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Codimension 2 bifurcation of twisted double homoclinic loops☆

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ABSTRACT

A local active coordinates approach is employed to obtain bifurcation equations of twisted double homoclinic loops. Under the condition of one twisted orbit, we obtain the existence and uniqueness and of the 1–1 double homoclinic loop, 2–1 double homoclinic loop, 2–1 right homoclinic loop, 1–1 large homoclinic loop, 2–1 large homoclinic loop and 2–1 large period orbit. For the case of double twisted orbits, we obtain the existence or non-existence of 1–1 double homoclinic loop, 1–2 double homoclinic loop, 2–1 double homoclinic loop, 2–2 double homoclinic loop, 2–1 large homoclinic loop, 1–2 large homoclinic loop, 2–2 large homoclinic loop, 2–2 right homoclinic loop, 2–2 large homoclinic loop, 2–2 left homoclinic loop and 2–2 large period orbit. Moreover, the bifurcation surfaces and their existence regions are given. Besides, bifurcation sets are presented on the 2 dimensional subspace spanned by the first two Melnikov vectors.

1. Introduction and hypotheses

During the last two decades, bifurcation problems of homoclinic and heteroclinic loops of planar systems have been investigated by many authors [1–15]. In [13], Shil'nikov studied the codimension 1 homoclinic bifurcation problem with two complex conjugated eigenvalues. He pointed that if the eigenvalues α and β verify $\operatorname{Re}\alpha = \operatorname{Re}\beta < 1$, then the dynamical behavior in a small neighborhood of the homoclinic orbit is chaotic. Recently, the bifurcation problems of homoclinic and heteroclinic loops in high dimensional systems have been comprehensively studied as well [16–22]. Among all these works, not many concern the bifurcation of double homoclinic loops. However, nowadays, there is an increasing interest in the subject, for example, see [23–32]. In [23,32], the attention mainly goes to the perturbation of the Hamiltonian planar vectors fields and their results focus on the number of limit cycles and their distributions by using method taken from bifurcation theory and qualitative analysis. In [24], by using normal form theory and Poincaré return maps, the stability of the planar double homoclinic loop is studied. In [31], one proved the existence of two families of homoclinic orbits in the small neighborhood of the double homoclinic loops of a planar system. In [30], the authors established the classification for the set of nonwandering points, homoclinic orbits and limit cycles, respectively. In [29], the author described the topological equivalence class of $X_\mu|_{\Omega_\mu}$ for a C^3-dynamical system X_μ in general position, where Ω_μ is a set of trajectories in a neighborhood of the double homoclinic loop. In [26,28], the author shows the existence of Lorenz attractors in the unfolding of a double homoclinic loop with a resonance condition on eigenvalues. While in [27], the author proves that perturbations of the initial stable double homoclinic loop can lead to the creation of a Lorenz attractor. In [25], with the same configuration below, the existence of an invariant set (shift type) in the variant center manifold (an intersection of a center stable manifold and a center unstable manifold) is obtained for conservative and reversible vector fields.

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For a fixed r, we consider the following C^r system

$$\dot{z} = f(z) + g(z, \mu), \tag{1}$$

where $z \in \mathbb{R}^{m+n+2}$, $m \geq 0$, $n \geq 0$, $m + n > 0$, $\mu \in \mathbb{R}^l$, $l \geq 2$, $0 \leq \|\mu\| \ll 1$, $f(0) = 0$, $g(z, 0) = 0$, here $\|\cdot\|$ denotes the scalar product. Conversely with these articles cited above, the degeneracy of the unperturbed vector field

$$\dot{z} = f(z), \tag{2}$$

comes from exclusively form the double homoclinicity, and various bifurcation manifolds and the corresponding existence regions are concretely given.

First of all, we assume that:

(H_1) System (2) has a hyperbolic equilibrium at the origin and the relevant linearization matrix $Df(0)$ has simple eigenvalues: λ_1, λ_{2i} $(i = 1, 2, \ldots, m)$, ρ_1, ρ_{2j} $(j = 1, 2, \ldots, n)$ satisfying

$$-\mathrm{Re}\,\rho_{2j} < -\rho_1 < 0 < \lambda_1 < \mathrm{Re}\,\lambda_{2i}.$$

With no strong resonance between ρ_1 and λ_1 being allowed, we can always assume that $\rho_1 > \lambda_1$ without loss of generality.

