n)$, if some $t_j\in F_\Sigma\{x_1,\ldots,x_k\}$ then we may drop t_j from T and obtain an equivalent system (see Lemmas 1 and 2 in Section 2.4), and so we may restrict our attention to those iterative systems T with no components in $F_\Sigma\{x_1,\ldots,x_k\}$. But for any such T and $a_1,\ldots,a_k\in F_\Sigma^* V$, $b_1=b_2=\cdots=b_n=*$ provides a solution to T with respect to $a_1,\ldots,a_k$.

Finally, we verify that this solution is unique. Suppose $b_1,\ldots,b_n$ is any solution for T with respect to $a_1,\ldots,a_k$ and choose $p_1\leqslant n$ such that $b_{p_1}\neq *$, and among all the $j\leqslant n$ with $b_j\neq *$, b_{p_1} has the minimum complexity. Since $t_{p_1}\notin F_\Sigma\{x_1,\ldots,x_k\}$, there exists p_2 with y_{p_2} in the image of t_{p_1}, and thus because of the way the operations are defined in $F_\Sigma^* V$, $b_{p_2}\neq *$, and so the complexity of b_{p_2} is $\geqslant$ the complexity of b_{p_1}. But $b_{p_1}=t_{p_1}(a_1,\ldots,a_k,b_1,\ldots,b_n)$ and hence $t_{p_1}=y_{p_2}$ and $b_{p_1}=b_{p_2}$. Now we may repeat the above argument with b_{p_2} replacing b_{p_1}, and eventually yield a cycle with $t_{p_1}=y_{p_2}, t_{p_2}=y_{p_3},\ldots,t_{p_m}=y_{p_1}$, contradicting the non-triviality of T. It follows that $b_1=b_2=\cdots=b_n=*$ is the only solution for T, as required.

Example 5. If S is a commutative semigroup containing a 'zero element' z such that $y=xy$ for $x, y\in S$ iff $y=z$, then every non-trivial iterative system, for the type

Σ consisting of a single binary operation, is uniquely solvable in S. This is not difficult to prove, using the results which will be established in Section 2.4. Moreover, every commutative semigroup S in which every non-trivial iterative system is uniquely solvable is of this form: since $y = yy$ is uniquely solvable in S, there is a unique element $z \in S$ with $z = zz$. Now for any $x \in S$ there is a unique $\bar{x} \in S$ with $\bar{x} = \bar{x}\bar{x}$, and then $\bar{x}\bar{x} = x\bar{x}\bar{x}$ which implies that $\bar{x}\bar{x} = \bar{x}$ (by the uniqueness of $\bar{x}$) and hence $\bar{x} = z$ (by the uniqueness of z). Thus, for all $x \in S$, z is the unique element of S with $z = xz$.

2.3. *Iterative algebras and free iterative algebras*

Let C be any set of iterative systems of type Σ, i.e. C is any subset of $\bigcup_{k,n\in\omega, n\geq 1} F_\Sigma(k,n)^n$.

An algebra A of type Σ is called *C-iterative* iff every $T \in C$ is uniquely solvable in A. A is called *iterative* iff every non-trivial iterative system is uniquely solvable in A.

A C-iterative algebra I is the *free C-iterative algebra over a set V* iff $V \subseteq I$, and each function from V into a C-iterative algebra A extends uniquely to a homomorphism from I into A. Standard arguments show that I is unique up to isomorphism over V; the following discussion yields the existence of such free algebras.

For any C-iterative Σ-algebra A, each $T \in C$ induces n k-ary operations $T_{iA}: A^k \to A$ $(1 \leq i \leq n)$ defined as follows: for $a_1, \dots, a_k \in A$, $(T_{1A}(a_1, \dots, a_k), \dots, T_{nA}(a_1, \dots, a_k))$ is the unique solution of T with respect to $(a_1, \dots, a_k)$. Moreover, if f is any homomorphism from A to a C-iterative Σ-algebra B, then, as mentioned above, f preserves *all* polynomials, and consequently $(f(T_{1A}(a_1, \dots, a_k)), \dots, f(T_{nA}(a_1, \dots, a_k))$ is a solution for T with respect to $(f(a_1), \dots, f(a_k))$ in B, and hence by uniqueness of the solution in B, $f(T_{iA}(a_1, \dots, a_k)) = T_{iB}(f(a_1), \dots, f(a_k))$ for $1 \leq i \leq n$. This means that each such $T \in C$ induces n k-ary operations on each C-iterative Σ-algebra, and that the Σ-homomorphisms also preserve these new operations.

Let $\Sigma_C \supseteq \Sigma$ be the type obtained by adjoining to Σ, for each $T \in C \cap F_\Sigma(k,n)^n$, n k-ary operations $T_1, \dots, T_n$. The above discussion shows that each C-iterative Σ-algebra can be made into a Σ_C-algebra in such a way that Σ-homomorphisms between C-iterative Σ-algebras are also Σ_C-homomorphisms. In fact, we can recognize the Σ_C-algebras so obtained:

Let K_C be the class of all Σ_C-algebras which satisfy the following identities and implications for all $T = (t_1, \dots, t_n) \in C$ and $1 \leq i \leq n$

$$(\#)\quad T_i(x_1, \dots, x_k) = t_i(x_1, \dots, x_k, T_1(x_1, \dots, x_k), \dots, T_n(x_1, \dots, x_k)),$$

$$(\#\#)\quad \Big(\bigwedge_{1\leq j\leq n} t_j(x_1, \dots, x_k, y_1, \dots, y_n) = y_j\Big) \to y_i = T_i(x_1, \dots, x_k).$$

Here, $\bigwedge$ stands for logical conjunction. The identities $(\#)$ ensure that, for an algebra $A \in K$, the elements $T_{iA}(a_1, \dots, a_k)$ provide a solution of T with respect to $(a_1, \ldots, a_k)$ and the implications (##) ensure that this solution is unique. Thus, the class of C-iterative Σ-algebras is precisely the class of underlying Σ-algebras of algebras in K_C, and the Σ-homomorphisms and Σ_c-homomorphisms coincide for these algebras.

However, K_C is a class of Σ_C-algebras which is closed under formation of subalgebras and products, and is non-trivial by Example 4 of Section 2.2, and hence has free algebras (Birkhoff [1]). In view of the preceding paragraph, we know that (the underlying Σ-algrebra of) the free Σ_C-algebra over a set V *is* the free iterative Σ-algebra over V, yielding the following result.

Theorem 1. *For any set V, and any set C of iterative systems, the free C-iterative algebra over V exists.*

Note that the same arguments prove a stronger result: we could, in addition to C-iterativeness, also ask that our algebras satisfy certain Σ-equations; this again will yield a class of Σ_C-algebras closed under subalgebras and products, which then has free algebras. Thus, *for any equational class K of algebras of type Σ and any set C of iterative systems of polynomial equations of type Σ, the class of all C-iterative algebras in K has free algebras.*

Let us conclude this section by considering algebras of type Σ which have solutions of all iterative systems of Σ-equations, but the solutions need not be unique. These algebras can also be realized as reducts of algebras of an expanded type; however since the solutions need not be unique, the Σ-homomorphisms need not preserve the added operations. Indeed, here the analogy breaks down: such classes of algebras need *not* have free algebras. For example, let Σ consist of one unary operation σ, and let $A = \omega \cup \{a, b\}$ where $a, b \notin \omega$, with the operation σ defined as the successor operation in ω and $\sigma(a) = a$, $\sigma(b) = b$.

This yields an algebra with solutions to all iterative systems of Σ-equations: if we have an iterative system $T = (t_1, \ldots, t_n) \in F_\Sigma(k, n)^n$ and some $t_j \in F_\Sigma\{x_1, \ldots, x_k\}$, then the jth equation is of the form $y_j = \sigma^p(x_i)$ and may be dropped to yield a shorter equivalent system. Thus we need only consider iterative systems with no x_i appearing at all, and then $y_j = a$ for all j yields a solution in A.

Now, let k be the class of all Σ-algebras with solutions of all non-trivial iterative systems of Σ-equations, and suppose K has a free algebra F over the singleton set $\{v\}$. Then there is a unique homomorphism $h: F \to A$ with $h(v) = 1$. Now define $g: A \to A$ by $g(n) = n$ for all $n \in \omega$, $g(a) = b$, $g(b) = a$. Then g is a homomorphism, so gh is a homomorphism from F to A. Since $gh(v) = 1$, it follows that $gh = h$. Now, there exists some element $u \in F$ with $u = \sigma(u)$, and hence $h(u) = \sigma(h(u))$ which implies that $h(u)$ is either a or b. However, since $gh = h$, it follows that $gh(u) = h(u)$, which contradicts the definition of g.

Thus, *if the solutions to iterative systems are not required to be unique, there need not be free algebras.*

2.4. *Reduction and other theorems for iterative systems*

Recall that an iterative system $T=(t_1,\dots,t_n)$ is degenerate iff it is uniquely solvable in every Σ-algebra, trivial iff there exists $i_1,\dots,i_m$ with $t_{i_1}=y_{i_2}$, $t_{i_2}=y_{i_3},\dots,t_{i_m}=y_{i_1}$, and ideal iff each t_i has complexity $\geqslant 1$. Further, for each n, let C_n be the set of all iterative systems such that each component has complexity n, and L_n consist of all iterative systems of length n (i.e. with n components).

Note first of all that permutation of the order of the equations in an iterative system does not affect the solvability. For example, if Σ has a binary operation $+$ and $T=(x_1+y_1, x_2+y_2, (y_1+y_2)+y_3)$ then for a_1, $a_2\in A$, (b_1,b_2,b_3) is a solution for T with respect to (a_1,a_2) iff

$$b_1=a_1+b_1,$$

$$b_2=a_2+b_2,$$

$$b_3=(b_1+b_2)+b_3.$$

Let π be the permutation of $\{1,2,3\}$ with $\pi(1)=3$, $\pi(2)=1$, $\pi(3)=2$, and let $T_\pi=((y_2+y_3)+y_1, x_1+y_2, x_2+y_3)$; then (c_1,c_2,c_3) is a solution for T_π with respect to (a_1,a_2) iff

$$c_1=(c_2+c_3)+c_1,$$

$$c_2=a_1+c_2,$$

$$c_3=a_2+c_3.$$

Clearly (b_1,b_2,b_3) solves T iff (b_3,b_1,b_2) solves T_π.

