A Limit Theorem for Perturbed Operator Semigroups
with Applications to Random Evolutions*

THOMAS G. KURTZ

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706
Communicated by Tosio Kato
Received February 24, 1972

Let $U(t)$ and $S(t)$ be strongly continuous contraction semigroups on a Banach space $L$ with infinitesimal operators $A$ and $B$, respectively. Suppose the closure of $A + \alpha B$ generates a semigroup $T_\alpha(t)$. The behavior of $T_\alpha(t)$ as $\alpha$ goes to infinity is examined. In particular, suppose $S(t)$ converges strongly to $P$. If the closure of $PA$ generates a semigroup $T(t)$ on $\mathcal{A}(P)$, then $T_\alpha(t)$ goes to $T(t)$ on $\mathcal{A}(P)$. If $PA = 0$ and $BF = -f$ for $f \in \mathcal{A}(P)$, conditions are given that imply $T_\alpha(\alpha t)$ converges on $\mathcal{A}(P)$ to a semigroup generated by the closure of $PA^A$.

The results are used to obtain new and known limit theorems for random evolutions, which in turn give approximation theorems for diffusion processes.

1. Introduction

The primary motivation behind the work presented here is the idea of a random evolution first introduced by Griego and Hersh [1] and developed further in [2–4, 10]. Intuitively, a random evolution describes a situation in which a Markov process $X(t)$ "controls" the development of another process, the other process being described by operators on a Banach space $L$.

In particular, let $X(t)$ be a right continuous, temporally homogeneous Markov process defined on a probability space $(\Omega, \mathcal{F}, P)$, taking values in a measurable state space $(E, \mathcal{B})$ with transition function $P(t, x, \cdot)$. Let $S(t)$ be the semigroup on $B(E, \mathcal{B})$, the space of bounded measurable functions on $E$, given by $S(t)f(x) = \int f(y) P(t, x, dy)$, and let $B$ denote its infinitesimal operator.

To each state $x \in E$, suppose there corresponds a strongly continuous semigroup of linear operators $T_x(t)$ on a Banach space $L$, with infinitesimal operator $A_x$. $(B(L)$ will denote the space of bounded

* Research supported in part by the National Science Foundation.
linear operators on \( L \). If in addition, \( X(t) \) is a pure jump process, \( \xi_0, \xi_1, \xi_2, \ldots \) is the sequence of states visited by \( X(t) \), \( \Delta_0 \Delta_1 \Delta_2 \cdots \) is the time spent at each of these visits, \( N(t) \) is the number of transitions before time \( t \), and

\[
\Delta_t = t - \sum_{k=0}^{N(t)-1} \Delta_k,
\]

then the random evolution

\[
T(t, \omega): [0, \infty) \times \Omega \to B(L)
\]

is given by

\[
T(t, \omega) = T_{\xi_0}(\Delta_0) T_{\xi_1}(\Delta_1) \cdots T_{\xi_{N(t)}}(\Delta_t).
\] (1.1)

Let \( \mathcal{L} \) be the Banach space of strongly measurable functions \( f: E \to L \) with \( \|f\| = \sup_x \|f(x)\|_L < \infty \). Then formally at least

\[
\mathcal{F}(t)f(x) = E(T(t, \omega) f(X(t, \omega)) \mid X(0) = x)
\] (1.2)

defines a semigroup of operators on \( \mathcal{L} \). (This is easily seen to be true if \( E \) is countable.)

The corresponding infinitesimal operator is (again formally)

\[
\mathcal{O}f(x) = A_x f(x) + B f(x).
\] (1.3)

(The meaning of this will be clarified below). If \( E \) is finite, say \( E = (1, 2, \ldots, n) \), then \( f(x) \) can be thought of as a vector \( (f_1 f_2 \cdots f_n) \) in \( L \times L \times \cdots \times L \equiv L^n \) and

\[
\mathcal{O}f = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & A_n & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} + Q \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix},
\] (1.4)

where \( Q \) is the matrix of infinitesimal parameters for \( X(t) \).

Proceeding a little more carefully, note that

\[
U(t)f(x) = T_{\omega}(t) f(x)
\] (1.5)

defines a semigroup on \( \mathcal{L} \) whose infinitesimal operator is a restriction of

\[
A f(x) = A_x f(x)
\] (1.6)
and \( S(t) \) can be considered as a semigroup on \( \mathcal{L} \) simply by defining
\[
S(t)f(x) = \int f(y) P(t, x, dy)
\]
(1.7)

for \( f \in \mathcal{L} \).

