ELSEVIER

Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE @ DIRECT

J. Math. Anal. Appl. 290 (2004) 291–301

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

Positive solutions for three-point boundary value problems with dependence on the first order derivative ☆

Yanping Guo [a,b,*] and Weigao Ge [b]

[a] *College of Sciences, Hebei University of Science and Technology, Shijiazhuang, Hebei 050018, PR China*
[b] *Department of Applied Mathematics, Beijing Institute of Technology, Beijing 100081, PR China*

Received 21 November 2002

Submitted by A.C. Peterson

Abstract

A new fixed point theorem in a cone is applied to obtain the existence of at least one positive solution for the second order three-point boundary value problem

$$\begin{cases} x'' + f(t, x, x') = 0, & 0 < t < 1, \\ x(0) = 0, \quad x(1) = \alpha x(\eta), \end{cases}$$

where f is a nonnegative continuous function, $\alpha > 0$, $\eta \in (0, 1)$, $\alpha\eta < 1$. The associated Green's function for the above problem is also used.

Keywords: Three-point boundary value problem; Fixed point theorem in a cone; Green's function; Positive solution

1. Introduction

In the past few years, there has been much attention focused on questions of positive solutions of three-point boundary value problems for nonlinear ordinary differential equa-

☆ The project is supported by the National Nature Science Foundation (10371030), and the Scientific Research Foundation of the Education Department of Hebei Province (2003216) and the Foundation of Hebei University of Science and Technology (XL200212).
* Corresponding author.
E-mail address: guoyanping65@sohu.com (Y. Guo).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2003.09.061

tions; see, to name a few [1–7]. Recently, Ma [1] used the Krasnoselskii's fixed point theorem in a cone [8] to prove the existence of positive solutions for the second order three-point boundary value problem

$$\begin{cases} u'' + a(t)g(u) = 0, & 0 < t < 1, \\ u(0) = 0, & u(1) = \alpha u(\eta), \end{cases} \tag{1}$$

where $\alpha > 0$, $\eta \in (0, 1)$, $\alpha\eta < 1$, $a \in C([0, 1], [0, \infty))$, and $g \in C([0, \infty), [0, \infty))$ is either superlinear or sublinear. Applying Leggett–Williams fixed point theorem [9], He and Ge [7] studied the existence of at least three positive solutions for the second order three-point boundary value problem

$$\begin{cases} u'' + f(t, u) = 0, & 0 < t < 1, \\ u(0) = 0, & u(1) = \alpha u(\eta). \end{cases} \tag{2}$$

All the above works were done under the assumption that the first order derivative x' is not involved explicitly in the nonlinear term f. In this paper, we are concerned with the existence of positive solutions for the second order three-point boundary value problem

$$\begin{cases} x'' + f(t, x, x') = 0, & 0 < t < 1, \\ x(0) = 0, & x(1) = \alpha x(\eta). \end{cases} \tag{3}$$

The following conditions are satisfied throughout this paper:

(H_1) $\alpha > 0$, $0 < \eta < 1$, and $1 - \alpha\eta > 0$;
(H_2) $f : [0, 1] \times [0, \infty) \times R \to [0, \infty]$ is continuous.

In [11], Avery and Anderson extended Krasnoselskii's fixed point theorem. In this paper, to show the existence of positive solutions to (3), a new fixed point theorem in a cone is proved at first, which can be also regarded as an extension of Krasnoselskii's fixed point theorem in a cone. Then some new criteria for the existence of positive solutions to (3) are given by applying the new theorem. This is the only work which allows f to depend on the first derivative of x to obtain positive solutions as far as we know.

2. Fixed point theorem in a cone

Let X be a Banach space and $K \subset X$ a cone. Suppose $\alpha, \beta : X \to R^+$ are two continuous convex functionals satisfying

$$\alpha(\lambda x) = |\lambda|\alpha(x), \quad \beta(\lambda x) = |\lambda|\beta(x), \quad \text{for } x \in X,\ \lambda \in R, \tag{4}$$

and

$$\begin{aligned} &\|x\| \leqslant M \max\{\alpha(x), \beta(x)\} \quad \text{for } x \in X \quad \text{and} \\ &\alpha(x) \leqslant \alpha(y) \quad \text{for } x, y \in K,\ x \leqslant y, \end{aligned} \tag{5}$$

where $M > 0$ is a constant.