The following lemma, which will be used several times in the remainder of this section, covers this phenomenon in full generality.

Lemma 1. *Suppose $T=(t_1,\dots,t_n)\in F_\Sigma(k,n)^n$ is an iterative system and π is a permutation of $\{1,2,\dots,n\}$. Let $h:F_\Sigma(k,n)\to F_\Sigma(k,n)$ be the homomorphism with $h(x_i)=x_i$ for all $i\leqslant k$, $h(y_j)=y_{\pi^{-1}(j)}$ for all $j\leqslant n$. Then for any Σ-algebra A and $a_1,\dots,a_k\in A$, $(b_1,\dots,b_n)$ is a solution for T with respect to $(a_1,\dots,a_k)$ iff $(b_{\pi(1)},\dots,b_{\pi(n)})$ is a solution for $T_\pi=(h(t_{\pi(1)}),\dots,h(t_{\pi(1)}))$, with respect to $(a_1,\dots,a_k)$.*

Proof. Let $f,g:F_\Sigma(k,n))\to A$ be the homomorphisms with $f(x_i)=g(x_i)=a_i$ for $1\leqslant i\leqslant k$ and $f(y_j)=b_j$, $g(y_j)=b_{\pi(j)}$. Then f and the composite gh coincide on all the generators of $F_\Sigma(k,n)$ and hence are equal. Thus, if $b_j=t_{jA}(a_1,\dots,a_k,b_1,\dots,b_n)$ for all $j\leqslant n$ then $b_j=f(t_j)=gh(t_j)=h(t_j)_A(a_1,\dots,a_k,b_{\pi(1)},\dots,b_{\pi(n)})$ for all $j\leqslant n$ and hence $b_{\pi(j)}=h(t_{\pi(j)})_A(a_1,\dots,a_k,b_{\pi(1)},\dots,b_{\pi(n)})$ for all $j\leqslant n$. The converse is also true, thus yielding the claim. $\square$

A similar, easy argument yields the following lemma, which will also be useful in the remainder of this section. It states formally the intuitively obvious fact that in dealing with iterative systems $T=(t_1,\ldots,t_n)$, if some t_j belongs to the subalgebra generated by $x_1,\ldots,x_k$, then the jth equation can be dropped to yield an equivalent, shorter, system.

Lemma 2. *Suppose $T=(t_1,\ldots,t_n)\in F_\Sigma(k,n)^n$ is an iterative system with $t_n\in F_\Sigma\{x_1,\ldots,x_k\}$. Let $h:F_\Sigma(k,n)\to F_\Sigma(k,n-1)$ be the homomorphism mapping all the x_i and all y_j with $j\leqslant n-1$ identically, with $h(y_n)=t_n$. Then for any Σ-algebra A and $(a_1,\ldots,a_k)\in A$, $(b_1,\ldots,b_{n-1})$ is a solution for $\bar{T}=(h(t),\ldots,h(t_{n-1}))$ with respect to $(a_1,\ldots,a_k)$ iff $(b_1,\ldots,b_{n-1},t_{nA}(a_1,\ldots,a_k))$ is a solution for T with respect to $(a_1,\ldots,a_k)$.*

For example, if $T=(y_1+y_2,x_1+x_2)$ then $\bar{T}=(y_1+(x_1+x_2))$; a solution to T in A with respect to (a_1,a_2) consists of $b_1,\ b_2\in A$ with $b_1=b_1+b_2$ and $b_2=a_1+a_2$, which is clearly equivalent to having $b_1=b_1+(a_1+a_2)$.

Recall that the algebra 2_Σ was defined in Example 3 in Section 2.2.

Proposition 1. *For an iterative system T the following are equivalent:*

(1) *T is uniquely solvable in 2_Σ;*

(2) *T is degenerate, i.e., uniquely solvable in every Σ-algebra;*

(3) *T is non-trivial and solvable (not necessarily uniquely) in every Σ-algebra.*

Proof. Let $T=(t_1,\ldots,t_n)\in F_\Sigma(k,n)^n$.

Note that if $t_j\in F_\Sigma\{x_1,\ldots,x_n\}$ for all $j\leqslant n$ then T is degenerate; in this case, for any algebra A and $a_1,\ldots,a_k\in A$, $b_j=t_j(a_1,\ldots,a_k)$ provide the required unique solutions.

(1)→(3). Assume T is uniquely solvable in 2_Σ. First we prove that T is solvable in every Σ-algebra. By Lemmas 1 and 2 we may assume $t_j\notin F_\Sigma\{x_1,\ldots,x_k\}$ for all $j\leqslant n$. But now, let $h:F_\Sigma(k,n)\to 2_\Sigma$ be the homomorphism with $h(x_i)=1$ for all $i\leqslant k$ and $h(y_j)=0$ for all $j\leqslant n$; it is easy to check using the definition of the operations in 2_Σ that $F_\Sigma\{x_1,\ldots,x_k\}\cup\{t\in F_\Sigma(k,n)\mid h(t)=0\}$ is closed under all the operations and hence, becuase it contains all the generators, must equal $F_\Sigma(k,n)$. From this it follows that $h(t_j)=0$ for all $j\leqslant n$ and hence $b_1=b_2=\cdots=b_n=0$ provides a solution for the equations $b_i=t_i(1,1,\ldots,b_1,\ldots,b_n)$. Since $b_1=b_2=\cdots=b_n=1$ also provides a solution, this means that $n=0$, i.e. that T is degenerate, and thus solvable in every Σ-algebra.

Next, suppose T is trivial. Then there exists $j_1,j_2,\ldots,j_m\in\{1,2,\ldots,n\}$ such that $t_{j_1}=y_{j_2},\ t_{j_2}=y_{j_3},\ldots,t_{j_m}=y_{j_1}$. Define sets $U_0\subseteq U_1\subseteq\cdots\subseteq\{1,2,\ldots,n\}$ as follows:

$$U_0=\{j_1,j_2,\ldots,j_m\}.$$

$j\in U_{p+1}$ iff $h_p(t_j)=0$ where $h_p:F_\Sigma(k,n)\to 2_\Sigma$ is the homomorphism with $h(x_i)=1$ for all $i\leqslant k$, $h_p(y_j)=0$ if $j\in U_p$, $h_p(y_j)=1$ otherwise.

Note that $U_0\subseteq U_1$ by the above identities, and $U_p\subseteq U_{p+1}$ implies $U_{p+1}\subseteq U_{p+2}$ because of the way the operations are defined in 2_Σ. Thus for some p, $U_p=U_{p+1}$.

But then, for this p, $h_p(t_j)=0$ implies $j\in U_{p+1}=U_p$ which implies $h_p(y_j)=0$, and conversely $h_p(y_j)=0$ implies $j\in U_p=U_{p+1}$ which implies $h_p(t_j)=0$. Thus, for all $j\leq n$ we have $h_p(t_j)=h_p(y_j)$ and conseequently $(h_p(y_1),\ldots,h_p(y_n))$ is a solution for T with respect to $a_1=a_2=\cdots=a_k=1$. Since $h_p(y_j)=0$ for all $j\in U$ this shows that T is not uniquely solvable in 2_Σ. Thus if T is uniquely solvable in 2_Σ it follows that T is solvable in every Σ-algebra and T is non-trivial.

(3)→(2). Suppose $T=(t_1,\ldots,t_n)\in F_\Sigma(k,n)^n$ is non-trivial, non-degenerate, and solvable in every Σ-algebra A. Using Lemmas 1 and 2, we may assume without loss of generality that no $t_j\in F_\Sigma\{x_1,\ldots,x_k\}$, and since T is non-degenerate, $n\geq 1$. Since T is solvable in every algebra, T is solvable in $F_\Sigma\{x_1,\ldots,x_k\}$, and hence there exists $b_1,\ldots,b_n\in F_\Sigma\{x_1,\ldots,x_k\}$ such that $b_j=t_j(x_1,\ldots,x_k,b_1,\ldots,b_n)$ for all $j\leq n$. Let j_1 have the property that b_{j_1} has minimum complexity among $b_1,b_2,\ldots,b_n$. Since $b_{j_1}=t_{j_1}(x_1,\ldots,x_k,b_1,\ldots,b_n)$ and $t_{j_1}\notin F_\Sigma\{x_1,\ldots,x_k\}$, it follows that t_j must really depend on one of its last n arguments, say j_2, and then because b_{j_1} has minimum complexity it follows that $t_{j_1}=y_{j_2}$ and hence $b_{j_1}=b_{j_2}$. But now b_{j_2} has minimum complexity too, and the argument can be repeated. Eventually we obtain a cycle $t_{j_1}=y_{j_2}$, $t_{j_2}=y_{j_3},\ldots,t_{j_m}=y_{j_1}$, which implies that T is trivial, a contradiction.

(2)→(1). This is trivial. □

Corollary. *There is an effective procedure to determine, given an iterative system T, whether or not T is degenerate.*

Proof. There are only finitely many cases to check to see whether T is uniquely solvable in 2_Σ.

Proposition 2. *There is an effective procedure which, given a non-trivial, non-degenerate iterative system $T=(t_1,\ldots,t_n)\in F_\Sigma(k,n)^n$, produces an ideal iterative system $\bar{T}=(\bar{t}_1,\ldots,\bar{t}_m)\in F_\Sigma(k,m)^m$ for some $m\leq n$ such that, for every Σ-algebra A and $a_1,\ldots,a_k\in A$, T is uniquely solvable in A with respect to $(a_1,\ldots,a_k)$ iff $\bar{T}$ is.*

Proof. Two procedures will be described which, when applied alternatively, will eventually produce the desired result.