Thanks to the Implicit Function Theorem, since the equilibrium of the unperturbed system (located at the origin for $\mu = 0$) is hyperbolic, this equilibrium persists and admits a continuation for small values of $\|\mu\|$. Up to a translation, one can assume that the equilibrium is always located at the origin.

Moreover we assume that $Df(0)$ satisfies the Sternberg condition of order Q with $Q = K([\frac{\lambda_{2m}}{\lambda_1}] + [\frac{\rho_{2n}}{\rho_1}]) + 2$, where K is the Q-smoothness of $Df(0)$, and $r \geq 3Q$, so that system (1) is uniformly C^K linearizable according to [33]. Hence, up to a C^K diffeomorphism, there exists a small neighborhood U of 0 in $\mathbb{R}^{m+n+2}$, such that for all $\mu \in \mathbb{R}^l$, $0 \leq \|\mu\| \ll 1$ and for all $(x, y, u, v) \in U$, system (1) has the following C^{K-1} $(K \geq 4)$ normal form:

$$\dot{x} = \lambda_1(\mu)x, \qquad \dot{y} = -\rho_1(\mu)y, \qquad \dot{u} = \lambda_2(\mu)u, \qquad \dot{v} = -\rho_2(\mu)v. \tag{3}$$

Here, $\lambda_2(\mu)$ is an $m \times m$ diagonal matrix with $\lambda_{21}, \lambda_{22}, \ldots, \lambda_{2m}$ as its diagonal elements and $\rho_2(\mu)$ is an $n \times n$ diagonal matrix with $\rho_{21}, \rho_{22}, \ldots, \rho_{2n}$ as its diagonal elements.

Besides, we make the following assumptions:

(H_2) System (2) has double homoclinic loops $\Gamma = \Gamma_1 \cup \Gamma_2$,

$$\Gamma_i = \{z = r_i(t) : t \in \mathbb{R}, r_i(\pm\infty) = 0\}$$

and $\dim(\mathrm{T}_{r_i(t)}\mathrm{W}^s \cap \mathrm{T}_{r_i(t)}\mathrm{W}^u) = 1$, $i = 1, 2$, where W^s and W^u designate the stable and unstable manifold respectively and $\mathrm{T}_A W$ is the tangent space of W at A.

(H_3) Let $e_i^{\pm} = \lim_{t\to\mp\infty} \frac{\dot{r}_i(t)}{|\dot{r}_i(t)|}$, then $e_i^+ \in \mathrm{T}_0\mathrm{W}^u$, $e_i^- \in \mathrm{T}_0\mathrm{W}^s$ are unit eigenvectors corresponding to λ_1 and $-\rho_1$, respectively. Moreover, $e_1^+ = -e_2^+$, $e_1^- = -e_2^-$.

(H_4) $\mathrm{Span}\{\mathrm{T}_{r_i(t)}\mathrm{W}^u, \mathrm{T}_{r_i(t)}\mathrm{W}^s, e_i^+\} = \mathbb{R}^{m+n+2}$ as $t \gg 1$, $\mathrm{Span}\{\mathrm{T}_{r_i(t)}\mathrm{W}^u, \mathrm{T}_{r_i(t)}\mathrm{W}^s, e_i^-\} = \mathbb{R}^{m+n+2}$ as $t \ll -1$. (see Fig. 2)

Remark 1.1. For the existing loop Γ, (H_3) is generic, which guarantees that Γ has no orbit flip. While (H_4) says that both homoclinic orbits are not of the inclination to flip. If both H_3 and H_4 hold, the orbit is called non-critically twisted.

With the above assumptions, the double homoclinic loops, say Γ_1, Γ_2, are of codimension 2. Besides, a non-degenerate homoclinic orbit Γ is called a non-twisted homoclinic orbit if the unstable manifold W^u has an even number of half twists along the homoclinic orbit. It is called a twisted homoclinic orbit if W^u has an odd number of half twists along Γ, see [34] for more details. We shall study the problems of p–q double homoclinic loops, p–q left (or right) homoclinic loop, p–q large homoclinic loop and p–q large period orbit bifurcated from the twisted double homoclinic loops in an arbitrarily high-dimensional system. Here, "left" or "right" means the corresponding orbit circulates in the small neighborhood of the original double homoclinic loops whereas it just takes infinite time in the neighborhood of one orbit of the double homoclinic loops, either Γ_1 or Γ_2. "Large" means that the corresponding orbit moves around in the small neighborhood of the original double homoclinic loops and it takes an infinite time in the neighborhood of each homoclinic orbit. In addition, "p–q" refers to the rounding number in each orbit's neighborhood. Precisely speaking, the p–q loop will round $\Gamma_2 p$ cycles, while it has winding number q in a small neighborhood of Γ_1. (see Fig. 1)

