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PERIOD-DOUBLING IN DISCRETE RELATIVE SPATIAL DYNAMICS AND THE FEIGENBAUM SEQUENCE

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Abstract—It is shown in this paper that although the period-doubling Feigenbaum sequence and the associated universal numbers in discrete maps of the logistic type hold over parameters, their true nature have them holding over slopes of the corresponding Poincaré maps. This finding enables one to find these Feigenbaum *slope* sequences in more complex maps. Further, it is demonstrated by an example in discrete relative growth spatial dynamics that a Feigenbaum sequence does *not* hold over the bifurcation parameter.

1. INTRODUCTION

Recently [1, 2] a discrete map has been presented for socio-spatial dynamic processes of the type

$$x(t+1) = \frac{1}{1 + AF[x(t)]},$$

$$0 < x(t+1) < 1,$$

$$A, F[x(t)] > 0,$$

where the relative accumulation of one stock is modeled between two regions in a temporally and spatially interdependent manner within a spatially prespecified (geographically defined) environment. According to this algorithm a one-time-period step memory in the temporal interdependence is assumed to govern the accumulation–decumulation of the (single) stock (for example, population) over the two-location space. The process is a function of some (comprehensive, phenomenological) environmental fluctuations depicted by parameter A; function $F[x(t)]$ specifies the comparative advantages of concentration–deconcentration of the stock at one location vs the other, at any time period t. Extensions of the algorithm to n locations and m stocks, as well as a more full exposition of this dynamic process are presented in Dendrinos and Sonis [3].

This algorithm unraveled certain innovative mathematical aspects associated with period-doubling cycles, turbulence and its (deterministically) chaotic motion, etc. Some of these findings are documented in Refs[1–3]. Empirical evidence regarding the applicability of the algorithm for the relative population dynamics of the U.S. regions over the period 1850–1983 together with implications for forecasting are presented in Dendrinos and Sonis[4].

Here, two major points are made: first, the generally attributed to *parameters* Feigenbaum sequence found in period-doubling cycles of discrete maps ought to be attributed to the *slopes* of their corresponding Poincaré maps. Second, the bifurcation parameter sequence for period-doubling cycles found in more complicated maps than the logistic one (like for example in the two-location, one-stock model of discrete relative spatial dynamics) does *not* obey the Feigenbaum sequence, although their slope sequence does.

These two findings have substantial and methodological implications for socio-spatial dynamics, far transcending their purely mathematical significance. Some of these implications are addressed at the end of this paper. In Section 2, the Feigenbaum sequence as extended by Helleman is revisited; then, in Section 3, the discrete relative spatial dynamics algorithm is discussed where it is shown that the Feigenbaum sequence is a slope sequence; in Section 4 a specific case is presented where the Feigenbaum sequence is shown not to hold over the ranges of the bifurcation parameter where period-doubling behavior is observed.

2. THE FEIGENBAUM–HELLEMAN MODEL

Feigenbaum considers a *one-parameter* family of maps $x(t+1) = F_a[x(t)]$, where F_a is the one-parameter function; in Feigenbaum[5, p. 135; 6, p. 5] it is shown that, when analyzing the "prototypic" family

$$F_a[x(t)] = 4ax(t)[1 - x(t)],$$

then there are *values of the parameter a*, denoted by a_n at which its period doubles for the nth time, at geometrically changing intervals. Feigenbaum shows that the presence of a universal constant (δ) such that

$$a_\infty - a_n \propto \delta^{-n},$$

or, put differently,

$$\delta_n \equiv \frac{a_{n+1} - a_n}{a_{n+2} - a_{n+1}},$$

which as $n \to \infty$ results in (numerically) $\delta = 4.6692016\ldots$

Helleman[7] extends Feigenbaum and summarizes his findings by stating that "as we vary some parameter which may be energy, or the Reynolds number, a period 2^k orbit 'bifurcates' from a period 2^{k-1} orbit." This is the so-called Feigenbaum sequence associated with *critical parameter values* (p. 232).