Let \( f_1 f_2 \cdots f_n \in \mathcal{L}, \eta_1 \eta_2 \cdots \eta_n \in \mathcal{D}(B) \subset B(E, \mathcal{B}) \), and
\[
f = \sum_{k=1}^{n} \eta_k f_k.
\]

Then
\[
\lim_{t \to 0} \frac{1}{t} (S(t)f - f) = \sum_{k=1}^{n} B\eta_k f_k.
\]

This indicates the relationship between \( B \) and the infinitesimal operator of \( S(t) \) considered as a semigroup on \( \mathcal{L} \), which we will also denote by \( B \).

The Trotter product formula [11] suggests that \( \mathcal{F}(t) \) (defined in (1.2)) should satisfy
\[
\mathcal{F}(t)f(x) = \lim_{n \to \infty} \left[ U(t/n) S(t/n) \right]^n f(x).
\]
(1.8)

Conditions can be stated under which this limit is justified (in particular if \( E \) is finite), but that is not our concern here.

In the case \( E \) finite Pinsky [9], Griego and Hersh [1], Hersh and Pinsky [4], and Hersh and Papanicolaou [3] have given limit theorems for the semigroups \( \mathcal{F}_a(t) \) corresponding to the infinitesimal operators

\[
\mathcal{A}_a f = \alpha \left( \begin{array}{ccc}
A_1 & & \\
A_2 & \ddots & \\
& \ddots & \ddots \\
0 & \cdots & A_n
\end{array} \right) \left( \begin{array}{c}
f_1 \\
f_2 \\
\cdots \\
f_n
\end{array} \right) + \alpha a Q \left( \begin{array}{c}
f_1 \\
f_2 \\
\cdots \\
f_n
\end{array} \right)
\]
(1.9)

with \( \alpha \) going to infinity.

In Section 2 we will prove an abstract version of these theorems, in Section 3 we will apply the theorem to a particular class of random evolutions obtaining a method of approximating a large class of diffusion processes, and finally in Section 4 we will introduce the notion of a random evolution with feedback and give an example indicating that the limit theorem is applicable in this more general context.
On occasion it will be convenient to consider an operator (possibly multivalued) \( A \) as a set of ordered pairs \( A = \{(x, y) : 4x = y\} \). Then \( \mathcal{D}(A) = \{x : (x, y) \in A\} \) and \( \mathcal{R}(A) = \{y : (x, y) \in A\} \). By \( \lim_{n \to \infty} x_n = x \) and \( \lim_{n \to \infty} y_n = y \).

We will need the following two theorems in Section 2. The first is a variation of a theorem in [6].

**Theorem 1.10.** Let \( \{T_n(t)\} \) be a sequence of strongly continuous contraction semigroups on a Banach space \( L \), with infinitesimal operators \( \lambda = A \).

Define

\[
A = \{(f, g) : \exists (f_n, g_n) \in A_n \text{ with } \lim_{n \to \infty} (f_n, g_n) = (f, g)\}.
\]

(A may be multivalued.) Then there exists a strongly continuous contraction semigroup \( T(t) \) on \( \mathcal{D}(A) \) such that \( T(t)f = \lim_{n \to \infty} T_{n}(t)f, \) all \( f \in \mathcal{D}(A) \), if and only if \( \mathcal{R}(A - A) \subset \mathcal{D}(A) \). The infinitesimal operator of \( T(t) \) is the closure of

\[
\{(f, g) : (f, g) \in A \text{ and } f, g \in \mathcal{D}(A)\}.
\]

**Theorem 1.11** [5, p. 516]. Let \( S(t) \) be a strongly continuous semigroup on \( L \) with infinitesimal operator \( B \). Suppose

\[
\lim_{\lambda \to 0} \int_0^\infty e^{-\lambda t} S(t)f \, dt = Pf
\]

exists for every \( f \in L \). Then

\[
P \text{ is a bounded linear projection, i.e., } P^2 = P; \quad (1.12)
\]

\[
S(t)P = PS(t) = P \quad \text{all } t > 0; \quad (1.13)
\]

\[
\mathcal{R}(P) = \mathcal{N}(B) \text{ (the null space of } B); \quad (1.14)
\]

\[
\mathcal{R}(B) \text{ is dense in } \mathcal{N}(P); \quad (1.15)
\]

\[
BPf = 0 \quad \text{all } f \in L, \quad PBf = 0 \quad \text{all } f \in \mathcal{D}(B). \quad (1.16)
\]