Lemma 2.1. *Let $r, L > 0$ be constants and*

$$\Omega = \{x \in X\colon\ \alpha(x) < r,\ \beta(x) < L\}.$$

Set

$$D = \{x \in X\colon\ \alpha(x) = r\}, \qquad E = \{x \in X\colon\ \alpha(x) \leqslant r,\ \beta(x) = L\}.$$

Assume $T: K \to K$ is a completely continuous operator satisfying

(A_1) $\alpha(Tu) < r$, $u \in D \cap K$;
(A_2) $\beta(Tu) < L$, $u \in E \cap K$.

Then

$$\deg\{I - T, \Omega \cap K, 0\} = 1.$$

Proof. Obviously $\partial\Omega \cap K \subset (D \cap K) \cup (E \cap K)$. According to Dugundji's extension theorem [10], $T|_{\overline{\Omega}\cap K}$ has a completely continuous extension $T^*: \overline{\Omega} \to K$ such that

$$T^*|_{\overline{\Omega}\cap K} = T|_{\overline{\Omega}\cap K}, \qquad T^*(\overline{\Omega}) \subset \overline{\mathrm{conv}}\, T(\overline{\Omega} \cap K) \subset K.$$

Let

$$h(x, \lambda) = x - \lambda T^* x, \quad x \in \overline{\Omega},\ \lambda \in [0, 1];$$

$h(x, \lambda): \overline{\Omega} \times [0, 1] \to K$ is completely continuous. We claim that $h(x, \lambda) \neq 0$ for $x \in \partial\Omega$ and $\lambda \in [0, 1]$. Suppose there is $x_0 \in \partial\Omega$, $\lambda_0 \in [0, 1]$ such that

$$h(x_0, \lambda_0) = 0.$$

Then $x_0 = \lambda_0 T^* x_0 \in K$. (A_1) and (A_2) implies $\lambda_0 \neq 1$.

If $x_0 \in D \cap K$, then

$$r = \alpha(x_0) = \alpha(\lambda_0 T^* x_0) = \lambda_0 \alpha(T x_0) \leqslant \lambda_0 r < r,$$

a contradiction. On the other hand, if $x_0 \in E \cap K$,

$$L = \beta(x_0) = \beta(\lambda_0 T^* x_0) = \beta(\lambda_0 T x_0) = \lambda_0 \beta(T x_0) \leqslant \lambda_0 L < L,$$

a contradiction, too. It follows that

$$\deg\{I - T, \Omega \cap K, 0\} = \deg\{I - T^*, \Omega, 0\} = \deg\{I, \Omega, 0\} = 1. \qquad \square$$

Lemma 2.2. *In Lemma* 2.1, *suppose (A_1) and (A_2) are replaced by*

(A_3) $\alpha(Tu) > r$, $u \in D \cap K$;
(A_4) $\beta(Tu) < L$, $u \in K$;

and there is a $p \in (\Omega \cap K) \setminus \{0\}$ such that $\alpha(p) \neq 0$ and $\alpha(x + \lambda p) \geqslant \alpha(x)$ for all $x \in K$ and $\lambda \geqslant 0$. Then

$$\deg\{I - T, \Omega \cap K, 0\} = 0.$$

Proof. Let $\eta = \max\{\alpha(u)\colon\ u \in \overline{\mathrm{conv}}\, T(\overline{\Omega}^* \cap K)\}$, $s = 1 + (r + \eta)/\alpha(p)$ and

$$\Omega^* = \{x \in X\colon\ \alpha(x) < r,\ \beta(x) < 1 + L + s\beta(p)\}.$$

According to Dugundji's extension theorem, $T|_{\overline{\Omega}^*\cap K}$ has a completely continuous extension $T^*:\overline{\Omega}^*\to K$ such that

$$T^*|_{\overline{\Omega}^*\cap K}=T|_{\overline{\Omega}^*\cap K},\qquad T^*(\overline{\Omega}^*)\subset\overline{\operatorname{conv}}\,T(\overline{\Omega}^*\cap K)\subset K. \tag{6}$$

Therefore

$$\alpha(x-T^*x)\leqslant\alpha(x)+\alpha(T^*x)\leqslant r+\eta,\quad \beta(T^*x)\leqslant L,\quad x\in\overline{\Omega}^*. \tag{7}$$

Let $H(x,\lambda)=x-T^*x-\lambda p$, $0\leqslant\lambda\leqslant s$. We claim that $H(x,\lambda)\neq 0$ for $x\in\partial\Omega^*$, $\lambda\in[0,s]$. Otherwise there are $x_0\in\partial\Omega^*$, $\lambda_0\in[0,s]$ such that

$$x_0-T^*x_0-\lambda_0 p=0.$$

Then $x_0=T^*x_0+\lambda_0 p\in K$. So $x_0\in\partial\Omega^*\cap K$.