Helleman dealt with conservative and dissipative systems of a more general form than Feigenbaum's prototypic family; specifically, he analyzed two-dimensional maps of the family

$$X(t+1) = f\{X(t), Y(t)\}, \qquad Y(t+1) = g\{X(t), Y(t)\}, \qquad t = 0, 1, 2, \ldots$$

where f and g are analytic nonlinear functions with a constant Jacobian

$$\det \frac{\partial\{X(t+1), Y(t+1)\}}{\partial\{X(t), Y(t)\}} = B = \text{constant}.$$

He analyzed conservative ($B = \pm 1$) and dissipative ($|B| < 1$) mappings. Examining the local behavior about a periodic orbit, due to the constancy of the determinant the Helleman system reduces to a "standard form":

$$y(t+1) + By(t-1) = 2Cy(t) + 2y(t)^2,$$

where $y(t)$ is the small deviation from the periodic orbit, and C is some new parameter. Helleman showed that, if $B = 0$ *and* $2C = a$ [7, footnote 38], then the above map reduces to the logistic model, first examined by May[8]:

$$x(t+1) = ax(t)[1 - x(t)].$$

We concentrate (without loss of generality) on the dissipative case (since as we demonstrate later, the basis for our contribution is $B = 0$).

As the bifurcation parameter C changes, period doubling occurs in such a manner that

$$C_n \underset{r}{=} -2C_{n+1}^2 + 2C_{n+1} + 2,$$

where the symbol $\underset{r}{=}$ implies "renormalization". The above results analytically in a constant toward which C_n converges geometrically (the dissipative Feigenbaum constant δ):

$$\delta \underset{r}{=} -4C_\infty + 2 = 1 + \sqrt{17} \underset{r}{\simeq} 5.12\ldots$$

while *numerically* $\delta \simeq 4.6692016091029909\ldots$ [7, p. 240]. For the logistic equation, the upper bound for period-doubling cycles is

$$a_\infty = 2 - 2C_\infty = 3.5699\ldots\ .$$

Point 1. The above discussion by both Feigenbaum and Helleman, on the values of the bifurcation parameters ought to be restated in terms of values in the changes of the *slope* of the Poincaré map. Both Feigenbaum and Helleman's analysis holds for the slopes as well as the values of the bifurcation parameters; but this is so *only* because slope and parameter are linearly connected for the maps they analyze, and because of the finding made later at the end of Section 3.

Point 2. We will show that, for more complex maps the Feigenbaum sequence holds for their slopes but *not* their bifurcation parameters.

Consider the general map $x(t+1) = F\{x(t)\}$, where $x(t)$ is a probability, F is strictly positive and it depends on any number of parameters. Then the Taylor series expansion in quadratic approximation is

$$x(t+1) = x^* + s^*\{x(t) - x^*\} + \tfrac{1}{2}S^*\{x(t) - x^*\}^2,$$

where s^* is the slope of the Poincaré map

$$s^* = \left.\frac{\partial x(t+1)}{\partial x(t)}\right|_*$$

and S^* is the second derivatives of the map at the equilibrium point

$$S^* = \left.\frac{\partial^2 x(t+1)}{\partial x(t)^2}\right|_*.$$

If $x(t) = x^* - [2z(t)/S^*]$, then the Taylor series approximation becomes

$$z(t+1) = s^* z(t)[1 - z(t)],$$

which is the logistic difference equation, the Feigenbaum's "prototypic" map. Thus it is the *slope* over which the universal numbers and Feigenbaum's sequence is connected. Q.E.D.

Thus, the above comment indicates that the period-doubling Feigenbaum sequence is universal for all types of maps, and not only for those for which the slope of the Poincaré map and the (bifurcation) parameter are linearly related.

3. THE DISCRETE RELATIVE DYNAMICS

Consider the two-dimensional dissipative map

$$x_1(t+1) = \frac{1}{1 + AF\{x_1(t), x_2(t)\}} > 0,$$

$$x_2(t+1) = 1 - x_1(t+1) > 0,$$

where A and F are strictly positive, which (without loss of generality) can be converted into the one-dimensional map

$$x(t+1) = \frac{1}{1 + AF\{x(t)\}} > 0,$$

$$A, F\{x(t)\} > 0,$$

where F is any arbitrary function of $x(t)$ and a parameter vector $\mathbf{a}$.
The Poincaré map's slope is at equilibrium:

$$s^* = Ax^{*2}\frac{\mathrm{d}F(x^*)}{\mathrm{d}x^*} = -x^*(1-x^*)\frac{\mathrm{d}}{\mathrm{d}x^*}\ln F(x^*),$$

with a determinant $B = 0$; the quadratic *approximate* map has the explicit solution for the 2-period cycles

$$x^*(0) = x^* - \frac{1 + s^* - \sqrt{(1-s^*)(s^*-3)}}{S^*},$$

$$x^*(1) = x^* - \frac{1 + s^* + \sqrt{(1+s^*)(s^*-3)}}{S^*},$$

with the conditions for its existence being $s^* < -1$ or $s^* > 3$. Its domain of stability is (see Appendix 1)