The present paper is organized as follows. After giving some preliminary results in Section 2, we obtain the bifurcation equations with a single twisted orbit in Section 3 and in Section 4, we give the bifurcation results in this case. In Section 5, we study the bifurcation of double twisted orbits.

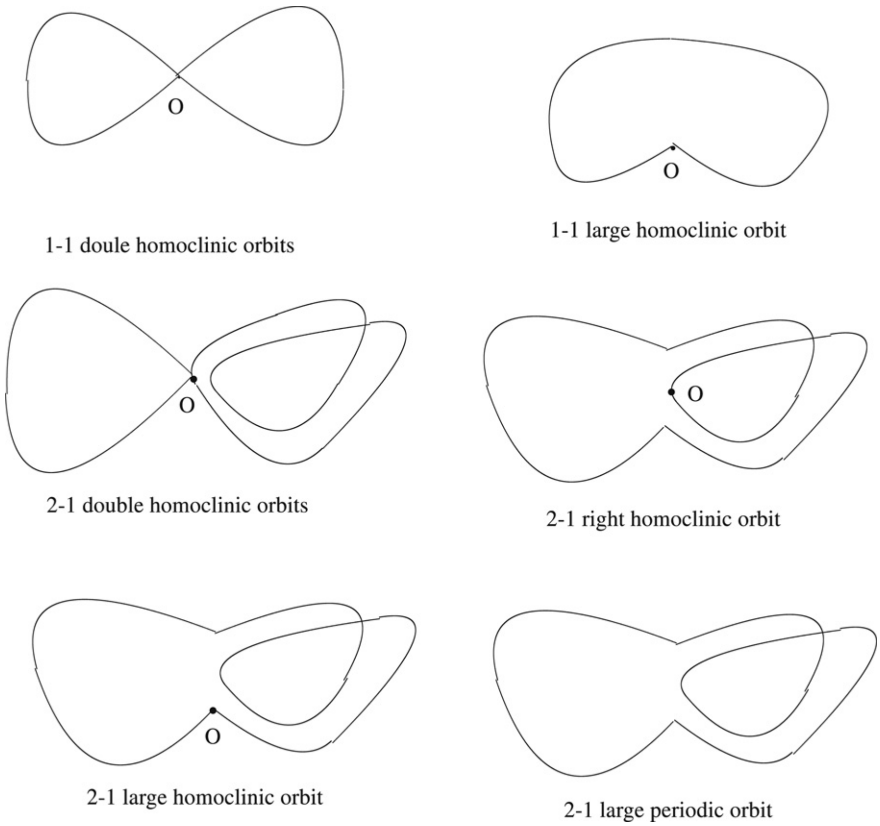

Fig. 1. Illustration of orbits definition.

2. Preliminaries

As in the meaning of the first approximation, the tangent vector bundles, situated in the tangent space bundles are confined on the homoclinic loops, which is the intersection of the stable manifold and the unstable manifold, inherit and exhibit sufficiently the properties (such as the geometry, the invariance, the contractibility, the expansiveness, etc.) of the system near the loop. Our aim is then to select carefully some tangent vector bundles along the loops and some others complement to them to form a moving frame so as to obtain the simplest form. Let us consider the linear variational system of (2)

$$\dot{z} = Df(r_i(t))z, \tag{4}$$

and its adjoint system

$$\dot{z} = -(Df(r_i(t)))^* z. \tag{5}$$

Denote $r_i(t) = (r_i^x(t), r_i^y(t), r_i^u(t), r_i^v(t))$ and take T_i^0, T_i^1 to be large enough such that

$$r_i(-T_i^1) = ((-1)^i\delta, 0, \delta_i^u, 0), \qquad r_i(T_i^0) = (0, (-1)^i\delta, 0, \delta_i^v),$$

where $\|\delta_i^u\|, \|\delta_i^v\| = O(\delta^\alpha)$, $i = 1, 2$, $\alpha = \min_{j,k}\{\text{Re}\,\rho_{2j}/\rho_1, \text{Re}\,\lambda_{2k}/\lambda_1\} > 1$, and δ is small enough so that

$$\{(x, y, u, v) : |x|, |y|, \|u\|, \|v\| < 4\delta\} \subset U.$$

We state the following lemma which can be found in [18,20].