**Remark.** If \( S(t) \) is the semigroup corresponding to a transition function \( P(t, x, \Gamma) \), then the projection \( P \) is given by

\[
Pf(x) = \int f(y) P(x, dy),
\]

where the measure \( P(x, \Gamma) \) is the limiting distribution of the process starting from \( x \), i.e., the weak limit as \( t \to \infty \) of \( P(t, x, \Gamma) \) (or \( 1/t \int_0^t P(s, x, \Gamma) \, ds \)).
2. Limit Theorems for Perturbed Semigroups

Let $U(t)$ and $S(t)$ be strongly continuous semigroups of linear contractions on a Banach space $L$ with infinitesimal operators $A$ and $B$, respectively. Suppose that for each sufficiently large $\alpha$, the closure of $A + \alpha B$ is the infinitesimal operator of a strongly continuous semigroup $T_\alpha(t)$ on $L$. (This implies $T_\alpha(t)f = \lim_{n\to\infty}[U(t/n)S(\alpha t/n)]^n f$ for all $f \in L$.) This is true, for example, if either $A$ or $B$ is bounded. For other conditions, see Trotter [11]. In addition assume that $B$ is the closure of $B$ restricted to $\mathcal{D}(A) \cap \mathcal{D}(B)$.

We are interested in what happens to $T_\alpha(t)$ as $\alpha$ goes to infinity.

**Theorem 2.1.** Let $U(t)$, $S(t)$ and $T_\alpha(t)$ be defined as above. Suppose $S(t)$ satisfies the conditions of Theorem 1.11. Let

$$D = \{f \in \mathcal{A}(P) : f \in \mathcal{D}(A)\},$$

and define $Cf = PAf$ for $f \in D$. Suppose $\mathcal{R}(\lambda - C) \subset D$ for some $\lambda > 0$. Then the closure of $C$ restricted so that $Cf \in D$ is the infinitesimal operator of a strongly continuous contraction semigroup $T(t)$ defined on $D$ and

$$\lim_{\alpha \to \infty} T_\alpha(t)f = T(t)f$$

for all $f \in D$.

**Proof.** Let $C_\alpha$ be the closure of $A + \alpha B$. (i.e., the infinitesimal operator of $T_\alpha(t)$.) By Theorem 1.10 it will be sufficient to show that

$$\{(f, g) : \exists (f_\alpha, g_\alpha) \in C_\alpha \text{ with } \lim_{\alpha \to \infty} (f_\alpha, g_\alpha) = (f, g)\}$$

contains the pairs $(f, Cf)$ all $f \in D$.

Since $B$ is the closure of $B$ restricted to $\mathcal{D}(A) \cap \mathcal{D}(B)$, for $g \in \overline{\mathcal{D}(B)}$ there exists $h_\alpha \in \mathcal{D}(A) \cap \mathcal{D}(B)$ such that $\lim_{\alpha \to \infty} B h_\alpha = g$. In addition, $h_\alpha$ can be selected so that $\|Ah_\alpha\| = o(\alpha)$, and hence

$$\lim_{\alpha \to \infty} (1/\alpha) C_\alpha h_\alpha = g.$$

For $f \in D$, (1.15) implies $PAf - Af$ is in $\overline{\mathcal{D}(B)}$. Let $h_\alpha \in \mathcal{D}(A) \cap \mathcal{D}(B)$ satisfy $\lim_{\alpha \to \infty} (1/\alpha) C_\alpha h_\alpha = PAf - Af$ and define

$$f_\alpha = f + (1/\alpha) h_\alpha.$$

Noting that $C_\alpha f = Af$, the theorem follows.

**Remark.** In the case of random evolutions a much stronger version of this theorem is given in [8].

Most of the work on random evolutions up to this point has been concerned with the case corresponding to $PAf = 0$ all $f \in D$. (In
this case the limiting semigroup is just the identity for all \( t \).) Limit theorems have been obtained under a renormalization of the time scale [1–4, 9], and an abstract version of these theorems is given below. The distinction between these two theorems is analogous to the distinction between the Law of Large Numbers and the Central Limit Theorem.