$$-1 < 4 + 2s^* - s^{*2} < 1$$

with boundaries found when $|4 + 2s^* - s^{*2}| = 1$. This results in the two intervals

$$s^*_-(1) = -1.4495\ldots \simeq 1 - \sqrt{6} < s^* < -1 = s^*_-(0),$$

$$s^*_+(0) = 3.4495\ldots \simeq 3 < s^* < 1 + \sqrt{6} = s^*_+(1).$$

The values $(1 - \sqrt{6})$ and $(1 + \sqrt{6})$ are the initial thresholds for the two Feigenbaum *slope* sequences corresponding to the slope being negative and positive, respectively. Notice that 3.4495... is the exact (because of the exactness involved in the Taylor series expansion) value of the *parameter a* of the May logistic model $x(t+1) = ax(t)[1 - x(t)]$. This is *not* necessarily due to the linear dependence of the parameter a with the slope at equilibrium, as one is the assume to justify Feigenbaum and Helleman's statements regarding the Feigenbaum sequence on the parameter.
From the above we further obtain the relationship between $s^*(0)$ and $s^*(1)$ as

$$\pm s^*_{\pm}(0) = 4 + 2s^*_{\pm}(1) - s^*_{\pm}(1)^2.$$

Renormalization produces for the quadratic approximation of our map

$$\pm s^*_\pm(n) = 4 + 2s^*_\pm(n+1) - s^*_\pm(n+1)^2,$$

or

$$s^*_+(n+1) = 1 + \sqrt{3 + s^*_+(n)}; \qquad s^*_- = 1 - \sqrt{5 - s^*_-(n)}.$$

The two Feigenbaum slope sequences, as a result, are

for $s^*_-(n)$: $-1,\ -1.4495\ldots,\ -1.5396\ldots,\ -1.5573\ldots$

for $s^*_+(n)$: $1,\ 3,\ 3.4495\ldots,\ 3.5396\ldots,\ 3.5573\ldots$

with their limits emerging out of the solution of the equation, as $n \to \infty$,

$$\pm \bar{s}_\pm = 4 + 2\bar{s}_\pm - \bar{s}^2_\pm$$

$$\bar{s}_+ = \frac{3 + \sqrt{17}}{2}, \qquad \bar{s}_+ \simeq 3.5616\ldots$$

$$\bar{s}_- = \frac{1 - \sqrt{17}}{2}, \qquad \bar{s}_- \simeq -1.5616\ldots$$

The two slope sequences are symmetric about the point 2:

$$s_+(n) = 2 - s_-(n).$$

In the May logistic model $x(t+1) = ax(t)[1 - x(t)]$ the linear dependence between slope and parameter a is $s^* = 2 - a$ which happens to resemble the above symmetry condition between $s_+(n)$ and $s_-(n)$. Thus, the period-doubling Feigenbaum *parameter* sequence, in the case of the logistic discrete map, happens to coincide exactly with (only) one *slope* sequence. This is the comment supplementing point 2 above.

The geometric Feigenbaum universal constant of *slopes* (as also produced by Helleman) is approximated from the above analysis by $\delta = 1 + \sqrt{17} \simeq 5.12\ldots$ while numerically it is exactly the Feigenbaum constant $\delta = 4.6692016\ldots$ This is shown in Appendix 2. May and Oster[9] present a better approximation for δ, analytically, than the value 5.12... They show that $\delta = 2 + \sqrt{2} = 4.43\ldots$ by renormalization over the first two successive bifurcations. One conceivably could improve our analysis by either replicating the May–Oster analysis, or by considering even closer bifurcations to the limit (i.e. close to the end of the period-doubling process).

4. THE NONGEOMETRIC SEQUENCE OF PARAMETER VALUES FOR PERIOD-DOUBLING CYCLES IN DISCRETE GENERAL MAPS

In this section we demonstrate, with the help of a specific example taken from a family of discrete relative log–linear maps, that the Feigenbaum sequence does not hold for parameter(s) during the period-doubling process. This finding strengthens further our case for associating the Feigenbaum sequences with slopes rather than parameters of discrete maps.