(1) If $\alpha(x_0)=r$, then $x_0\in D\cap K$. From (A_3), we have $\alpha(T^*x_0)=\alpha(Tx_0)>r$. However,

$$\alpha(T^*x_0)\leqslant\alpha(T^*x_0+\lambda_0 p)=\alpha(x_0)=r,$$

a contradiction.

(2) If $\beta(x_0)=1+L+s\beta(p)$, then

$$1+L+s\beta(p)=\beta(x_0)=\beta(T^*x_0+\lambda_0 p)\leqslant\beta(T^*x_0)+\lambda_0\beta(p)\leqslant L+s\beta(p),$$

a contradiction. So

$$\deg\{I-T^*,\Omega^*,0\}=\deg\{I-T^*-sp,\Omega^*,0\}=\deg\{I-T^*,\Omega^*,sp\}.$$

If there is $x\in\Omega^*$ such that $x-T^*x=sp$, then $\alpha(x-T^*x)=s\alpha(p)$ and

$$s=\frac{\alpha(x-T^*x)}{\alpha(p)}\leqslant\frac{r+\eta}{\alpha(p)}<1+\frac{r+\eta}{\alpha(p)}=s,$$

a contradiction. So $\deg\{I-T^*,\Omega^*,0\}=0$.

In addition, if $x\in\Omega^*$, $x=T^*x$, (A_4) implies

$$\beta(x)=\beta(T^*x)<L,$$

and then $x\in\Omega$. So

$$\deg\{I-T,\Omega\cap K,0\}=\deg\{I-T^*,\Omega,0\}=\deg\{I-T^*,\Omega^*,0\}=0. \qquad\square$$

Theorem 2.1. *Let $r_2>r_1>0$, $L>0$ be constants and*

$$\Omega_i=\{x\in X:\ \alpha(x)<r_i,\ \beta(x)<L\},\quad i=1,2,$$

two bounded open sets in X. Set

$$D_i=\{x\in X:\ \alpha(x)=r_i\}.$$

Assume $T:K\to K$ is a completely continuous operator satisfying

(A_5) $\alpha(Tu)<r_1$, $u\in D_1\cap K$; $\alpha(Tu)>r_2$, $u\in D_2\cap K$;
(A_6) $\beta(Tu)<L$, $u\in K$;

(A_7) *there is a* $p \in (\Omega \cap K) \setminus \{0\}$ *such that* $\alpha(p) \neq 0$ *and* $\alpha(x + \lambda p) \geqslant \alpha(x)$ *for all* $x \in K$ *and* $\lambda \geqslant 0$.

Then T *has at least one fixed point in* $(\Omega_2 \setminus \overline{\Omega}_1) \cap K$.

Proof. Applying Lemmas 2.1 and 2.2,

$$\deg\{I - T, \Omega_1 \cap K, 0\} = 1, \qquad \deg\{I - T, \Omega_2 \cap K, 0\} = 0,$$

and then

$$\begin{aligned}\deg\bigl\{I - T, (\Omega_2 \setminus \overline{\Omega}_1) \cap K, 0\bigr\} &= \deg\{I - T, \Omega_2 \cap K, 0\} - \deg\{I - T, \Omega_1 \cap K, 0\} \\ &= -1.\end{aligned}$$

So T has at least one fixed point in $(\Omega_2 \setminus \overline{\Omega}_1) \cap K$. □

3. The main results

Lemma 3.1 [1]. *Let* $\alpha\eta \neq 1$; *then for* $y \in C[0, 1]$, *the problem*

$$\begin{cases} x'' + y(t) = 0, & 0 < t < 1, \\ x(0) = 0, & x(1) = \alpha x(\eta) \end{cases} \tag{8}$$

has a unique solution

$$x(t) = -\int_0^t (t - s)y(s)\,ds + \frac{t}{1 - \alpha\eta}\int_0^1 (1 - s)y(s)\,ds - \frac{\alpha t}{1 - \alpha\eta}\int_0^\eta (\eta - s)y(s)\,ds.$$

Lemma 3.2 [1]. *Let* $0 < \alpha < 1/\eta$. *If* $y \in C[0, 1]$ *and* $y \geqslant 0$, *then the unique solution* x *of problem* (8) *satisfies*

$$\min_{t \in [\eta, 1]} x(t) \geqslant \gamma \|x\|,$$

where $\gamma = \min\{\alpha\eta, \alpha(1 - \eta)/(1 - \alpha\eta), \eta\}$.