Lemma 2.1. *There exists a fundamental solution matrix* $Z_i(t) = (z_i^1(t), z_i^2(t), z_i^3(t), z_i^4(t))$ *for system* (4) *with*

$$z_i^1(t) \in (T_{r_i(t)}W^u)^c \cap (T_{r_i(t)}W^s)^c,$$
$$z_i^2(t) = -\dot{r}_i(t)/|\dot{r}_i^y(T_i^0)| \in T_{r_i(t)}W^u \cap T_{r_i(t)}W^s,$$
$$z_i^3(t) \in T_{r_i(t)}W^{uu},$$
$$z_i^4(t) \in T_{r_i(t)}W^{ss}$$

satisfying

$$Z_i(-T_i^1)=\begin{pmatrix}\omega_i^{11} & \omega_i^{21} & 0 & \omega_i^{41}\\ \omega_i^{12} & 0 & 0 & \omega_i^{42}\\ \omega_i^{13} & \omega_i^{23} & I_{m\times m} & \omega_i^{43}\\ 0 & 0 & 0 & \omega_i^{44}\end{pmatrix},\qquad Z_i(T_i^0)=\begin{pmatrix}1 & 0 & \omega_i^{31} & 0\\ 0 & (-1)^i & \omega_i^{32} & 0\\ 0 & 0 & \omega_i^{33} & 0\\ \bar{\omega}_i^{14} & \omega_i^{24} & \omega_i^{34} & I_{n\times n}\end{pmatrix},$$

where, as $T_i^j \gg 1$ $(i=1,2, j=0,1)$, $\omega_i^{12}\det\omega_i^{33}\det\omega_i^{44}\neq 0$, $(-1)^i\omega_i^{21}<0$, $\|\omega_i^{24}\|\ll 1$, $\|\bar{\omega}_i^{14}\|\ll 1$, $|(\omega_i^{12})^{-1}\omega_i^{11}|\ll 1$, $\|(\omega_i^{12})^{-1}\omega_i^{13}\|\ll 1$; $\|(\omega_i^{21})^{-1}\omega_i^{23}\|\ll 1$, $\|(\det\omega_i^{44})^{-1}\omega_i^{4k}\|\ll 1$, *for* $k\neq 4$; $\|(\det\omega_i^{33})^{-1}\omega_i^{3k}\|\ll 1$, *for* $k\neq 3$.

Remark 2.1. In the above lemma, W^{uu} stands for the strong unstable manifold while W^{ss} stands for the strong stable manifold.

As is well known from the matrix theory, system (5) has a fundamental solution matrix $\Phi_i(t) = (Z_i^{-1}(t))^* = (\phi_i^1(t), \phi_i^2(t), \phi_i^3(t), \phi_i^4(t))$. Introduce the local active coordinates $N_i = (n_i^1, 0, n_i^3, n_i^4)$, then we parametrized a point $z=(x,y,u,v)$ near the orbits Γ_i in the section $S_i(t)$ by the coordinates (n_i^1, n_i^3, n_i^4). And the section $S_i(t)$ can be written as

$$S_i(t)=\{z=r_i(t)+Z_i(t)N_i^*=r_i(t)+z_i^1(t)n_i^1+z_i^3(t)n_i^3+z_i^4(t)n_i^4\}. \tag{6}$$

Choose the cross sections, for $i=1,2$,

$$S_i^0=\{z=S_i(T_i^0): |x|,|y|,|u|,|v|<2\delta\}\subset U,\qquad S_i^1=\{z=S_i(-T_i^1): |x|,|y|,|u|,|v|<2\delta\}\subset U.$$