**THEOREM 2.2.** Let \( U(t) \), \( S(t) \) and \( T_\alpha(t) \) be defined as above. Suppose \( S(t) \) satisfies the conditions of Theorem 1.11. Let \( D \) and \( C \) be defined as in Theorem 2.1 and suppose \( Cf = 0 \) all \( f \in D \). Let

\[
D_0 = \{ f \in D : \exists h \in \mathcal{D}(A) \cap \mathcal{D}(B) \text{ with } Bh = -Af \}. \tag{2.3}
\]

Define \( C_0f = PAh \) for \( f \in D_0 \). Suppose \( \mathcal{D}(\lambda - C_0) \supset D_0 \) for some \( \lambda > 0 \). Then the closure of \( C_0 \) restricted so that \( C_0f \in D_0 \) is the infinitesimal operator of a strongly continuous contraction semigroup \( T(t) \) defined on \( D_0 \) and \( \lim_{t \to \infty} T_\alpha(at)f = T(t)f \) for all \( f \in D_0 \).

**Proof.** As before, let \( C_\alpha \) denote the infinitesimal operator for \( T_\alpha(t) \). Then \( \alpha C_\alpha \) is the infinitesimal operator for \( T_\alpha(at) \). We must show that \( \{ (f, g) : \exists (f_\alpha, g_\alpha) \in C_\alpha \text{ with } \lim_{\alpha \to \infty} (f_\alpha, g_\alpha) = (f, g) \} \) contains the pairs \( (f, C_0f) \) for all \( f \in D_0 \).

Let \( f \in D_0 \) and let \( h \in \mathcal{D}(A) \cap \mathcal{D}(B) \) satisfy \( Bh = -Af \). Let \( h_\alpha \in \mathcal{D}(A) \cap \mathcal{D}(B) \) satisfy \( \lim_{\alpha \to \infty} C_\alpha h_\alpha = PAh - Ah \). (The existence of the \( h_\alpha \) was verified in the proof of Theorem (2.1).) Let

\[
f = f + (1/\alpha)h + (1/\alpha^2)h_\alpha.
\]

Then

\[
\alpha C_\alpha f_\alpha = \alpha Af + Ah + \alpha Bh + (1/\alpha) C_\alpha h_\alpha = Ah + (1/\alpha) C_\alpha h_\alpha.
\]

Therefore \( \lim_{\alpha \to \infty} \alpha C_\alpha f_\alpha = PAh \).

**Remark.** Suppose \( \int_0^\infty \| (S(t) - P)f \| \, dt < \infty \) for all \( f \in L \). Then \( Pg = 0 \) implies the solution \( Bh = -g \) is given by \( h = \int_0^\infty (S(t) - P)g \, dt \). As far as the definition of \( D_0 \) is concerned, this indicates how to solve \( Bh = -Af \), but the requirement that \( h \in \mathcal{D}(A) \) must still be satisfied.

## 3. Random Evolutions

Let \( X(t) \) be a temporally homogeneous Markov process with measurable state space \( (E, \mathcal{B}) \) and transition function \( P(t, x, \Gamma) \).

Suppose for each \( x \in E \), \( T_\alpha(t) \) is a strongly continuous contraction
semigroup on a Banach space \( L \). Let \( \mathcal{L} \) be the Banach space of bounded strongly measurable functions \( f: E \to L \) with \( \| f \| = \sup_x \| f(x) \|_L \). Let \( \mathcal{L}_0 \) be a subspace of \( \mathcal{L} \) and suppose

\[
U(t) f(x) = T_x(t) f(x)
\]

and

\[
S(t) f(x) = \int_E f(y) P(t, x, dy)
\]

define strongly continuous semigroups on \( \mathcal{L}_0 \), with infinitesimal operators \( A \) and \( B \), respectively. Suppose the closure of \( A + \alpha B \) is the infinitesimal operator of a strongly continuous semigroup \( \mathcal{T}_a(t) \) on \( \mathcal{L}_0 \) and that \( R \) is the closure of \( R \) restricted to \( \mathcal{D}(A) \cap \mathcal{D}(B) \).

The results of Section 2 are of course applicable in this situation. Unfortunately, for many processes

\[
\lim_{\lambda \to 0} \lambda \int_0^\infty e^{-\lambda t} \int_E f(y) P(t, x, dy) \, dt
\]

exists pointwise or (if \( E \) is locally compact) uniformly on compact sets, but not uniformly on all of \( E \). Therefore, the following generalizations of Theorems 2.1 and 2.2 may be useful. They are proved using the results in [7] in place of Theorem 1.10. Convergence of bounded sequences, uniformly on compact sets will be denoted by \( \text{buc–lim} \). Similar theorems could be proved for bounded pointwise convergence, but the conditions would be more complicated. Theorem 1.11 generalizes easily to bounded pointwise convergence or to convergence of bounded sequences, uniformly on compact sets.