Lemma 3.3. *Let* $(1 - \alpha\eta) \neq 0$ *and* $y \in C[0, 1]$. *The Green function for the boundary value problem*

$$\begin{cases} -x'' = 0, & 0 < t < 1, \\ x(0) = 0, & x(1) = \alpha x(\eta) \end{cases} \tag{9}$$

is given by

$$G(t, s) = \begin{cases} \frac{s[(1-t)-\alpha(\eta-t)]}{1-\alpha\eta}, & s \leqslant t,\ s \leqslant \eta, \\ \frac{s(1-t)+\alpha\eta(t-s)}{1-\alpha\eta}, & \eta \leqslant s \leqslant t, \\ \frac{t[(1-s)-\alpha(\eta-s)]}{1-\alpha\eta}, & t \leqslant s \leqslant \eta, \\ \frac{t(1-s)}{1-\alpha\eta}, & t \leqslant s,\ s \geqslant \eta. \end{cases}$$

Proof. For $t \leqslant \eta$, the unique solution of (8) can be expressed as

$$\begin{aligned}
x(t) &= -\int_0^t (t-s)y(s)\,ds \\
&\quad + \frac{t}{1-\alpha\eta}\left[\int_0^t (1-s)y(s)\,ds + \int_t^\eta (1-s)y(s)\,ds + \int_\eta^1 (1-s)y(s)\,ds\right] \\
&\quad - \frac{\alpha t}{1-\alpha\eta}\left[\int_0^t (\eta-s)y(s)\,ds + \int_t^\eta (\eta-s)y(s)\,ds\right] \\
&= \int_0^t \frac{s[(1-t)-\alpha(\eta-t)]}{1-\alpha\eta}y(s)\,ds + \int_t^\eta \frac{t[(1-s)-\alpha(\eta-s)]}{1-\alpha\eta}y(s)\,ds \\
&\quad + \int_\eta^1 \frac{t(1-s)}{1-\alpha\eta}y(s)\,ds \\
&= \int_0^1 G(t,s)y(s)\,ds.
\end{aligned}$$

For $t \geqslant \eta$, the unique solution of (8) can be expressed as

$$\begin{aligned}
x(t) &= -\left[\int_0^\eta (t-s)y(s)\,ds + \int_\eta^t (t-s)y(s)\,ds\right] \\
&\quad + \frac{t}{1-\alpha\eta}\left[\int_0^\eta (1-s)y(s)\,ds + \int_\eta^t (1-s)y(s)\,ds + \int_t^1 (1-s)y(s)\,ds\right] \\
&\quad - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta-s)y(s)\,ds \\
&= \int_0^\eta \frac{s[(1-t)-\alpha(\eta-t)]}{1-\alpha\eta}y(s)\,ds + \int_\eta^t \frac{s(1-t)+\alpha\eta(t-s)}{1-\alpha\eta}y(s)\,ds \\
&\quad + \int_t^1 \frac{t(1-s)}{1-\alpha\eta}y(s)\,ds \\
&= \int_0^1 G(t,s)y(s)\,ds.
\end{aligned}$$

Therefore, the unique solution of (8) can be expressed as $x(t) = \int_0^1 G(t,s)y(s)\,ds$. Lemma 3.3 is now proved. □

Let $X = C^1([0,1], R)$ with $\|x\| = \max_{0\leqslant t\leqslant 1}[x^2(t) + (x'(t))^2]^{1/2}$, and $K = \{x \in X\colon x(t) \geqslant 0,\ x \text{ is concave on } [0,1]\}$. Define functionals $\alpha(x) = \max_{0\leqslant t\leqslant 1}|x(t)|$ and $\beta(x) = \max_{0\leqslant t\leqslant 1}|x'(t)|$ for each $x \in X$, then $\|x\| \leqslant \sqrt{2}\max\{\alpha(x), \beta(x)\}$ and

$$\alpha(\lambda x) = |\lambda|\alpha(x), \quad \beta(\lambda x) = |\lambda|\beta(x), \quad x \in X,\ \lambda \in R,$$
$$\alpha(x) \leqslant \alpha(y) \quad \text{for } x, y \in K,\ x \leqslant y.$$