Suppose $T=(t_1,\ldots,t_n)\in F_\Sigma(k,n)^n$ is non-trivial and non-degenerate.

Procedure 1. Apply when some $t_j\in\{x_1,\ldots,x_k\}$.

In this case $n\neq 1$ since otherwise T would be degenerate. Thus we may apply Lemmas 1 and 2 to get an iterative system $\bar{T}$ with one fewer component than T, whose solvability is equivalent to that of T.

Procedure 2. Apply when some $t_j\in\{y_1,\ldots,y_n\}$.

By Lemma 1, we may assume $j=n$, so that $t_n = y_p$ for $p \leq n-1$. Now let $h: F_\Sigma(k,n) \to F_\Sigma(k,n-1)$ be the homorphism with $h(x_i) = x_i$ for $i \leq k$, $h(y_j) = y_j$ for $j \leq n-1$ and $h(y_n) = y_p$. Then for any algebra A and $a_1, \ldots, a_k, b_1, \ldots, b_{n-1} \in A$, $(b_1, \ldots, b_{n-1})$ is a solution for $T' = (h(t_1), \ldots, h(t_{n-1}))$ with respect to $(a_1, \ldots, a_k)$ iff $(b_1, \ldots, b_{n-1}, b_p)$ is a solution for T with respect to $(a_1, \ldots, a_k)$. In particular T' is non-degenerate.

Now simply apply procedures 1 and 2 until neither apply any more to yield the desired $\bar{T}$. □

Corollary. *For any Σ-algebra A, if every ideal iterative system is uniqely solvable in A then every non-trivial iterative system is uniquely solvable in A.*

Remark. The proof of Proposition 3 actually produces more than is claimed: given $T = (t_1, \ldots, t_n)$ the procedure produces $\bar{T} = (\bar{t}_1, \ldots, \bar{t}_m)$, distinct $j_1, \ldots, j_m \leq n$ and for each $j \notin \{j_1, \ldots, j_m\}$ a polynomial $s_j \in F_\Sigma(k,m)$ (which always has complexity zero) such that for any algebra A and $a_1, \ldots, a_k \in A$:

(i) If $(b_1, \ldots, b_m)$ is a solution for $\bar{T}$ with respect to $(a_1, \ldots, a_k)$ then $(c_1, \ldots, c_n)$ is a solution for T with respect to $(a_1, \ldots, a_k)$ where $c_{j_i} = b_i$ for $i \leq m$ and $c_j = s_j(a, \ldots, a_k, b_1, \ldots, b_m)$ for $j \notin \{j_1, \ldots, j_m\}$, and

(ii) if $(c_1, \ldots, c_n)$ provides a solution for T with respect to $(a_1, \ldots, a_k)$ then $(c_{j_1}, \ldots, c_{j_m})$ is a solution for $\bar{T}$ with respect to $(a_1, \ldots, a_k)$ and moreover $c_j = s_j(a_1, \ldots, a_k, c_{j_1}, \ldots, c_{j_m})$ for $j \notin \{j_1, \ldots, j_m\}$.

This will be used in Proposition 5.

Proposition 3. *There is an effective procedure which, given an ideal $T = (t_1, \ldots, t_n) \in F_\Sigma(k,n)^n$, produces $\bar{T} = (\bar{t}_1, \ldots, \bar{t}_m) \in F_\Sigma(k,m)^m$ for some $m \geq n$ and polynomials $s_1, \ldots, s_{m-n} \in F_\Sigma(k,n)$ such that each $\bar{t}_i$ has complexity one, and for every Σ-algebra A and $a_1, \ldots, a_k \in A$,*

(i) If $(b_1, \ldots, b_m)$ is a solution for T with respect to $(a_1, \ldots, a_k)$ then $(b_1, \ldots, b_n, s_{1A}(a_1, \ldots, a_k, b_1, \ldots, b_n), \ldots, s_{(m-n)A}(a_1, \ldots, a_k, b_1, \ldots, b_n))$ is a solution for $\bar{T}$ with respect to $(a_1, \ldots, a_k)$, and conversely,

(ii) If $(b_1, \ldots, b_m)$ is a solution for $\bar{T}$ with respect to $(a_1, \ldots, a_k)$ then $(b_1, \ldots, b_n)$ is solution for T with respect to $(a_1, \ldots, a_k)$ and in addition $b_{n+i} = s_{iA}(a_1, \ldots, a_k, b_1, \ldots, b_n)$ for all $i \leq m$.

Proof. For any ideal $T = (t_1, \ldots, t_n) \in F_\Sigma(k,n)^n$, define the overflow of T by

$$\mathrm{OV}(T) = \left(\sum_{j=1}^{n} \text{complexity of } t_j \right) - n.$$

We will construct $\bar{T}$ by induction on $\mathrm{OV}(T)$

If $\mathrm{OV}(T) = 0$ then take $\bar{T} = T$.

Assume now that $\mathrm{OV}(T) \geqslant 1$. Then for some $j \leqslant n$, the complexity of t_j is at least 2, and so $t_j = \sigma(s_1, \ldots, s_{|\sigma|})$ for some $\sigma \in \Sigma$ and $s_i \in F_\Sigma(k, n)$, and for some $p \leqslant |\sigma|$, s_p has complexity $\geqslant 1$.

Define $\bar{t}_i \in F_\Sigma(k, n+1)$ as follows:

$$\bar{t}_i = t_i \quad \text{for } j \neq i \leqslant n,$$

$$\bar{t}_j = \sigma(s_1, \ldots, s_{p-1}, y_{n+1}, s_{p+1}, \ldots, s_{|\sigma|}),$$

$$\vdots$$

$$\bar{t}_{n+1} = s_p.$$

Then the total complexity of $\bar{T} = (\bar{t}_1, \ldots, \bar{t}_{n+1})$ is the same as that of T, and hence $\mathrm{OV}(\bar{T}) = \mathrm{OV}(T) - 1$.

Now, let A be any Σ-algebra and suppose that $(b_1, \ldots, b_{n+1})$ is a solution for $\bar{T}$ with respect to $(a_1, \ldots, a_k)$. Then

$$\begin{aligned} b_{n+1} &= \bar{t}_{(n+1)A}(a_1, \ldots, a_k, b_1, \ldots, b_{n+1}) \\ &= s_{pA}(a_1, \ldots, a_k, b_1, \ldots, b_n) \end{aligned}$$

and so

$$\begin{aligned} b_j &= \sigma_A(s_{1A}(a_1, \ldots, a_k, b_1, \ldots, b_n), \ldots, b_{n+1}, \ldots, \\ &\qquad s_{|\sigma|A}(a_1, \ldots, a_k, b_1, \ldots, b_n)) \\ &= \sigma_A(s_{1A}(a_1, \ldots, a_k, b_1, \ldots, b_n), \ldots, s_{pA}(a_1, \ldots, a_k, b_1, \ldots, b_n), \ldots, \\ &\qquad s_{|\sigma|A}(a_1, \ldots, a_k, b_1, \ldots, b_n)) \\ &= t_{jA}(a_1, \ldots, a_k, b_1, \ldots, b_n). \end{aligned}$$

Also for $j \neq i \leqslant n$

$$b_i = \bar{t}_{iA}(a_1, \ldots, a_k, b_1, \ldots, b_{n+1}) = t_{iA}(a_1, \ldots a_k, b_1, \ldots, b_n).$$

Thus $(b_1, \ldots, b_n)$ is a solution for T with respect to $(a_1, \ldots, a_k)$.

Conversely, if $(b_1, \ldots, b_n)$ is a solution for T with respect to $(a_1, \ldots, a_k)$ then $(b_1, \ldots, b_n, s_{pA}(a_1, \ldots, a_k, b_1, \ldots, b_n))$ is a solution for $\bar{T}$ with respect to $(a_1, \ldots, a_k)$.

Now if the above process is repeated, after finitely many steps we obtain a $\bar{T}$ with overflow equal to zero, as required. $\square$

Corollary. *An algebra of type Σ is iterative iff it is C_1-iterative.*

Remark. The process, outlined in Section 2.3, of adding operations to provide solutions of iterative systems can of course be repeated. However, if we add solutions for *all* non-trivial iterative systems of type Σ, then in any iterative Σ-algebra we already have unique solutions to all non-trivial iterative systems of the expanded type, and moreover these solutions are provided by polynomials of that type. This fact is explicitly laid out in the proposition below; first an example.

Example. Suppose Σ consists of one binary operation $+$, and $\bar{\Sigma} \supseteq \Sigma$ is the type obtained by adding to Σ, for each non-trivial iterative system $T \in F_\Sigma(k, n)^n$, n new operations $T_1, \ldots, T_n$ which provide the solutions to T in every iterative Σ-algebra. Consider the iterative system $T = (t_1, t_2) = (x_1 + y_2, x_2 + y_1)$. The $\bar{\Sigma}$ contains binary operations T_1 and T_2 such that in every iterative algebra A of type Σ, for all a_1, $a_2 \in A$,

$$T_{1A}(a_1, a_2) = a_1 + T_{2A}(a_1, a_2),$$

$$T_{2A}(a_1, a_2) = a_2 + T_{1A}(a_1, a_2).$$

Now consider the $\bar{\Sigma}$-iterative system $T_1(x_1, y_1) \in F_{\bar{\Sigma}}(1, 1)$. Suppose A is an iterative algebra of type Σ and $a_1 \in A$. If the latter system is solvable in A then there is an element $b_1 \in A$ with $b_1 = T_{1A}(a_1, b_1)$, and consequently we have

$$b_1 = T_{1A}(a_1, b_1) = a_1 + T_{2A}(a_1, b_1),$$

$$T_{2A}(a_1, b_1) = b_1 + T_{1A}(a_1, b_1) = b_1 + b_1.$$

Thus $(b_1, T_{2A}(a_1, b_1))$ provides a solution to the iterative system $S = (x_1 + y_2, y_1 + y_1)$ with respect to a_1. Conversely, if (b_1, b_2) is a solution to S with respect to a_1, then $b_1 = a_1 + b_2$, $b_2 = b_1 + b_1$ and hence $b_1 = T_1(a_1, b_1)$.