With the above notation, system (1) has the following form

$$\dot{n}_i^j=(\phi_i^j(t))^* g_\mu(r_i(t),0)\mu+o(\|\mu\|),\quad i=1,2; j=1,3,4, \tag{7}$$

which is C^{K-2} and produces the transition maps $P_i^1: S_i^1\to S_i^0$, $i=1,2$. Here, g_μ is the derivative of g with respect to μ. Integrating both sides of (7) from $-T_i^1$ to T_i^0, we have

$$n_i^j(T_i^0)=n_i^j(-T_i^1)+M_i^j\mu+o(\|\mu\|),\quad i=1,2; j=1,3,4,$$

where $N_i(T_i^0)=(n_i^1(T_i^0),0,n_i^3(T_i^0),n_i^4(T_i^0))$, $N_i(-T_i^1)=(n_i^1(-T_i^1),0,n_i^3(-T_i^1),n_i^4(-T_i^1))$, and $M_i^j=\int_{-T_i^1}^{T_i^0}(\phi_i^j(t))^* g_\mu(r_i(t),0)\,\mathrm{d}t$, $i=1,2; j=1,3,4$ are Melnikov vectors (see for example [17,18,20–22]).

Remark 2.2. The Melnikov vectors in the case $j=1$ are given by

$$M_i^1=\int_{-T_i^1}^{T_i^0}(\phi_i^1(t))^* g_\mu(r_i(t),0)\,\mathrm{d}t=\int_{-\infty}^{+\infty}(\phi_i^1(t))^* g_\mu(r_i(t),0)\,\mathrm{d}t\quad \text{for } i=1,2.$$

3. Bifurcation equations with single twisted orbit

We now study the case of a single twisted orbit which means that the following hypothesis is satisfied.

(H_5) The orbit Γ_1 is nontwisted and Γ_2 is twisted, that is, $\omega_1^{12}>0$ and $\omega_2^{12}<0$.

Consider the map $P_1^0: S_1^0\to S_2^1$, $q_1^0\longmapsto q_2^1$, $\overline{P}_2^0: S_2^0\to S_2^1$, $\bar{q}_2^0\longmapsto \bar{q}_2^1$ and $P_2^0: S_2^0\to S_1^1$, $q_2^0\longmapsto q_1^1$ induced by the flow of (3) in the neighborhood U of $z=0$. Set the flying time from q_1^0 to q_2^1 as τ_1, $\bar{q}_2^0$ to $\bar{q}_2^1$ as τ_2, q_2^0 to q_1^1 as τ_3 and the Shilnikov time $s_k=e^{-\lambda_1\tau_k}$, $k=1,2,3$ (see Fig. 2). Then we have

$$\begin{aligned}
P_1^0 &: q_1^0(x_1^0,y_1^0,u_1^0,v_1^0)\to q_2^1(x_2^1,y_2^1,u_2^1,v_2^1),\\
&\quad x_1^0=s_1x_2^1,\qquad y_2^1=s_1^{\rho_1/\lambda_1}y_1^0,\qquad u_1^0=s_1^{\lambda_2/\lambda_1}u_2^1,\qquad v_2^1=s_1^{\rho_2/\lambda_1}v_1^0,\\
\overline{P}_2^0 &: \bar{q}_2^0(\bar{x}_2^0,\bar{y}_2^0,\bar{u}_2^0,\bar{v}_2^0)\to \bar{q}_2^1(\bar{x}_2^1,\bar{y}_2^1,\bar{u}_2^1,\bar{v}_2^1),\\
&\quad \bar{x}_2^0=s_2\bar{x}_2^1,\qquad \bar{y}_2^1=s_2^{\rho_1/\lambda_1}\bar{y}_2^0,\qquad \bar{u}_2^0=s_2^{\lambda_2/\lambda_1}\bar{u}_2^1,\qquad \bar{v}_2^1=s_2^{\rho_2/\lambda_1}\bar{v}_2^0,\\
P_2^0 &: q_2^0(x_2^0,y_2^0,u_2^0,v_2^0)\to q_1^1(x_1^1,y_1^1,u_1^1,v_1^1),\\
&\quad x_2^0=s_3x_1^1,\qquad y_1^1=s_3^{\rho_1/\lambda_1}y_2^0,\qquad u_2^0=s_3^{\lambda_2/\lambda_1}u_1^1,\qquad v_1^1=s_3^{\rho_2/\lambda_1}v_2^0,
\end{aligned}$$

where for $k=1,2,3$, $s_k^{\lambda_2/\lambda_1}=\operatorname{diag}(s_k^{\lambda_{21}/\lambda_1}, s_k^{\lambda_{22}/\lambda_1},\ldots,s_k^{\lambda_{2m}/\lambda_1})$, $s_k^{\rho_2/\lambda_1}=\operatorname{diag}(s_k^{\rho_{21}/\lambda_1}, s_k^{\rho_{22}/\lambda_1},\ldots,s_k^{\rho_{2n}/\lambda_1})$ are diagonal matrices of order m and n respectively.