**Theorem 3.1.** Let \( U(t) \), \( S(t) \) and \( \mathcal{T}_a(t) \) be defined as above. Suppose \( E \) is locally compact and that for every compact \( K \subseteq E \) and every \( \varepsilon > 0 \), there is a compact set \( K_\varepsilon \subseteq E \) such that

\[
\sup_t \sup_{x \in K} P(t, x, K_\varepsilon) < \varepsilon.
\]

Suppose

\[
\text{buc–lim} \lambda \int_0^\infty e^{-\lambda t} S(t) f \, dt = Pf
\]

exists for all \( f \in \mathcal{L}_0 \). Let

\[
D = \{ f \in \mathcal{R}(P) : f \in \mathcal{D}(A) \},
\]
and define \( C_f = P A f \) for \( f \in D \). Suppose \( \mathcal{R}(\lambda - C) \supset \bar{D} \) for some \( \lambda > 0 \). Then the closure of \( C \) restricted so that \( C_f \in \bar{D} \) is the infinitesimal operator of a strongly continuous contraction semigroup \( \mathcal{T}(t) \) defined on \( D \) and \( \text{buc-lim}_{\lambda \to \infty} \mathcal{T}_{\lambda}(\alpha t) f = \mathcal{T}(t)f \) for all \( f \in \bar{D} \).

**Theorem 3.2.** Under the conditions of Theorem 3.1, suppose \( C_f = 0 \) all \( f \in D \). Let

\[
D_0 = \{ f \in D : \exists h \in \mathcal{D}(A) \cap \mathcal{D}(B) \text{ with } Bh = -Af \}.
\]

Define \( C_0 f = PAh \) for \( f \in D_0 \). Suppose \( \mathcal{R}(\lambda - C_0) \supset \bar{D}_0 \) for some \( \lambda > 0 \). Then the closure of \( C_0 \) restricted so that \( C_0 f \in \bar{D}_0 \) is the infinitesimal operator of a strongly continuous contraction semigroup \( \mathcal{T}(t) \) defined on \( \bar{D}_0 \) and \( \text{buc-lim}_{\lambda \to \infty} \mathcal{T}_{\lambda}(\alpha t) f = \mathcal{T}(t)f \) for all \( f \in \bar{D}_0 \).

**Remark.** In the above theorems \( \mathcal{L}_0 \) can be taken to be a space on which \( U(t) \) and \( S(t) \) are "buc-continuous," and \( A \) and \( B \) the operators defined by

\[
Af = \text{buc-lim}_{t \to 0} [(U(t)f - f)/t], \quad Af \in \mathcal{L}_0,
\]

and

\[
Bf = \text{buc-lim}_{t \to 0} [(S(t)f - f)/t], \quad Bf \in \mathcal{L}_0,
\]

provided

\[
\int_0^\infty e^{-M} S(t)f \, dt \quad \text{and} \quad \int_0^\infty e^{-M} U(t)f \, dt
\]

are in \( \mathcal{L}_0 \) for all \( f \) in \( \mathcal{L}_0 \).

**Example 3.3.** Let \( E \) be a compact topological space and assume \( \mathcal{B} \) is the \( \sigma \)-algebra of Borel sets. For each fixed \( t > 0 \) assume that \( \{P(t, x, \Gamma) : \Gamma \in \mathcal{B}\} \) is an equicontinuous family of functions. This implies that \( S(t) \) is a compact operator on the space of bounded measurable real valued functions, and assuming further that \( \lim_{t \to \infty} S(t)f = Pf \) exists and is independent of \( x \) for every bounded, measurable, real valued \( f \) (and hence for every \( f \in \mathcal{L} \)),

\[
\int_0^\infty \| S(t)f - Pf \| \, dt < \infty \quad \text{for all } f \in \mathcal{L}.
\]

Assume that \( \lim_{t \to 0} P(t, x, \Gamma) = 1 \) for every open set \( \Gamma \) with \( x \in \Gamma \). This implies \( S(t) \) is strongly continuous on the subspace of bounded continuous \( L \)-valued functions. Let this space be \( \mathcal{L}_0 \).
Assume that $T_x(t)f$ is a continuous function of $x$ for each $f \in L$, 
(this implies $U(t)$ is strongly continuous on $L_0$) and that $A_xf$ is a
continuous function of $x$ for all $f \in \mathcal{D}(A_x)$.