Now, in $\bar{\Sigma}$ there are unary operations S_1 and S_2 providing the unique solutions of S in every iterative Σ-algebra, and we see from the above that $b_1 = S_{1A}(a_1)$. Thus the unique solution to the iterative $\bar{\Sigma}$-system $y_1 = T_1(x_1, y_1)$ is given, in any iterative Σ-algebra, by $y_1 = S_1(x_1)$.

Proposition 4. *Let $\bar{\Sigma}$ be the type obtained from Σ by adding operations to give solutions for all non-trivial systems of type Σ. Then for any non-trivial iterative system $T \in F_{\bar{\Sigma}}(k, n)^n$ there exist $\bar{\Sigma}$-polynomials $u_1, \ldots, u_n \in F_{\bar{\Sigma}}\{x_1, \ldots, x_k\}$ such that, for any iterative Σ-algebra A and any $a_1, \ldots, a_k \in A$, $(u_1(a_1, \ldots, a_k), \ldots, u_n(a_1, \ldots, a_k))$ is the unique solution to T with respect to $(a_1, \ldots, a_k)$.*

Proof. Let $T = (t_1, \ldots, t_n)$.

If T is degenerate then it is uniquely solvable in $F_{\bar{\Sigma}}\{x_1, \ldots, x_k\}$ and hence there exist $u_1, \ldots, u_n \in F_{\bar{\Sigma}}\{x_1, \ldots, x_k\}$ with

$$u_i(x_1, \ldots, x_k) = t_i(x_1, \ldots, x_k, u_1(x_1, \ldots, x_k), \ldots, u_n(x_1, \ldots, x_k))$$

and then these are the required polynomials.

Assume T is non-degenerate. It follows from Proposition 4 and the remark following Proposition 3, applied to $\bar{\Sigma}$ rather than Σ, that it is enough to prove the claim in the case that all t_i have complexity 1.

Define the 'overcomplexity' $\mathrm{OC}(T)$ of T to be the total number of operations appearing in T which are not Σ-operations. We prove the claim by induction on $\mathrm{OC}(T)$.

If $\mathrm{OC}(T)=0$ then $t_j \in F_\Sigma(k,n)$ for each j, and the result follows from the definiton of $\bar{\Sigma}$.

Suppose $t_n \notin F_\Sigma(k,n)$; then t_n involves one of the new $\bar{\Sigma}$-operations, and hence there exists a non-trivial iterative system $T^{\#}=(t_1^{\#},\dots,t_m^{\#}) \in F_\Sigma(p,m)^m$ such that $t_n = T_j^{\#}(z_1,\dots,z_p)$ for some $j \leq m$ and some $z_1,\dots,z_p \in \{x_1,\dots,x_k,y_1,\dots,y_n\}$. By suitably permuting the equations in $T^{\#}$ we may assume $j=1$, so $t_n = T_1^{\#}(z_1,\dots,z_p)$. Let $h: F_\Sigma(p,m) \to F_\Sigma(k,n-1+m)$ be the homomorphism with $h(x_i)=z_i$ for $i \leq p$ and $h(y_j)=y_{n-1+j}$ for $j \leq m$. Consider the system $S=(t_1,\dots,t_{n-1},h(t_1^{\#}),\dots,h(t_m^{\#}))$, whose solutions are of the form

$$
\begin{aligned}
y_1 &= t_1(x_1,\dots,x_k,y_1,\dots,y_n)\\
&\ \ \vdots\\
y_{n-1} &= t_{n-1}(x_1,\dots,x_k,y_1,\dots,y_n)\\
y_n &= h(t_1^{\#})(x_1,\dots,x_k,y_1,\dots,y_n)\\
&= t_1^{\#}(z_1,\dots,z_p,y_n,y_{n+1},\dots,y_{n-1+m})\\
&\ \ \vdots\\
y_m &= h(t_m^{\#})(x_1,\dots,x_k,y_1,\dots,y_n)\\
&= t_m^{\#}(z_1,\dots,z_p,y_n,y_{n+1},\dots,y_{n-1+m}).
\end{aligned}
$$

Note that $\mathrm{OC}(S)=\mathrm{OC}(T)-1$. Also, in an iterative Σ-algebra A, if $(b_1,\dots,b_{n-1+m})$ is a solution for S with respect to $(a_1,\dots,a_k)$ then $(b_1,\dots,b_n)$ is a solution for T with respect to $(a_1,\dots,a_k)$. Now we may use the techniques of Propositions 3 and 4 to produce a $\bar{\Sigma}$-system equivalent to S, with the same overcomplexity, each of whose components has complexity 1; the inductive hypothesis may now be applied to this system, yielding the desired result. □

It follows from this result that $\bar{\Sigma}$ has the kind of homogeneity found in the 'iterative theories' of Elgot [4]: everything has iterative solutions within the system. It is precisely this homogeneity which allows Bloom, Ginali and Rutledge [3] to prove that "scalar iteration implies vector iteration". The analogous result for algebras would be that every L_1-iterative algebra is iterative. If Proposition 5 were still true when $\bar{\Sigma}$ is replaced by Σ_{L_1} then the proof in [3] would carry over. However, this analogue is not true: we will see in the next section that there are L_1-iterative algebras which are not iterative when Σ consists of a single binary operation. The farthest one can go in this direction is the following.

Proposition 5. *If Σ consists exclusively of unary operations then every L_1-iterative Σ-algebra is iterative.*

Proof. Assume A is L_1-iterative, and let $T=(t_1,\ldots,t_n)\in F_\Sigma(k,n)^n$ be a non-degenerate iterative system where each t_j has complexity 1. By Lemmas 1 and 2 we may assume $t_n\notin F_\Sigma\{x_1,\ldots,x_k\}$. Thus $t_n=\sigma(y_j)$ for some $j\leq n$, and some $\sigma\in\Sigma$.

If $j\leq n-1$ then we may use Procedure 2 of Proposition 3 to reduce T to a shorter, equivalent system.

If $j=n$, then $t_n=\sigma(y_n)$. Now, A is L_1-iterative and hence the equation $y=\sigma(y)$ has a unique solution $b\in A$. Let $h:F_\Sigma(k,n)\to F_\Sigma(k+1,n-1)$ be the homomorphism with $h(x_i)=x_i$ for $i\leq n$, $h(y_j)=y_j$ for $j\leq n-1$ and $h(y_n)=x_{k+1}$ and let $\bar{T}=(h(t_1),\ldots,h(t_{n-1}))$. Then $(b_1,\ldots,b_n)$ is a solution for T with respect to $(a_1,\ldots,a_k)$ iff $b_n=b$ and $(b_1,\ldots,b_{n-1})$ is a solution for $\bar{T}$ with respect to $(a_1,\ldots,a_k,b)$.

Thus we reduce T to an equivalent system of length 1 which is uniquely solvable by the L_1-iterativeness of A. □

3. Tree constructions of free iterative algebras

In the preceding section, standard results from universal algebra were used to prove the existence of free C-iterative algebras. Here we will give concrete descriptions of free C-iterative algebras, for certain C, as algebras of trees.

3.1. Regular trees

For a type Σ, the algebra $T_\Sigma V$ of all V-labelled Σ-trees was defined in Section 2.1; here we will deal with those trees in $T_\Sigma V$ that have essentially only finitely many subtrees.

For a tree $t\in T_\Sigma V$ and $u\in\mathrm{dom}(t)$, define $t|u$, *the subtree of t at u*, by

$$(t|u)(v)=t(uv)$$

Here, for sequences $u,v\in\omega^*$, uv is the concatenation product of u and v.

A tree t is called *regular* iff $\{t|u \mid u\in\mathrm{dom}(t)\}$ is finite. Alternatively, we may define an equivalence relation on dom(t) by: $u\sim v$ iff $t|u=t|v$ (iff $t(ux)=t(vx)$ for all $x\in\omega^*$); then t is regular iff the equivalence relation $\sim$ has finite index. The regular trees are exactly the same as the regular = algebraic trees of Ginali [5], but the above definition is simpler and easier to work with.

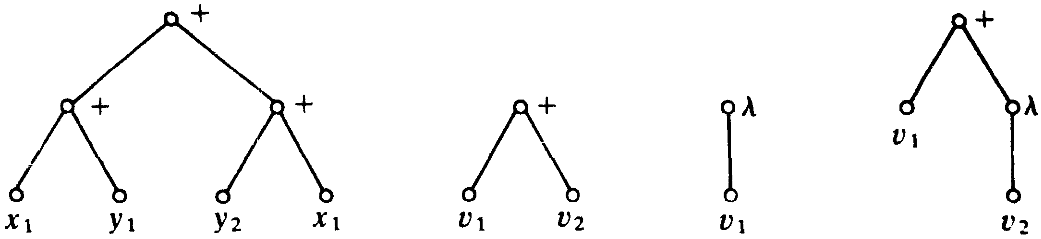
Let $R_\Sigma V$ be the set of all regular trees in $T_\Sigma V$; then $R_\Sigma V$ is clearly closed under all the Σ-operations as defined in $T_\Sigma V$, and contains $F_\Sigma V$. We are going to prove that $R_\Sigma V$ is the free iterative Σ-algebra over V.