If (H_1) holds, the Green function $G(t,s) \geqslant 0$ for the problem (9). Let $y(t) = 1$, we have

$$\int_0^1 G(t,s)\,ds = -\frac{1}{2}t^2 + \frac{t(1-\alpha\eta^2)}{2(1-\alpha\eta)}.$$

In the following, we denote

$$M = \max_{t\in[0,1]}\int_0^1 G(t,s)\,ds, \qquad m = \max_{t\in[0,1]}\int_\eta^1 G(t,s)\,ds, \qquad Q = \frac{3-\alpha\eta+\alpha\eta^2}{2(1-\alpha\eta)}.$$

We will suppose that there are $L > b > \gamma b > c > 0$ such that $f(t,u,v)$ satisfies the following growth conditions:

(H_3) $f(t,u,v) < c/M$ for $(t,u,v) \in [0,1]\times[0,c]\times[-L,L]$;
(H_4) $f(t,u,v) \geqslant b/m$ for $(t,u,v) \in [0,1]\times[\gamma b,b]\times[-L,L]$;
(H_5) $f(t,u,v) < L/Q$ for $(t,u,v) \in [0,1]\times[0,b]\times[-L,L]$;
(H_6) $f(t,u,v) < L^2/(2b)$ for $(t,u,v) \in [0,1]\times[0,b]\times[-L,L]$.

Let

$$f^*(t,u,v) = \begin{cases} f(t,u,v), & (t,u,v) \in [0,1]\times[0,b]\times(-\infty,\infty), \\ f(t,b,v), & (t,u,v) \in [0,1]\times(b,\infty)\times(-\infty,\infty), \end{cases} \tag{10}$$

and

$$f_1(t,u,v) = \begin{cases} f^*(t,u,v), & (t,u,v) \in [0,1]\times[0,\infty)\times[-L,L], \\ f^*(t,u,-L), & (t,u,v) \in [0,1]\times[0,\infty)\times(-\infty,-L], \\ f^*(t,u,L), & (t,u,v) \in [0,1]\times[0,\infty)\times[L,\infty). \end{cases} \tag{11}$$

Then $f_1 \in C([0,1]\times[0,\infty)\times R, R^+)$. Define

$$(Tx)(t) = \int_0^1 G(t,s)f_1\big(s, x(s), x'(s)\big)\,ds.$$

Theorem 3.1. *Suppose (H_1)–(H_5) hold. Then BVP (3) has at least one positive solution $y(t)$ satisfying*

$$c < \alpha(x) < b, \qquad \big|y'(t)\big| < L.$$

Proof. Take

$$\Omega_1 = \{x \in X\colon |x(t)| < c,\ |x'(t)| < L\}, \qquad \Omega_2 = \{x \in X\colon |x(t)| < b,\ |x'(t)| < L\},$$

two bounded open sets in X, and

$$D_1 = \{x \in X\colon \alpha(x) = c\}, \qquad D_2 = \{x \in X\colon \alpha(x) = b\}.$$

Obviously, $T : K \to K$ is completely continuous, and there is a $p \in (\Omega_2 \cap K) \setminus \{0\}$ such that $\alpha(x + \lambda p) \geqslant \alpha(x)$ for all $x \in K$ and $\lambda \geqslant 0$. For $x \in D_1 \cap K$, $\alpha(x) = c$. From (H_3), we get

$$\begin{aligned}\alpha(Tx) &= \max_{t\in[0,1]} \left| \int_0^1 G(t,s) f_1\big(s, x(s), x'(s)\big)\, ds \right| \\ &< \max_{t\in[0,1]} \int_0^1 G(t,s) \frac{c}{M}\, ds = \frac{c}{M} \max_{t\in[0,1]} \int_0^1 G(t,s)\, ds = c.\end{aligned}$$

Whereas for $x \in D_2 \cap K$, $\alpha(x) = b$. From Lemma 3.2, we have $x(t) \geqslant \gamma\alpha(x) = \gamma b$ for $t \in [\eta, 1]$. So, from (H_4), we get

$$\begin{aligned}\alpha(Tx) &= \max_{t\in[0,1]} \left| \int_0^1 G(t,s) f_1\big(s, x(s), x'(s)\big)\, ds \right| \\ &> \max_{t\in[0,1]} \left| \int_\eta^1 G(t,s) f_1\big(s, x(s), x'(s)\big)\, ds \right| \\ &> \max_{t\in[0,1]} \int_\eta^1 G(t,s) \frac{b}{m}\, ds = \frac{b}{m} \max_{t\in[0,1]} \int_\eta^1 G(t,s)\, ds = b.\end{aligned}$$