Consider the log–linear specifications of the previously presented discrete relative dynamics:

$$F\{x(t)\} = x(t)^a\{1 - x(t)\}^b,$$

$$-\infty \leqslant a, b \leqslant +\infty.$$

Interpretation of a, b (elasticities of accumulation) is found in Dendrinos and Sonis[3]. In this case the following conditions hold:

$$s^* = (a+b)x^* - a \rightarrow x^* = \frac{s^* + a}{a+b},$$

$$A = (1 - x^*)^{1-b} x^{*-(a+1)}.$$

The complexity of these formulas prevents us from arriving at a general statement. However, the above conditions for specific points in the (a, b) space can be solved for analytically to make the point. For example, consider the case where $a = 2 + b$, located in the region of the parameters space where period-doubling cycles occur; it results in the following 2-period cycle equilibria:

$$A_{(2)}^{-1/b+1} = x_{(2)}^*(1 - x_{(2)}^*),$$

$$x_{(2)}^*(0) = \frac{1}{2} - \sqrt{\frac{1}{4} - A_{(2)}^{-1/b+1}},$$

$$x_{(2)}^*(1) = \frac{1}{2} + \sqrt{\frac{1}{4} - A_{(2)}^{-1/b+1}},$$

where $A_{(2)}$ is the critical threshold of the positive parameter A where the 2-period cycle commences; $x_{(2)}^*(0), x_{(2)}^*(1)$ are the two equilibria of the state variable comprising the 2-period cycle. The associated slopes are given by

$$s_{(2)}^* = \{(a+b)x_{(2)}^*(0) - a\}\{(a+b)x_{(1)}^*(1) - a\},$$

$$s_{(2)}^* = 1 - (a+b)^2\Delta = 1 - 4(b+1)^2\Delta,$$

where Δ is given by

$$\Delta = \frac{1}{4} - A_{(2)}^{-1/b+1}.$$

For a full derivation see Dendrinos and Sonis[3, Chapter II, Section B.6].

From the above one obtains directly that at this point in the two-parameter space (a, b) the correspondence between the bifurcation parameter A and the slope $s_{(n)}^*$ at any 2^n-period cycle is

$$A_{(n)} = \left\{\frac{1}{4}\frac{(b+1)^2 + s_{(n)}^* - 1}{(b+1)^2}\right\}^{-(b+1)},$$

identifying a nonlinear relationship between the two, which is *not* geometric. Since this is only one specific point in the parameter space (a, b) the qualitative property emerges which has the (whatever) sequence applicable at any point in the (a, b) space to be altered as one moves around the neighborhood of this point. Although not derived by the authors, it might be of interest to identify (analytically or numerically) the transition properties among various kinds of turbulence found in this algorithm's parameter (a, b) space.

5. IMPLICATIONS AND CONCLUSIONS

Since the bifurcation parameter A of the discrete relative spatial dynamics is interpreted as an environmentally fluctuating factor; and since the slope of the Poincaré map $\{[\partial x(t+1)]/[\partial x(t)]\}$ of the discrete relative spatial concentration onto one location (and simultaneous deconcentration from the other) of the (single in this case) stock; then the direct implications of the above analysis are that:

(1) there is a connection between environmental fluctuations and speeds in relative concentration–deconcentration; it is at least log–linear (in the simplest case) and in general of a transcendental form; and

(2) the simple geometric progression found to be applicable over the speeds of relative concentration–deconcentration during period-doubling phases in the turbulent regions of discrete relative dynamics may be the outcome of much more complex changes in environmental (fluctuating parameter A) behavior. The inference here is that very complex environmental processes may result in rather simple patterns in the speeds of concentration–deconcentration in relative distribution of various stocks over space and time; the opposite also holds, so that small changes in the environmental fluctuations may result in drastic changes in the speeds of relative accumulation–decumulation of stocks.

Concluding, we point out to the various extensions possible in the above analysis, particularly in improving the analytical approximation to the period-doubling Feigenbaum constant of *slopes*; and at the search for analytically tracking the connection between A and s^* at other points in the (a, b) space in an effort toward deriving deterministic (analytic or numerical) rules governing transition from one kind of turbulence to another in various regions of the *very slow* changing (a, b) parameters.

These findings may also be of interest to physicists, theoretical and experimental, in that they identify variance in the behavior of these two entities, environmental fluctuations (slow moving parameter) and their resulting gradients (slopes, or speeds of motion, i.e. the fast moving variables).