To keep notation to a minimum, we will write, for each polynomial $t\in F_\Sigma(k,n)$, simply 't' instead of '$t_{R_\Sigma V}$' for the polynomial function induced by t in $R_\Sigma V$. Note that, for each $t\in F_\Sigma(k,n)$ and $r_1,\ldots,r_k,s_1,\ldots,s_n\in R_\Sigma V$, we have

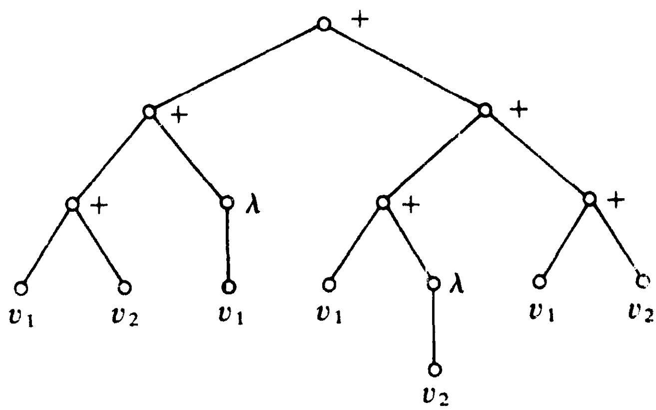
$$(*)\qquad t(r_1,\ldots,r_k,s_1,\ldots,s_n)(u)=\begin{cases}t(u) & \text{if } t(u)\in\Sigma,\\ r_i(v) & \text{if } u=wv \text{ and } t(w)=x_i,\\ s_j(v) & \text{if } u=wv \text{ and } t(w)=y_j.\end{cases}$$

Note that if $t(w)=x_i$ or y_j for some w such that $u=wv$ then w is uniquely determined, and is a 'leaf' in $\mathrm{dom}(t)$. The proof of the above equality is a straightforward induction on the complexity of t.

Example. Suppose Σ has a binary operation $+$ and one unary operation λ.



$t(x_1, y_1, y_2) = (x_1+y_1)+(y_2+x_1)$ $\quad r_1 = v_1+v_2$ $\quad s_1 = \lambda(v_1)$ $\quad s_2 = v_1+\lambda(v_2)$



$t(r_1, s_1, s_2)$

Proposition 6. *$R_\Sigma V$ is an iterative Σ-algebra.*

Proof. By Propositions 3 and 4 it is enough to prove that every non-degenerate iterative system in C_1 is uniquely solvable in $R_\Sigma V$.

Let $T=(t_1, \ldots, t_n) \in F_\Sigma(k,n)^n$ where each t_j has complexity 1; then for each j, $t_j(\emptyset)=\sigma_j \in \Sigma$. Let $r_1, \ldots, r_k \in R_\Sigma V$.

Define $s_j \in R_\Sigma V$ for $1 \leqslant j \leqslant n$, specifying $s_j(u)$ by induction on the length of u, as follows:

$$s_j(\emptyset) = \sigma_j,$$

$$s_j(mu) = \begin{cases} r_j(u) & \text{if } m \leqslant |\sigma_j| \text{ and } t_j(m) = x_i, \\ s_p(u) & \text{if } m \leqslant |\sigma_j| \text{ and } t_j(m) = y_p. \end{cases}$$

Then for all $j \leqslant n$, each subtree of s_j is either a subtree of some r_i $(i \leqslant k)$ or is equal to some s_p, and hence each s_j is regular.

Moreover, by the identities $(*)$ preceding the proposition, we have $s_j = t_j(r_1, \ldots, r_k, s_1, \ldots, s_n)$ for each $j \leqslant n$. Thus $(s_1, \ldots, s_n)$ is a solution to T with respect to $(r_1, \ldots, r_k)$ in $R_\Sigma V$. Moreover, a straightforward induction using the identities (*) to calculate $t_j(r_1, \ldots, r_k, s_1, \ldots, s_n)(u)$ for $u \in \omega^*$ show that the s_j are uniquely determined. □

Theorem 2. *$R_\Sigma V$ is the free iterative Σ-algebra over V, for any set V.*

Proof. Let $f: V \to B$ be any function into an iterative Σ-algebra B; we must show that f extends uniquely to a homomorphism $\bar{f}: R_\Sigma V \to B$.

Let $t \in R_\Sigma V$; then there exist $v_1, \ldots, v_k$ with $t \in R_\Sigma\{v_1, \ldots, v_k\}$. We may assume $t \notin V$. Let $u_1, \ldots, u_n$ be a set of respresentatives for the non-trivial subtrees of t (i.e. those with complexity ≥ 1), so that for all $u \in \operatorname{dom}(t)$ either $t|u = t|u_j$ for some $j \leq n$ or $t(u) = v_i$ for some $i \leq k$. For each $j \leq n$, let $\sigma_j = t(u_j) \in \Sigma$.

Define $t_1, \ldots, t_n \in F_\Sigma(k, n)$ as follows:

$$t_j(\emptyset) = \sigma j,$$

$$t_j(m) = \begin{cases} y_p & \text{if } t|u_j m = t|u_p, \\ x_i & \text{if } t(u_j m) = v_i \end{cases} \quad \text{for all } m \leq |\sigma_j|.$$

Note that if we choose $u'_1, \ldots, u'_n \in \omega^*$ such that $t|u_j = t|u'_j$ for each $j \leq n$, and substitute u'_j in the above definition of t_j we get exactly the same tree t_j.

In B, there exist (unique) $b_1, \ldots, b_n$ such that $b_j = t_j(f(v_1), \ldots, f(v_k), b_1, \ldots, b_n)$.

Define $\bar{f}(t) = b_p$ where p is the unique number $\leq n$ with $t = t|u_p$.

It may seem that this definition depends on the order chosen for the representatives $u_1, \ldots, u_n$ of the different subtrees of t. However, it follows from Lemma 2 that even if we order them differently, since this only amounts to permuting the equations, we still end up with the same element of B for $\bar{f}(t)$, and hence $\bar{f}: R_\Sigma V \to B$ is well-defined.

Moreover, if $\bar{f}$ is a homomorphism, then the following discussion shows that it is unique: For each $j \leq n$ define $s_j = t|u_j \in R_\Sigma\{v_1, \ldots, v_k\}$; then $t = s_p$. Moreover, for each $j \leq n$, $s_j = t_j(v_1, \ldots, v_k, s_1, \ldots, s_n)$; this follows from the identities (*). Thus $(s_1, \ldots, s_n)$ is the solution for $T = (t_1, \ldots, t_n)$ with respect to $(v_1, \ldots, v_k)$ in $R_\Sigma V$. Since homomorphisms necessarily preserve such solutions, $\bar{f}(t) = \bar{f}(s_p)$ *must* be b_p.

Thus all that remains is to verify that $\bar{f}$ is a homomorphism.

Suppose $t = \sigma(r_1, \ldots, r_{|\sigma|})$ for some $\sigma \in \Sigma$ and $r_i \in R_\Sigma V$. Let $u_1, \ldots, u_n$ be a set of representatives for the non-trivial subtrees of t, and $T = (t_1, \ldots, t_n) \in F_\Sigma(k, n)^n$ the corresponding iterative systems as defined above. Let $b_1, \ldots, b_n$ be the solution in B for T with respect to $f(v_1), \ldots, f(v_k)$, so that $\bar{f}(t) = b_p$ for the $p \leq n$ with $t = t|u_p$.

Since r_1 is a subtree of t, we known that among the u_i are a set of representatives for the non-trivial subtrees of r_1. If r_1 is itself nontrivial, then we may assume without loss of generality that $u_j = 1w_j$ for $j \leq m$ and that $w_1, \ldots, w_m \in \operatorname{dom}(r_1)$ are a set of representatives for the non-trivial subtrees of r_1. Let $\bar{T} = (\bar{t}_1, \ldots, \bar{t}_m)$ be the iterative system produced as above from r_1 and these $w_1, \ldots, w_m$. Since $r_1(w) = t(1w)$ for all $w \in \omega^*$, it follows that $\bar{t}_j = t_j$ for all $j \leq m$ and hence $(b_1, \ldots, b_m)$ is a solution for $\bar{T}$ with respect to $(f(v_1), \ldots, f(v_k))$ in B.

All together, this shows that if r_1 is non-trivial then $\bar{f}(r_1)=b_{p_1}$ for the unique $p_1 \leq n$ with $r_1 = t|u_{p_1}$. Of course if r_1 is trivial then $r_1 = v_i$ for some i and then $\bar{f}(r_1)=f(v_i)$.

Similarly, for those other r_m that are non-trivial, $\bar{f}(r_m)=b_{p_m}$ for the unique $p_m \leq n$ with $r_m = t|u_{p_m}$.

Now suppose $t = t|u_p$. Then $t_p(\emptyset)=\sigma$, and for each $m \leq |\sigma|$, $t|u_p m = r_m$. Thus $t_p = \sigma(z_1, \ldots, z_{|\sigma|})$ where for $m \leq |\sigma|$,

$$z_m = \begin{cases} y_{p_m} & \text{if } r_m = t|u_p, \\ x_i & \text{if } r_m = v_i. \end{cases}$$

Since $(b_1, \ldots, b_n)$ is a solution for T in B with respect to $f(v_1), \ldots, f(v_k)$ it follows that $b_p = \sigma(c_1, \ldots, c_{|\sigma|})$ where

$$c_m = \begin{cases} b_{p_m} & \text{if } z_m = y_{p_m}, \\ f(v_i) & \text{if } z_m = x_i. \end{cases}$$

But by the above remarks, $c_m = \bar{f}(r_m)$ for each $m \leq |\sigma|$ and hence $\bar{f}(t) = \sigma(\bar{f}(r_1), \ldots, \bar{f}(r_{|\sigma|}))$, as required. □

3.2. *Application to context-free grammars*

For certain sets C of iterative systems, the free C-iterative algebra can be described as a specific subalgebra of $R_\Sigma V$. In this section we will describe a collection CF of iterative systems that come from context-free grammars, and the appropriate trees making up the free CF-iterative algebra.

Consider a context-free grammar $G=(N, V, P, S)$ where $N=\{y_1, \ldots, y_n\}$ is the set of variables (non-terminals), V is the terminal alphabet, $S \in N$ is the 'start' or 'axiom' symbol, and $P \subseteq N \times (N \cup V)^+$ is the set of productions (rewrite rules). Here, $(N \cup V)^+$ is the set of non-empty strings of letters from $N \cup V$, so we do not allow erasing productions. G is called *cycle-free* if there is no G-derivation of the form

$$y_{i_1} \to y_{i_2} \to \cdots \to y_{i_m} \to y_{i_1}.$$

As is well known, every context-free grammar is effectively equivalent to one which is cycle-free.