Finally assume that $\mathcal{D}(A_x)$ is dense in $L$ and that for sufficiently
large $x > 0$, $\mathcal{R}(\lambda - (A + \alpha B))$ is dense in $L_0$ for some $\lambda > 0$. (Note
that for $f \in \mathcal{D}(A)$ $Af(x) = A_xf(x)$.) The density of $\mathcal{D}(A_x)$ in $L$
implies $A + \alpha B$ is densely defined for all $\alpha > 0$ and that $B$ is
the closure of $B$ restricted to $\mathcal{D}(A) \cap \mathcal{D}(B)$. The fact that $A + \alpha B$ is
densely defined and $\mathcal{R}(\lambda - (A + \alpha B))$ is dense in $L_0$ some $\lambda > 0$
implies the closure of $A + \alpha B$ generates a strongly continuous
contraction semigroup $T_x(t)$ on $L_0$.

The operators $Pf$ and $Vf = \int_0^\infty (S(t) - P)f$ can be written as
integrals:

$$Pf = \int f(y) \mu(dy)$$

where $\mu$ is the stationary measure for $X(t)$;

$$Vf(x) = \int f(y) \nu(x, dy),$$

where $\nu(x, \Gamma) = \int_0^\infty (P(t, x, \Gamma) - \mu(\Gamma)) dt$.

Identifying $f \in L$ with the function in $\mathcal{L}$ that is identically $f$,
$\mathcal{R}(P) = L$ and $Cf = \int A_xf \mu(dx)$ for $f \in \mathcal{D}(A_x) = D$.

If $Cf = 0$ all $f \in D$ then $C_0f$ is given by

$$C_0f = \int A_x \left[ \int A_yf(x, dy) \right] \mu(dx)$$

for all $f \in D$ such that

$$\int A_yf(x, dy) \in \mathcal{D}(A).$$

In particular let $L = \mathcal{C}(\mathbf{R})$ be the space of continuous functions
on $\mathbf{R}$ that vanish at infinity with the sup norm, and assume that

$$A_xf = v(x)(d/dx)f, \quad \text{if } f \text{ and } (d/dx)f \in \mathcal{C}(\mathbf{R}),$$

where $v$ is a continuous real valued function on $E$. Then

$$\int v(x) \mu(dx) = 0$$

implies $C_0$ is defined for all $f \in \mathcal{C}(\mathbf{R})$ such that

$$(d^2/dx^2)f \in \mathcal{C}(\mathbf{R})$$

and

$$C_0f(x) = \int v(x) \int v(y) v(x, dy) \mu(dx) f''(x).$$
In this case $C_0$ satisfies the conditions of Theorem 2.2. The semigroup $\mathcal{F}_a(t)$ corresponds to the Markov process $(X(\alpha t), Z_a(t))$ where $Z_a(t)$ is given by

$$Z_a(t) = Z_a(0) + \int_0^t \nu(X(\alpha s)) \, ds.$$ 

The conclusion of Theorem 2.2 implies $Z_a(\alpha t)$ converges in law to Brownian motion with variance parameter

$$\sigma^2 = 2 \int v(x) \int v(y) \nu(x, dy) \mu(dx).$$

(This generalizes a result of Pinsky [9].)

More generally, let $v(z, x)$ and $(d/dx) v(z, x)$ be bounded and jointly continuous, and assume $\int v(z, x) \mu(dx) = 0$. Let

$$A_x f(z) = v(z, x)(d/dx) f(z).$$

Then for $f \in \mathcal{C}(\mathbb{R})$ with $(d^2/dx^2) f \in \mathcal{C}(\mathbb{R})$

$$C_0 f(z) = \int v(z, x) \int v(z, y) \nu(x, dy) \mu(dx) \frac{d^2}{dx^2} f(z)$$