REFERENCES

1. M. Sonis and D. S. Dendrinos, A discrete relative growth model: switching, role reversal and turbulence. In *International Perspectives of Regional Decentralization* (Edited by P. Friedrich and I. Messer). Nomos, Baden-Baden (1987).
2. D. S. Dendrinos and M. Sonis, The onset of turbulence in discrete relative multiple spatial dynamics. *J. appl. Math. Comput.* **22,** 25–44 (1987).
3. D. S. Dendrinos and M. Sonis, *Turbulence and Socio-Spatial Dynamics: Toward a Structural Theory of Social Systems Evolution—Series in Synergetics.* Springer, New York (submitted).
4. D. S. Dendrinos and M. Sonis, Nonlinear discrete relative population dynamics of the U.S. regions. *J. appl. Math. Comput.* (in press).
5. M. Feigenbaum, Low dimensional dynamics and the period-doubling scenario. In *Dynamical Systems and Chaos* (Edited by H. Araki *et al.*) *Lecture Notes in Physics*, Vol. 179, pp. 131–148. Springer, New York (1983).
6. M. Feigenbaum, Universal behavior in nonlinear systems. *Los Alamos Sci.* Summer, 4–26 (1980).
7. R. H. G. Helleman, Feigenbaum sequences in conservative and dissipative systems. In *Chaos and Order in Nature* (Edited by H. Haken)—*Series in Synergetics*, Vol. 11, pp. 232–248. Springer, New York (1981).
8. R. May, Simple mathematical models with very complicated dynamics. *Nature, Lond.* **261,** 459–407 (1976).
9. R. M. May and G. F. Oster, Period-doubling and the onset of turbulence, an analytical estimate of the Feigenbaum ratio. *Phys. Lett.* **78A,** 1–3 (1980).

APPENDIX 1

The Approximate Domains of Stability for the 2-Period Cycle

From the approximation

$$x(t+1) = x^* + s^*\{x(t) - x^*\} + \frac{1}{2}S^*\{x(t) - x^*\}\}^2$$

and the formulas

$$x^*(0, 1) = x^* - \frac{1 + s^* \pm \sqrt{(1 + s^*)(s^* - 3)}}{S^*}$$

we obtain

$$x(t+1) - x^*(0) = (s^* + S^*u_1)\{x(t) - x^*(1)\} + \frac{1}{2}S^*\{x(t) - x^*(1)\}^2,$$

$$x(t+1) - x^*(0) = (s^* + S^*u_0)\{x(t) - x^*(0)\} + \frac{1}{2}S^*\{x(t) - x^*(0)\}^2,$$

where

$$u_0 = x^*(0) - x^*, \qquad u_1 = x^*(1) - x^*.$$

Thus, the second iterate is

$$x(t+2) - x^*(1) = (s^* + S^*u_0)(s^* + S^*u_1)\{(x(t) - x^*(1)\}$$
$$+ \frac{1}{2}S^*[(s^* + S^*u_0) + (s^*u_1)^2]\{x(t) - x^*(1)\}^2 + \cdots \text{higher order terms.}$$

Analogously, one obtains

$$x(t+2) - x^*(0) = (s^* + S^*u_0)(s^* + S^*u_1)\{x(t) - x^*(0)\}$$
$$+ \frac{1}{2}S^*[(s^* + S^*u_1) + (s^* + S^*u_0)^2]\{x(t) - x^*(0)\}^2 + \cdots \text{higher order terms.}$$

The coefficient, $(s^* + S^*u_0)\ (s^* + S^*u_1)$, of the linear term of the above two expressions must be the slope of the second iterate of their Taylor series expansion. Thus, the slopes of the second iterates at equilibrium are

$$s_2^*\{x^*(0)\} = s_2^*\{x^*(1)\} = 4 + 2s^* - s^{*2}. \qquad \text{Q.E.D.}$$

APPENDIX 2

The Derivative of the Universal Slope Sequence Number

The geometric rate of conversion to the upper (lower) *slope* bound is given by

$$\lim_{n\to\infty} \delta_n \equiv \lim_{n\to\infty} \frac{s_\pm(n) - s_\pm(n-1)}{s_\pm(n+1) - s_\pm(n)} = \delta,$$

where asymptotically

$$s_\pm(n) \underset{n\to\infty}{\simeq} \bar{s}_\pm + \{s_\pm(0) - \bar{s}_\pm\}\delta^{-n}.$$

This, if inserted in the original quadratic equation on slopes

$$\pm s_\pm^*(n) = 4 + 2s_\pm^*(n+1) - s_\pm^*(n+1)^2$$

produces:

$$\delta = \pm 2(\bar{s}_\pm - 1) = 1 + \sqrt{17} = 5.12\ldots$$

which is identical to Helleman's finding from his (and our) approximation.