With each such grammar G we associate iterative systems as follows. For each $u = v_1 v_2 \cdots v_k \in (N \cup V)^+$ let $\bar{u} = (\cdots((v_1 \times v_2) \times v_3) \times \cdots \times v_k)$. Then, for each choice of $j \leq n$, suppose $u_1, \ldots, u_m \in (N \cup V)^+$ are all sequences u with $(y_j, u) \in P$, and let

$$t_j = (\cdots(\bar{u}_1 + \bar{u}_2) + \cdots + \bar{u}_m).$$

Then $T=(t_1, \ldots, t_n)$ is an iterative system of type Σ where Σ consists of two binary operations $+$ and $\times$. Of course the definition of T depends on the order we chose for the u_i and so with each G we associate all iterative systems obtained as above.

Let CF be the set of all iterative systems obtained in this way from all cycle-free context-free grammars over the terminal alphabet $V=\{v_1,\ldots,v_k\}$.

Let PV^+ be the set of all subsets of V^+, furnished with the structure of a Σ-algebra by defining + as set union, and $\times$ by

$$S\times T=\{uv\,|\,u\in S, v\in T\}$$

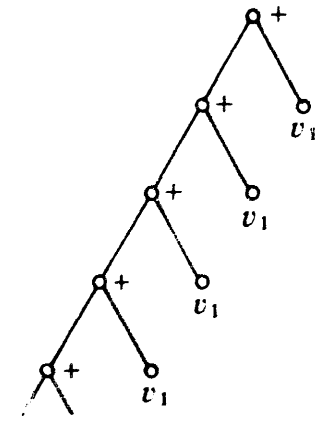
Then the following discussion shows that PV^+ is a CF-iterative algebra.

For $G=(N,V,P,S)$ with $N=\{y_1,\ldots,y_n\}$, define $T=(t_1,\ldots,t_n)$ as described above. For each $j\leq n$, let $U_j=\{u\in V^+\,|\,y_j\to_G^* u\}$, i.e. U_j consists of all words in the terminal alphabet that are derivable from y_j using the rewrite rules in G. Then $(U_1,\ldots,U_n)$ is a solution for T with respect to $(\{v_1\},\ldots,\{v_k\})$. Moreover this solution is unique: if $(U'_1,\ldots,U'_n)$ is any other solution, then an inductive argument based on the length of $u\in V^+$ and using the fact that G is cycle-free, shows that $u\in U'_j$ iff $u\in U_j$ for all $j\leq n$.

Next, we will see that a suitable subset of $R_\Sigma V$ forms the free CF-iterative algebra: let $G_\Sigma V\subseteq R_\Sigma V$ consist of all trees $t\in R_\Sigma V$ such that:

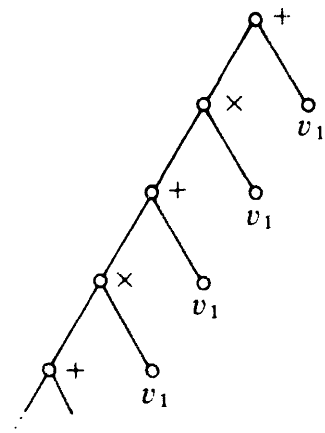
(#) for all $v\neq\emptyset$ and $u\in\omega^*$, if $t|u=t|uv$ then $t(uw)=\times$ for some prefix w of v.

Examples.



$t|\emptyset=t|1=t|11$

t is not in $G_\Sigma V$



$s|\emptyset=s|11=s|1111$

s is in $G_\Sigma V$.

Theorem 3. *$G_\Sigma V$ is the free CF-iterative algebra over V.*

Proof. It is clear that $G_\Sigma V$ is closed under the operation $\times$. Suppose $r,s\in G_\Sigma V$, $t=r+s$ and $t|u=t|uv$; we must show $t(uw)=\times$ for some prefix w of v. If $u\neq\emptyset$, say $u=1u'$, then $r|u'=t|u=t|uv=r|u'v$ and the result follows from the fact that $r\in G_\Sigma V$. If $u=\emptyset$ then we may assume without loss of generality that $v=1v'$, so $t=t|1v'=r|v'$, and hence $r=t|1=r|v'1$. Since $r\in G_\Sigma V$, there is a prefix w' of $v'1$ with $r(w')=\times$. If $w'=v'1$ then replace w' with $\emptyset$, so that w' is a prefix of v' and $r(w')=\times$. But then $w=1w'$ is a prefix of v, and $t(w)=r(w')=\times$, as required.

Next we will see that, for any $T=(t_1,\ldots,t_n)\in F_\Sigma(k,n)^n$ with $T\in$ CF and any $r_1,\ldots,r_k\in G_\Sigma V$, the solution $(s_1,\ldots,s_n)$ to T with respect to $(r_1,\ldots,r_k)$ in $R_\Sigma V$ actually consists of members of $G_\Sigma V$, which implies $G_\Sigma V$ is iterative.

Using either standard arguments from language theory, or the techniques of Propositions 2 and 3, we can verify that it is enough to prove the claim for $T=(t_1,\ldots,t_n)\in \mathrm{CF}$ with each t_j having complexity one (the reduction to such a system will not introduce cycles if the original grammar is cycle-free, and so will itself be cycle free). Now, let $r_1,\ldots,r_k\in G_\Sigma V$, and consider $s_1,\ldots,s_n$, as defined in Proposition 6, the solution to T in $R_\Sigma V$ with respect to $(r_1,\ldots,r_k)$.

Suppose for some $j\leqslant n$ and $u,v\in\omega^*$ that $s_j|u=s_j|uv$; we must show for some prefix v' of v that $s_j(uv')=\times$. Because the $r_i\in G_\Sigma V$ and the t_j have complexity 1, it is enough to deal with the case $u=\emptyset$.

Suppose $v=m_1m_2\cdots m_q\in\omega^+$, $j_1\leqslant n$ and $s_{j_1}=s_{j_1}|v$, but $s_{j_1}(m_1\cdots m_{q'})\neq\times$ for any $q'\leqslant q$. We may assume that $q\geqslant n+1$.

If $t_{j_1}(m_1)=x_i$ for some i then $s_{j_1}|m_1=r_i$ and hence $r_i=r_i|m_2\cdots m_qm_1$. Since $r_i\in G_\Sigma V$ this implies that $r_i(m_2\cdots m_{q'})=\times$ for some $q'\geqslant q$ and then $s_{j_1}(m_1,\ldots,m_{q'})=\times$, a contradiction.

Thus $t_{j_1}(m_1)=y_{j_2}$ for some $j_2\leqslant n$ and hence in the grammar G from which T was derived there is a production $y_{j_1}\to y_{j_2}$. Moreover we have $s_{j_1}|m_1=s_{j_2}$ and hence $s_{j_2}=s_{j_2}|m_2\cdots m_qm_1$. Using the argument in the preceding paragraph again, we see that $t_{j_2}(m_2)=y_{j_3}$ for some $j_3\leqslant n$ and hence G has a production $y_{j_2}\to y_{j_3}$. Proceeding in this way, because $q\leqslant n+1$, we eventually produce a cycle in G, which is a contradiction. $\square$

Remark. The image of $G_\Sigma V$, under the homomorphism to PV^+ which maps each $v\in V$ to $\{v\}$, is the set of all context-free languages over the alphabet V which do not contain $\emptyset$.

3.3. Irredundant iterative systems

In this section we consider algebras with unique solutions to systems $T=(t_1,\ldots,t_n)\in F_\Sigma(k,n)^n$ where each 'variable' y_j is linked to at least one of the 'constants' x_i. Such systems of equations arise when considering context-free grammars in which each 'variable' or 'non-terminal' has a derivation to a terminal word. Thus for example we do not insist on either the existence or uniqueness of solutions to systems such as

$$y=y+y$$

but for each element a of the algebra in question, there is a unique b with $b=b+a$.

More precisely, for an iterative system $T=(t_1,\ldots,t_n)$, define a binary relation $\to_T$ on $Z=\{x_1,\ldots,x_k,y_1,\ldots,y_n\}\cup\{\sigma\in\Sigma\mid|\sigma|=0\}$ as follows: $y_j\to_T z$ if $z=t_j(u)$ for some $u\in\omega^*$. Let $\to_T^*$ be the transitive closure of $\to_T$; then we say that T is irredundant iff for all $j\leqslant n$, $y_j\to_T^* z$ for some $z\in Z_0=\{x_1,\ldots x_n\}\cup\{\sigma\in\Sigma\mid|\sigma|=0\}$.

Let IR be the set of all irredundant iterative systems. We will describe the free IR-iterative algebra: let $I_\Sigma V\subseteq R_\Sigma V$ consist of those trees $t\in R_\Sigma V$ such that

for all $u\in\mathrm{dom}(t)$ there exists $v\in\omega^*$ with $t(uv)\in Z_0$.

Theorem 4. *$I_\Sigma V$ is the free IR-iterative algebra over V.*

Proof. The fact that $I_\Sigma V$ is closed under the Σ-operations is clear, and that it is IR-iterative comes from the construction of iterative solutions in $T_\Sigma X$ and the definition of IR. The fact that set maps from V into IR-iterative algebras extend uniquely to homomorphisms follows from the same arguments as in Theorem 2, the point being that, given a tree $t \in I_\Sigma V$, the corresponding iterative system constructed from it is in IR. □

3.4. L_1-iterative algebras

In this section we will describe a subalgebra of $R_\Sigma V$ which is L_1-iterative but not iterative, as promised in Section 2.4.

Let $V=\{v\}$ be a singleton, and define subsets $A_n \subseteq R_\Sigma V$ as follows:

$A_0 = F_\Sigma V$,

A_{n+1} is the subalgebra of $R_\Sigma V$ generated by the elements of A_n together with all solutions in $R_\Sigma V$ to single iterative equations with constants from A_n.

Then define $A = \bigcup A_n$ $(n \in \omega)$; then A is an L_1-iterative subalgebra of $R_\Sigma V$.

Suppose Σ consists of a single binary operation $+$. We will show that A is not iterative.