$$+ \int v(z, x) \int d/dx v(z, y) \nu(x, dy) \mu(dx) \frac{d}{dx} f(z).$$

If $\sigma^2(z) = 2 \int v(z, x) \int v(z, y) \nu(x, dy) \mu(dx)$ is strictly positive and Hölder continuous and $m(z) = \int v(z, x) \int (d/dx) v(z, y) \nu(x, dy) \mu(dx)$ is Hölder continuous (both functions are bounded) then the conditions of Theorem 2.2 are satisfied. In this case $\mathcal{F}_a(t)$ corresponds to the Markov process $(X(\alpha t), Z_a(t))$ where $Z_a(t)$ satisfies

$$Z_a(t) = Z_a(0) + \int_0^t \nu(Z_a(s), X(\alpha s)) \, ds.$$ 

The conclusion of Theorem 2.2 implies $Z_a(\alpha t)$ converges in law to the diffusion process with differential generator

$$\frac{1}{2} \sigma^2(x)(d^2/dx^2) + m(x) \frac{d}{dx}.$$ 

**Example 3.4.** Let $E = \{1, 2, 3, \ldots\}$ and let $X(t)$ be a positive recurrent pure jump Markov chain with transition matrix $(|p_{ij}(t)|)$. Define

$$p_i = \lim_{t \to \infty} p_{ii}(t)$$
and assume \( \int_0^\infty |p_{ij}(t) - p_j| \, dt < \infty \). This is true if the first hitting time of a state has a finite second moment. Let
\[
\nu_{ij} = \int_0^\infty (p_{ij}(t) - p_j) \, dt.
\]
Let \( D = \{ f : f \in \bigcap_i D(A_i) \text{ sup}_i \| A_i f \| < \infty \} \), and assume \( \sum_i p_i A_i f = 0 \). If \( \sup_i \sum_j \| \nu_{ij} A_j f \| < \infty \),
\[
\sum_j \nu_{ij} A_j f \in D(A_i)
\]
and
\[
\sup_i \| A_i \sum_j \nu_{ij} A_j f \| < \infty,
\]
then
\[
C_0 f = \sum_i p_i A_i \sum_j \nu_{ij} A_j f.
\]

Note \( \mathcal{F}_a(t) \) is given by Eq. (1.2) (with \( X(t) \) replaced by \( X_a(t) \equiv X(\alpha t) \)) and is "buc-continuous" on all of \( \mathcal{L} \). Theorem 3.2 is applicable if \( \mathcal{B}(\lambda - C_0) \supset \mathcal{D}(C_0) \) for some \( \lambda > 0 \).

4. RANDOM EVOLUTIONS WITH FEEDBACK

The diffusion approximation example in Section 3 is a special case of a more general class of random evolutions. Suppose for each \( x \in E \), the semigroup \( T_x(t) \) corresponds to a Markov transition function \( P_x(t, x, \Gamma) \) defined on a measurable state space \( (S, \mathcal{F}) \). Then the semigroup \( \mathcal{F}_a(t) \) should correspond to a Markov process \( (Z_a(t), X_a(t)) \) where intuitively \( X_a(t) \) is the Markov process corresponding to \( a \) and \( Z_a(t) \) develops like a Markov process with transition function \( P_x(t, x, \Gamma) \) when \( X_a(t) = x \).

In this case the infinitesimal operator looks like
\[
C_a f(x, x) = A_x f(x, x) + \alpha B f(x, x),
\]
where for fixed \( x \) \( A_x \) acts on \( f(x, x) \) as a function of \( x \) alone and for fixed \( z \) \( B \) operates on \( f(z, x) \) as a function of \( x \) alone.

The process \( X_a(t) \) can be thought of as "controlling" the process \( Z_a(t) \), but it is a control that operates in ignorance of \( Z_a(t) \). This difficulty can be remedied by allowing \( B \) to depend on \( z \). That is, consider
\[
C_a f(x, x) = A_x f(x, x) + \alpha B_z f(x, x),
\]
where for fixed \( x \) \( A_x \) acts on \( f(x, x) \) as a function of \( x \) alone and for fixed \( z \) \( B_z \) operates on \( f(z, x) \) as a function of \( x \) alone.
where $A_x$ acts on $f$ as a function of $z$ alone, and $B_z$ acts on $f$ as a function of $x$ alone.

The processes that correspond to infinitesimal operators of this form should be useful in modeling such things as population growth ($Z_a(t)$ would be the population size and $X_a(t)$ would be the environmental factors that affect the rate of growth of the population, which in turn would be affected by the size of the population) as well as in control theory. These possibilities will be considered in future papers. In particular, a compactness theorem of the kind of interest in control theory will be given. Here we will give only one simple example indicating that the results in Section 2 are applicable in this more general setting.