For $x \in K$, from (H_5), we get

$$\begin{aligned}\beta(Tx) &= \max_{t\in[0,1]} \left| -\int_0^t f_1\big(s, x(s), x'(s)\big)\, ds + \frac{1}{1-\alpha\eta} \int_0^1 (1-s) f_1\big(s, x(s) x'(s)\big)\, ds \right. \\ &\qquad \left. - \frac{\alpha}{1-\alpha\eta} \int_0^\eta (\eta - s) f_1\big(s, x(s) x'(s)\big)\, ds \right| \\ &< \left[1 + \frac{1}{1-\alpha\eta} \int_0^1 (1-s)\, ds + \frac{\alpha}{1-\alpha\eta} \int_0^\eta (\eta - s)\, ds \right] \frac{L}{Q} \\ &= \frac{3 - \alpha\eta + \alpha\eta^2}{2(1-\alpha\eta)} \frac{L}{Q} = L.\end{aligned}$$

Theorem 2.1 implies there is $y \in (\Omega_2 \setminus \overline{\Omega}_1) \cap K$ such that $y = Ty$. So, y is a positive solution for BVP (3) satisfying

$$c < \alpha(y) < b, \qquad \left|y'(t)\right| < L.$$

Thus, Theorem 3.1 is completed. □

If $\alpha \leqslant 1$, then $(Tu)(1) = \alpha(Tu)(\eta) \leqslant (Tu)(\eta)$. Let $L_1 > 0$ such that $\max\{f(t,u,v), (t,u,v) \in [0,1] \times [0,b] \times [-L,L]\} = L_1^2/(2b) < L^2/(2b)$. So, there is $\sigma \in (0,1)$ such that $\max_{0\leqslant t\leqslant 1}(Tx)(t) = (Tx)(\sigma)$. For $b > a$, define

$$[A]_a^b = \begin{cases} b, & A > b, \\ A, & a \leqslant A \leqslant b, \\ a, & A < a, \end{cases}$$

$$(T^*x)(t) = \int_0^t \Bigg[-\int_0^s f_1\big(r, x(r), x'(r)\big)\, dr + \frac{1}{1-\alpha\eta} \int_0^1 (1-r) f_1\big(r, x(r), x'(r)\big)\, dr - \frac{\alpha}{1-\alpha\eta} \int_0^\eta (\eta - r) f_1\big(r, x(r), x'(r)\big)\, dr \Bigg]_{-L_1}^{L_1} ds + d_x,$$

where

$$d_x = \int_0^\sigma \Bigg\{ \Bigg[-\int_0^s f_1\big(r, x(r), x'(r)\big)\, dr + \frac{1}{1-\alpha\eta} \int_0^1 (1-r) f_1\big(r, x(r), x'(r)\big)\, dr - \frac{\alpha}{1-\alpha\eta} \int_0^\eta (\eta - r) f_1\big(r, x(r), x'(r)\big)\, dr \Bigg] - \Bigg[-\int_0^s f_1\big(r, x(r), x'(r)\big)\, dr + \frac{1}{1-\alpha\eta} \int_0^1 (1-r) f_1\big(r, x(r), x'(r)\big)\, dr - \frac{\alpha}{1-\alpha\eta} \int_0^\eta (\eta - r) f_1\big(r, x(r), x'(r)\big) dr \Bigg]_{-L_1}^{L_1} \Bigg\} ds.$$

Obviously, T^*x is well defined. It is easy to see that $T^* : K \to K$ is a completely continuous operator. And clearly

$$\alpha(T^*x) = \alpha(Tx) = \int_0^\sigma \Bigg[-\int_0^s f_1\big(r, x(r), x'(r)\big)\, dr + \frac{1}{1-\alpha\eta} \int_0^1 (1-r) f_1\big(r, x(r), x'(r)\big)\, dr - \frac{\alpha}{1-\alpha\eta} \int_0^\eta (\eta - r) f_1\big(r, x(r), x'(r)\big)\, dr \Bigg] ds$$

$$= \int_0^1 G(\sigma, r) f_1\big(r, x(r), x'(r)\big)\, dr,$$

$$\beta(T^*x) \leqslant L_1 \quad \text{for } x \in K, \qquad (T^*x)(t) \geqslant (Tx)(t), \quad t \in [0,1].$$