If A is iterative, then there exist $s_1, s_2, s_3 \in A$ with

$$s_1 = s_1 + s_2,$$

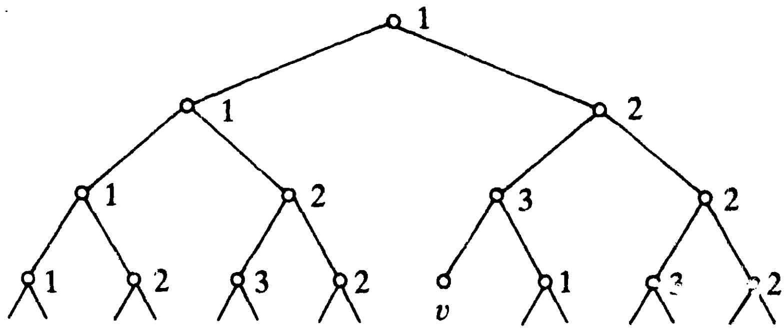
$$s_2 = s_3 + s_2,$$

$$s_3 = v + s_1.$$

Since A is a subalgebra of $R_\Sigma V$, and the latter has unique solutions to iterative systems, it follows that s_1, s_2 and s_3 are precisely the solution to this system in $R_\Sigma V$.

Let n be the smallest natural number such that one of s_1, s_2 or s_3 is a subtree of a tree in A_n. Since each of s_1, s_2, s_3 is a subtree of each of the others, it follows that they all occur for the first time in A_n.

Thus there exists $t \in F_\Sigma(k, 1)$ and $r_1, \ldots, r_k \in A_{n-1}$ with $s_1 = t(r_1, \ldots, r_k, s_1)$. Since each r_i is a subtree of s_1 and belongs to A_{n-1}, and since the only non-trivial subtrees of s_1 are s_1, s_2 and s_3, it follows that $r_i = v$ for all $i \leq k$, and hence $s_1 = t(v, v, \ldots, v, s_1)$. But then $\mathrm{dom}(t)$ is a subset of $\mathrm{dom}(s_1)$, and $u1 \in \mathrm{dom}(t)$ iff $u2 \in \mathrm{dom}\,t$, and for all $u \in \mathrm{dom}(t)$, either $s_1|u$ is trivial or $s_1|u = s_1$. However, s_1 is pictured below; the labels 1, 2, 3 indicate the nodes, which are all labelled $+$, whose subtree is s_1, s_2 or s_3 respectively, and it is easy to see that the above is impossible.



This leaves open the (fairly plausible) conjecture that A is actually the free L_1-iterative algebra on V.

4. Iterative theories

4.1. Existence of free iterative theories

In this section, we will see how the existence of free iterative theories, proved by Bloom and Elgot [2] and Ginali [5], follows from the existence of free iterative algebras. The explicit description of free iterative algebras given in the preceding chapter is not needed at all here; we will only use Theorem 1 and Propositions 2 and 4 of Section 2. However, Ginali's explicit description of the free iterative theory can also be accomplished using the arguments given here together with the results of Section 3.

The definitions of an algebraic theory and iterative algebraic theory can be found in Elgot [4], but are repeated here for the convenience of the reader. Briefly, an *algebraic theory* is a category $\boldsymbol{T}$, with countably many objects $[0], [1], [2], \ldots,$ in which $[n]$ is the nth copower of $[1]$. The coproduct injections $[1]\to[n]$ (called 'base' morphisms by Elgot) are ambiguously denoted $1, 2, \ldots, n$, and $[n]$ being the nth copower of $[1]$ means that for all k and morphisms $f_1, \ldots, f_n:[1]\to[k]$ there is a unique morphism $f_1f_2\sqcup\cdots\sqcup f_n:[n]\to[k]$ such that $(f_1\sqcup\cdots\sqcup f_n)\cdot j = f_j$ for all $j\leq n$. (Elgot writes $(f_1, \ldots, f_n)$ instead of $f_1\sqcup\cdots\sqcup f_n$.) A morphism $f:[m]\to[n]$ in $\boldsymbol{T}$ is called *ideal* iff for all $i\leq m, f\cdot i:[1]\to[n]$ is not a coproduct injection. An *ideal theory* is an algebraic theory in which the composite $g\cdot f$ is ideal whenever f is ideal. Finally, an *iterative algebraic theory* is an ideal theory $\boldsymbol{T}$ in which, for every ideal morphism $f:[n]\to[n+m]$ in $\boldsymbol{T}$ there is a unique morphism $f^{\#}:[n]\to[m]$ such that $f^{\#}$ is the composite

$$[n]\xrightarrow{f}[m+n]\xrightarrow{id_m\sqcup f^{\#}}[m]$$

(Here, id_m is the identity morphism on $[m]$.)

Now, suppose $\boldsymbol{T}$ is an algebraic theory, and that for each $\sigma\in\Sigma$ we are given a morphism $\bar{\sigma}:[1]\to[|\sigma|]$ in $\boldsymbol{T}$. For each natural number m, we make $\boldsymbol{Tm}$, the set of all $\boldsymbol{T}$-morphisms from $[1]$ to $[m]$, into a Σ-algebra in the obvious way, namely:

$$\text{for } f_1, \ldots, f_{|\sigma|}\in \boldsymbol{Tm},\quad \sigma_{\boldsymbol{Tm}}(f_1, \ldots, f_{|\sigma|}) = (f_1\sqcup f_2\sqcup\cdots\sqcup f_{|\sigma|})\cdot\bar{\sigma}.$$

For each natural number m, there is a unique Σ-homomorphism from $F_\Sigma\{v_1, \ldots, v_m\}$ to Tm which maps v_i to the ith coproduct injection $[1] \to [m]$; for each $t \in F_\Sigma\{v_1, \ldots, v_m\}$ let $\bar{t} \in T$ be the image of t under this homomorphism. A simple inductive argument on the complexity t then shows that for all $t \in F_\Sigma\{v_1, \ldots, v_n\}$ and all $f_1, \ldots, f_n \in Tm$

$$t_{Tm}(f_1, \ldots, f_n) = (f_1 \sqcup \cdots \sqcup f_n) \cdot \bar{t}.$$

Proposition 7. *If T is an iterative algebraic theory and if for each $\sigma \in \Sigma$, $\bar{\sigma} : [1] \to [|\sigma|]$ is an ideal morphism in Tm, then Tm is an iterative Σ-algebra for all natural numbers m.*

Proof. If T is an iterative algebraic theory then it is ideal, and then an inductive argument based on the complexity of t shows that if $\bar{\sigma}$ is ideal for each $\sigma \in \Sigma$ then $\bar{t}$ is ideal for all $t \in F_\Sigma\{v_1, \ldots, v_n\}$ of complexity ≥ 1.

Let $T = (t_1, \ldots, t_n) \in F_\Sigma(k, n)^n$ be an ideal iterative system, and let $f_1, \ldots, f_k \in T_m$. Consider the morphism $t : [n] \to [n + m]$ which is the following composite:

$$t = [n] \xrightarrow{\bar{t}_1 \sqcup \cdots \sqcup \bar{t}_n} [k + n] \xrightarrow{(f_1 \sqcup \cdots \sqcup f_k) + id_n} [m + n]$$

where id_n is the identity morphism on $[n]$.

Since $\bar{t}_1 \sqcup \cdots \sqcup \bar{t}_n$ is an ideal morphism, and T is ideal, it follows that t is ideal, and hence there exists a unique morphism $t^\# : [n] \to [m]$ with $t^\# = (id_m \sqcup t^\#) \cdot t$. But then $t^\# = t_1^\# \sqcup \cdots \sqcup t_n^\#$ for unique $t_1^\#, \ldots, t_n^\# \in Tm$, and for each $j \leq n$

$$\begin{aligned} t_{jTm}(f_1^\#, \ldots, f_k, t_1^\#, \ldots, t_n^\#) &= (f_1 \sqcup \cdots \sqcup f_k \sqcup t_1^\# \sqcup \cdots \sqcup t_n^\#) \cdot \bar{t}_j \\ &= (id_m \sqcup t^\#) \cdot ((f_1 \sqcup \cdots \sqcup f_k) + id_n) \cdot \bar{t}_j \\ &= (id_m \sqcup t^\#) \cdot t \cdot j \\ &= t^\# \cdot j \\ &= t_j^\#. \end{aligned}$$

Thus $t_1^\#, \ldots, t_n^\#$ provide a solution in Tm for T with respect to $(f_1, \ldots, f_k)$. A similar computation to the one above, using the uniqueness of $t^\#$, shows that this solution is unique. Hence, by the Corollary to Proposition 2, Tm is an iterative Σ-algebra.

Now, for $V = \{v_1, v_2, \ldots\}$ a countable set, suppose that $A_\Sigma V$ is the free iterative Σ-algebra over V, whose existence is provided by Theorem 1. For each natural number n, let $A_\Sigma n$ be the sub-iterative algebra of $A_\Sigma V$ generated by $\{v_1, \ldots, v_n\}$; then $A_\Sigma n$ is the free iterative algebra over $\{v_1, v_2, \ldots, v_n\}$. Construct an algebraic theory A from $A_\Sigma V$ as follows: for each m, the A-morphisms from $[1]$ to $[m]$ are the elements of $A_\Sigma m$, and the morphisms $[n]\to[m]$ are n-tuples of morphisms from $[1]$ to $[m]$. For $\boldsymbol{A}$-morphisms $s_1,\ldots,s_m:[1]\to[k]$ and $t_1,\ldots,t_n:[1]\to[m]$, the composite $(s_1,\ldots,s_m)\cdot(t_1,\ldots,t_n):[n]\to[k]$ is the n-tuple $(r_1,\ldots,r_n)$ where r_j is the image of t_j under the unique homorphism from $A_\Sigma n\to A_\Sigma k$ which maps v_i to s_i for $1\leq i\leq m$. As in Section 1.1 we write $r_j=t_{jA_\Sigma k}(s_1,\ldots,s_m)$.