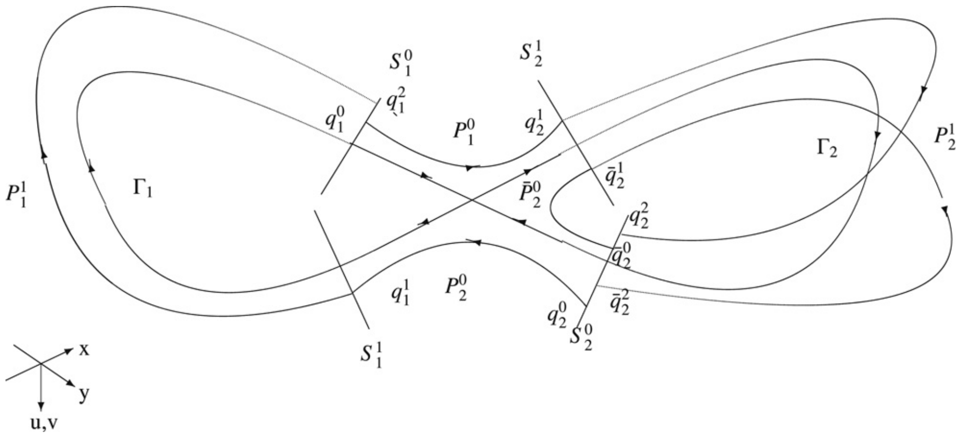

Fig. 2. Poincaré map with single twisted orbit.

To be more precise, let

$$S_{i+}^1 = \{q \in S_i^1 \mid y_q > 0\}, \qquad S_{i-}^1 = \{q \in S_i^1 \mid y_q < 0\},$$
$$S_{i+}^0 = \{q \in S_i^0 \mid x_q > 0\}, \qquad S_{i-}^0 = \{q \in S_i^0 \mid x_q < 0\}, \quad i = 1, 2.$$

Then,

$$P_1^0 : S_{1+}^0 \to S_{2-}^1, \qquad \overline{P}_1^0 : S_{1-}^0 \to S_{1-}^1, \qquad P_2^0 : S_{2-}^0 \to S_{1+}^1, \qquad \overline{P}_2^0 : S_{2+}^0 \to S_{2+}^1.$$

Equipped with these formulae, we calculate the relations between

$$q_i^{2j}(x_i^{2j}, y_i^{2j}, u_i^{2j}, v_i^{2j}), \qquad q_i^{2j+1}(x_i^{2j+1}, y_i^{2j+1}, u_i^{2j+1}, v_i^{2j+1}), \qquad P_i^0(q_i^{2j}) = q_{i+1}^{2j+1}$$

and their new coordinates $N_i^{2j}(n_i^{2j,1}, 0, n_i^{2j,3}, n_i^{2j,4})$, $N_i^{2j+1}(n_i^{2j+1,1}, 0, n_i^{2j+1,3}, n_i^{2j+1,4})$ for $i = 1, 2$, where $q_3^1 = q_1^1$, and similar relations for $\bar{q}_2^{2j}$ and $\bar{q}_2^{2j+1} = \overline{P}_2^0(\bar{q}_2^{2j})$. Using (6) and according to the expressions of $Z_i(-T_i^1)$ and $Z_i(T_i^0)$, we obtain