Theorem 3.2. *Suppose* (H_2)*–*(H_4) *and* (H_6) *hold. If in addition* $0 < \alpha \leqslant 1$, $\eta \in (0,1)$*, then BVP* (3) *has at least one positive solution* $y(t)$ *satisfying*

$$c < \alpha(y) < b, \qquad \big|y'(t)\big| < L.$$

Proof. From (H_6), we have $\beta(T^*x) \leqslant L_1 < L$ for $x \in K$. Essentially using the same reasoning as in Theorem 3.1, we may obtain T^* has a fixed point $y \in (\Omega_2 \setminus \overline{\Omega}_1) \cap K$.

For all $x \in K$, clearly $(Tx)'(\sigma) = (T^*x)'(\sigma) = 0$. We now show

$$\big|(T^*y)'(t)\big| < L_1. \tag{12}$$

If (12) fails to be true, from the concavity of $(T^*y)(t)$ there is a $t_0 \in [0,\sigma) \cup (\sigma, 1]$, say, for example, $t_0 \in [0,\sigma)$, such that

$$(T^*y)'(t_0) = (Ty)'(t_0) = L_1,$$
$$0 < (T^*y)'(t) = (Ty)'(t) < L_1, \quad t \in (t_0, \sigma]. \tag{13}$$

Then $(T^*y)(t)$ is a solution of the equation $x'' + f_1(t, x, x') = 0$ on the interval $[t_0, \sigma]$ with $0 \leqslant (T^*y)(t) \leqslant b$. From the definition of L_1, then

$$(T^*y)''(s) \geqslant -\frac{L_1^2}{2b}, \quad s \in [t_0, \sigma]. \tag{14}$$

Multiplying $(T^*y)'(s)$ on both sides of (14), one has

$$(T^*y)''(s)(T^*y)'(s) \geqslant -\frac{L_1^2}{2b}(T^*y)'(s), \quad s \in [t_0, \sigma). \tag{15}$$

Integrating (15) on $[t_0, \sigma]$, it follows that

$$-\frac{1}{2}\big[(T^*y)'(t_0)\big]^2 \geqslant -\frac{L_1^2}{2b}\big((T^*y)(\sigma) - (T^*y)(t_0)\big) > -\frac{L_1^2}{2},$$

and then $0 < (T^*y)'(t_0) < L_1$, a contradiction. Now it follows that

$$(T^*y)(t) = (Ty)(t), \quad 0 \leqslant t \leqslant 1,$$

which implies in turn $y(t) = (Ty)(t)$ and then $y(t)$ is a positive solution to BVP (3) satisfying

$$c < \alpha(y) < b, \qquad \big|y'(t)\big| < L. \qquad \square$$

References

[1] R. Ma, Positive solutions of a nonlinear three-point boundary value problem, Electron. J. Differential Equations 34 (1999) 1–8.

[2] R. Ma, Positive solutions for second order three-point boundary value problems, Appl. Math. Lett. 14 (2001) 1–5.
[3] R. Ma, Existence theorems for a second order m-point boundary value problem, J. Math. Anal. Appl. 211 (1997) 545–555.
[4] R. Ma, Existence of solutions of nonlinear m-point boundary value problems, J. Math. Anal. Appl. 256 (2001) 556–567.
[5] R. Ma, Positive solutions of a nonlinear m-point boundary value problem, Comput. Math. Appl. 42 (2001) 755–765.
[6] D. Anderson, Multiple positive solutions for a three-point boundary value problem, Math. Comput. Modelling 27 (1998) 49–57.
[7] X. He, W. Ge, Triple solutions for second order three-point boundary value problems, J. Math. Anal. Appl. 268 (2002) 256–265.
[8] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, San Diego, 1988.
[9] R.W. Leggett, L.R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, Indiana Univ. Math. J. 28 (1979) 673–688.
[10] J. Dugundji, An extension of Tietze theorem, Pacific J. Math. 1 (1951) 353–367.
[11] R.I. Avery, D.R. Anderson, Fixed point theorem of cone expansion and compression of functional type, J. Difference Equations Appl. 8 (2002) 1073–1083.