It is straightforward to check that this yields an algebraic theory, in which the ith coproduct injection $[1]\to[n]$ is just v_i. Moreover, judicious use of homomorphisms from $A_\Sigma V$ into the algebra described in Example 3 of Section 2.4 will show that this theory is ideal. It follows from Proposition 4 that $\boldsymbol{A}$ is an iterative theory. $\square$

Theorem 5. *$\boldsymbol{A}$ is the free iterative theory generated by Σ.*

Proof. Suppose T is an iterative algebraic theory and for each $\sigma\in\Sigma$, $\bar\sigma:[1]\to[|\sigma|]$ is an ideal morphsim in $\boldsymbol{T}$. By Proposition 7, for each m, $\boldsymbol{T}m$ is an iterative Σ-algebra and hence there is a unique homomorphism ϕ_m from $A_\Sigma m=\boldsymbol{A}m$ into $\boldsymbol{T}m$ such that $\phi_m(v_i)$ is the ith corproduct injection $[1]\to[m]$ in $\boldsymbol{T}$. But then $\phi=(\phi_m)_{m\in\omega}$ provides the desired functor from $\boldsymbol{A}$ to $\boldsymbol{T}$. $\square$

4.2. *Further examples*

In this section, Elgot's examples [4] of iterative theories are discussed, and put into the framework of iterative algebras. Since morphisms $[n]\to[p]$ in an algebraic theory are essentially just n-tuples of morphisms $[1]\to[p]$, only the latter will be described.

The first example concerns 'timed terminal behaviour' of machines. Here, X is the set of external states (tape configuration plus position of reading head for a Turing machine, etc.) and a morphism $[1]\to[n]$ in the theory $[\![X\cdot N,\square]\!]$ is a function $f:X\to(X\times N\times[p])\cup\{\square\}$ (where $[p]=\{1,2,3,\ldots,p\}$) such that

$$f(x)=(y,0,j)\text{ for some }x\in X\text{ implies }f(z)=(z,0,j)\text{ for all }z\in X.$$

In the latter case, f is the jth coproduct injection. Such a function f is interpreted as the timed terminal behaviour of a machine with one entrance and p exits, $f(x)=\square$ means the machine has no output on input x, and $f(x)=(y,k,j)$ means the machine, given input x, outputs y at the jth exit in time k. Note that Elgot's definition [4, p. 184] of a morphism $[1]\to[p]$ is a function $f:(X\times N\times[1])\cup\{\square\}\to(X\times N\times[p])\cup\{\square\}$ such that $f(\square)=\square$ and for each k, $f(x,k,1)$ determines and is completely determined by $f(x,0,1)$ and hence we need only consider the values of f on triples $(x,0,1)$.

The algebra A of machines is now defined as follows: the elements of A are all the functions $f:X\to(X\times X\times[p])\cup\{\square\}$ as described in the preceding paragraph.

Also, each function $g: X \to (X \times N^+ \times [n]) \cup \{\square\}$ (where $N^+ = \{1, 2, 3, \ldots\}$) induces an n-ary operation σ_g on A as follows: for $f_i: X \to (X \times N \times [p_i]) \cup \{\square\}$ $(1 \leq i \leq n)$,

$$\sigma_g(f_1, \ldots, f_n): X \to (X \times N \times [p]) \cup \{\square\}$$

where p = maximum of $p_1, p_2, \ldots, p_n$ and for each $x \in X$,

$$\sigma_g(f_1, \ldots, f_n)(x) = \begin{cases} \square & \text{if } g(x) = \square, \\ \square & \text{if } g(x) = (y, k, i) \text{ and } f_i(y) = \square, \\ (z, k+r, j) & \text{if } g(x) = (y, k, i) \text{ and } f_i(y) = (z, r, j). \end{cases}$$

This corresponds to connecting up the machines so that given an input x, first the machine g acts on it; if this produces an output at the ith exit then this is fed into the machine f_i.

It is straightforward to check that A, with all these operations, is an iterative algebra, analogous to Elgot's result that the theory $[\![X \cdot N, \square]\!]$ is iterative.

Elgot's second example is that of matrices of subsets of a monoid M which has a 'length function', i.e. a function $l: M \to N$ such that $l(xy) = l(x) + l(y)$ and $l(x) = 0$ iff $x = 0$. His theory $[M]$ has as morphisms $[1] \to [p]$ all p-tuples $(U_1, \ldots, U_p)$ of subsets of M such that $1 \in U_j$ implies $U_j = \{1\}$ and $U_k = \emptyset$ for $k \neq j$. Thus, besides the coproduct injections $(\emptyset, \emptyset, \ldots, \{1\}, \emptyset, \ldots, \emptyset)$, the morphisms are p-tuples of subsets of M which do not contain 1. Alternatively, we may consider the algebra $\bar{M}$ whose elements are all subsets of M not containing 1, and which has, for each n-tuple $S = (S_1, \ldots, S_n) \in \bar{M}^n$, an n-ary operation which maps $(X_1, \ldots, X_n)$ to $S_1X_1 + S_2X_2 + \cdots + S_nX_n$. Then $\bar{M}$ with these operations is an iterative algebra. As a further alternative, we may consider $\bar{M}$ as an algebra of type $\Sigma = \{+, \times\}$ where $+$, $\times$ are binary operations; $+$ is set union and $U \times V = \{uv \mid u \in U, v \in V\}$. Then techniques similar to those in Section 3.2 show that $\bar{M}$ has unique solutions to all iterative systems $T = (t_1, \ldots, t_n)$ of type Σ such that for each $i \leq n$, if $t_i(u) = +$ then there exists ι with $t_i(uv) = \times$.

The third example from Elgot [4] which we will discuss is that of sequacious functions, which essentially keep track of the sequence of external states attained in a computation. For Elgot's sequencious functions, x is always a prefix of $f(x)$; here we replace f with the function g uniquely determined by $xg(x) = f(x)$. With this reduction, the theory has, as morphisms $[1] \to [p]$, all functions $f: X \to (X^* \times [p]) \cup X^\infty$ (where X^∞ is the set of countable sequences from X) such that

$$\text{if } f(x) = (\emptyset, i) \text{ for some } x \in X \text{ then } f(z) = (\emptyset, i) \text{ for all } z \in X.$$

Thus, besides the coproduct injections which are constant with value $(\emptyset, i)$, the morphisms from $[1]$ to $[p]$ are all functions $f: X \to (X^+ \times [p]) \cup X^\infty$. The algebraic approach is to consider the algebra S whose elements are all functions $f: X \to (X^+ \times [p]) \cup X^\infty$ as above. Also, for each function $g: X \to (X^+ \times [n]) \cup X^\infty$, S has a n-ary operation σ_g defined as follows: for $f_i: X \to (X^* \times [p_i] \cup X^\infty (1 \leq i \leq n)$,

$$\sigma_g(f_1, \ldots, f_n): X \to (X^* \times [p]) \cup X^\infty$$

where $p = \text{maximum}\,\{p_1, p_2, \ldots, p_n\}$, and for each $x \in X$,

$$\sigma_g(f_1, \ldots, f_n)(x) = \begin{cases} g(x) & \text{if } g(x) \in X^{\infty}, \\ uyf_i(y) & \text{if } g(x) = (uy, i) \text{ for } u \in X^*, y \in X \text{ and} \\ & \quad f_i(y) \in X^{\infty}, \\ (uyw, j) & \text{if } g(x) = (uy, i) \text{ for } u \in X^*, u \in X \text{ and} \\ & \quad f_i(y) = (ws, j). \end{cases}$$

Just as in the example concerning timed terminal behaviour, this models the glueing together of the machines. It can be verified that S, with all these operations, is an iterative algebra.

References

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Note added in proof

Iterative algebras have also been studied in Tiuryn (Unique fixed points vs. least fixed points, *Theoret. Comput. Sci.* **13** (1981) 229–254). Tiuryn uses the results of Ginali et al. on free iterative theories to show that the algebra of all regular trees is the free iterative algebra (with respect to solutions of *all* ideal iterative equations), which is the reverse of the approach taken here, in Section 4. In addition, he explores the connection between iterative algebras and regular algebras, the latter being ordered algebras where solutions of iterative equations are obtained as joins of certain ω-chains.

A related topic has been developed by Benson and Guessarian (Algebraic solutions to recursion schemes, manuscript CS-81-079, Washington State University), namely that of algebras with solutions to recursive equations, these being more general than iterative. In their approach, solutions are not required to be unique, but are explicitly added as extra operations. A brief description follows.

A recursion scheme over a set Σ of basic operations is a set S of equations

$$G_i(x_1, \ldots, x_{n_i}) = t_i(x_1, \ldots, x_{n_i}) \qquad (1 \leq i \leq n)$$

where the G_i are new operations, and the t_i are polynomials in both the Σ-operations and these new ones, i.e. elements of the free $\Sigma \cup \{G_1, \ldots, G_n\}$-algebra on an appropriate set of generators. For such a recursion scheme S, let $\Phi = \{G_1, \ldots, G_n\}$, then we may consider the class of all $\Sigma \cup \Phi$-algebras satisfying the above equations and all $\Sigma \cup \Phi$-homomorphisms between them. Since this is just an equationally defined class of (universal) algebras, the existence of free algebras is a standard result (Birkhoff [1]). The approach of Benson and Guessarian is essentially to construct this free algebra as a quotient of the absolutely free $\Sigma \cup \Phi$-algebra; neither approach yields a canonical form, or description, for the elements of the free algebra. Similarly, by adding a set of operations for *each* recursion scheme over Σ, one can obtain an algebra which has solutions to all recursion schemes and is free, or universal, with this property. In the same way one can also obtain free algebras with solutions to recursive equations satisfying some additional equations, as in the last section of their paper. The essential difference between their approach and the one presented in the present paper is that the solutions to recursion schemes need not be unique; however, since the homomorphisms are stipulated to preserve these solutions, because they are added as operations, this does fall within the usual framework of universal algebra. As the example at the end of Section 2.3 above shows, if neither of these approaches is followed (i.e. solutions need not be unique and are also not added as operations which must be preserved by morphisms) then free algebras need not exist.