Let $E = \{1, 2\}$ and $S = (-\infty, \infty)$, and consider the operator

$$C_a \left( \begin{array}{c} f_1(z) \\ f_2(z) \end{array} \right) = \left( \begin{array}{cc} \frac{d}{dz} & 0 \\ 0 & -\frac{d}{dz} \end{array} \right) \left( \begin{array}{c} f_1(z) \\ f_2(z) \end{array} \right) + a \left( \begin{array}{cc} -(1 + z^2) & (1 + z^2) \\ (1 + z^2) & -(1 + z^2) \end{array} \right) \left( \begin{array}{c} f_1(z) \\ f_2(z) \end{array} \right).$$

(4.3)

The process $Z_a(t)$ moves to the right with unit velocity if $X_a(t) = 1$ and to the left if $X_a(t) = 2$, and $P\{X_a(t + \Delta t) \neq X_a(t) \mid Z_a(t) = z\} = \alpha(1 + z^2) \Delta t + o(\Delta t)$.

The projection is

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

and clearly $f \in \mathcal{D}(A) \cap \mathcal{R}(P)$ implies $PAf = 0$. A solution of $Bh = -Af$ for $f \in \mathcal{D}(A) \cap \mathcal{R}(P)$ is given by

$$h = \begin{pmatrix} 0 \\ -\frac{1}{1 + z^2} f'(z) \end{pmatrix}.$$  

For $f$ twice continuously differentiable with compact support we have

$$PAh = \left( \frac{1}{2(1 + z^2)} f''(z) - \frac{z}{(1 + z^2)^2} f'(z) \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$  

It follows, that as $\alpha$ goes to infinity $Z_a(\alpha t)$ converges in law to the diffusion process with generator

$$\frac{1}{2(1 + z^2)} \frac{d^2}{dz^2} - \frac{z}{(1 + z^2)^2} \frac{d}{dz},$$

where $A_x$ acts on $f$ as a function of $z$ alone, and $B_z$ acts on $f$ as a function of $x$ alone.

The processes that correspond to infinitesimal operators of this form should be useful in modeling such things as population growth ($Z_a(t)$ would be the population size and $X_a(t)$ would be the environmental factors that affect the rate of growth of the population, which in turn would be affected by the size of the population) as well as in control theory. These possibilities will be considered in future papers. In particular, a compactness theorem of the kind of interest in control theory will be given. Here we will give only one simple example indicating that the results in Section 2 are applicable in this more general setting.

Let $E = \{1, 2\}$ and $S = (-\infty, \infty)$, and consider the operator

$$C_a \left( \begin{array}{c} f_1(z) \\ f_2(z) \end{array} \right) = \left( \begin{array}{cc} \frac{d}{dz} & 0 \\ 0 & -\frac{d}{dz} \end{array} \right) \left( \begin{array}{c} f_1(z) \\ f_2(z) \end{array} \right) + a \left( \begin{array}{cc} -(1 + z^2) & (1 + z^2) \\ (1 + z^2) & -(1 + z^2) \end{array} \right) \left( \begin{array}{c} f_1(z) \\ f_2(z) \end{array} \right).$$

(4.3)

The process $Z_a(t)$ moves to the right with unit velocity if $X_a(t) = 1$ and to the left if $X_a(t) = 2$, and $P\{X_a(t + \Delta t) \neq X_a(t) \mid Z_a(t) = z\} = \alpha(1 + z^2) \Delta t + o(\Delta t)$.

The projection is

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

and clearly $f \in \mathcal{D}(A) \cap \mathcal{R}(P)$ implies $PAf = 0$. A solution of $Bh = -Af$ for $f \in \mathcal{D}(A) \cap \mathcal{R}(P)$ is given by

$$h = \begin{pmatrix} 0 \\ -\frac{1}{1 + z^2} f'(z) \end{pmatrix}.$$  

For $f$ twice continuously differentiable with compact support we have

$$PAh = \left( \frac{1}{2(1 + z^2)} f''(z) - \frac{z}{(1 + z^2)^2} f'(z) \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$  

It follows, that as $\alpha$ goes to infinity $Z_a(\alpha t)$ converges in law to the diffusion process with generator

$$\frac{1}{2(1 + z^2)} \frac{d^2}{dz^2} - \frac{z}{(1 + z^2)^2} \frac{d}{dz},$$
REFERENCES