$$\bar{n}_2^{2j,1} = \bar{x}_2^{2j} - \omega_2^{31}(\omega_2^{33})^{-1}\bar{u}_2^{2j}, \qquad \bar{n}_2^{2j,3} = (\omega_2^{33})^{-1}\bar{u}_2^{2j},$$
$$\bar{n}_2^{2j,4} = \bar{v}_2^{2j} - \delta_2^v - \bar{\omega}_2^{14}\bar{x}_2^{2j} + (\bar{\omega}_2^{14}\omega_2^{31} - \omega_2^{34})(\omega_2^{33})^{-1}\bar{u}_2^{2j},$$
$$\bar{n}_2^{2j+1,1} = (\omega_2^{12})^{-1}\bar{y}_i^{2j+1} - (\omega_2^{12})^{-1}\omega_2^{42}(\omega_2^{44})^{-1}\bar{v}_2^{2j+1},$$
$$\bar{n}_2^{2j+1,3} = \bar{u}_2^{2j+1} - \delta_2^u - \omega_2^{13}(\omega_2^{12})^{-1}\bar{y}_i^{2j+1} + [\omega_2^{13}(\omega_2^{12})^{-1}\omega_2^{42} - \omega_2^{43}](\omega_2^{44})^{-1}\bar{v}_2^{2j+1},$$
$$\bar{n}_2^{2j+1,4} = (\omega_2^{44})^{-1}\bar{v}_2^{2j+1},$$
$$n_i^{2j,1} = x_i^{2j} - \omega_i^{31}(\omega_i^{33})^{-1}u_i^{2j}, \qquad n_i^{2j,3} = (\omega_i^{33})^{-1}u_i^{2j},$$
$$n_i^{2j,4} = v_i^{2j} - \delta_i^v - \bar{\omega}_i^{14}x_i^{2j} + (\bar{\omega}_i^{14}\omega_i^{31} - \omega_i^{34})(\omega_i^{33})^{-1}u_i^{2j},$$
$$n_i^{2j+1,1} = (\omega_i^{12})^{-1}y_i^{2j+1} - (\omega_i^{12})^{-1}\omega_i^{42}(\omega_i^{44})^{-1}v_i^{2j+1},$$
$$n_i^{2j+1,3} = u_i^{2j+1} - \delta_i^u - \omega_i^{13}(\omega_i^{12})^{-1}y_i^{2j+1} + [\omega_i^{13}(\omega_i^{12})^{-1}\omega_i^{42} - \omega_i^{43}](\omega_i^{44})^{-1}v_i^{2j+1},$$
$$n_i^{2j+1,4} = (\omega_i^{44})^{-1}v_i^{2j+1},$$

and

$$x_i^{2j+1} \approx (-1)^i\delta, \bar{x}_2^{2j+1} \approx \delta, \qquad y_i^{2j} \approx (-1)^i\delta, \bar{y}_2^{2j} \approx \delta, \quad i = 1, 2.$$

From the above, we obtain the following Poincaré maps:

$$F_1 = P_2^1 \circ P_1^0 : S_1^0 \to S_2^0,$$
$$n_2^{2,1} = -(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1}\delta - (\omega_2^{12})^{-1}\omega_2^{42}(\omega_2^{44})^{-1}s_1^{\rho_2/\lambda_1}v_1^0 + M_2^1\mu + o(\|\mu\|),$$
$$n_2^{2,3} = u_2^1 - \delta_2^u + \omega_2^{13}(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1}\delta + [\omega_2^{13}(\omega_2^{12})^{-1}\omega_2^{42} - \omega_2^{43}](\omega_2^{44})^{-1}s_1^{\rho_2/\lambda_1}v_1^0 + M_2^3\mu + o(\|\mu\|),$$
$$n_2^{2,4} = (\omega_2^{44})^{-1}s_1^{\rho_2/\lambda_1}v_1^0 + M_2^4\mu + o(\|\mu\|),$$
$$\overline{F}_2 = P_2^1 \circ \overline{P}_2^0 : S_2^0 \to S_2^0,$$
$$\bar{n}_2^{2,1} = (\omega_2^{12})^{-1}s_2^{\rho_1/\lambda_1}\delta - (\omega_2^{12})^{-1}\omega_2^{42}(\omega_2^{44})^{-1}s_2^{\rho_2/\lambda_1}\bar{v}_2^0 + M_2^1\mu + o(\|\mu\|),$$

$$\bar{n}_2^{2,3} = \bar{u}_2^1 - \delta_2^u - \omega_2^{13}(\omega_2^{12})^{-1}s_2^{\rho_1/\lambda_1}\delta + [\omega_2^{13}(\omega_2^{12})^{-1}\omega_2^{42} - \omega_2^{43}](\omega_2^{44})^{-1}s_2^{\rho_2/\lambda_1}\bar{v}_2^0 + M_2^3\mu + o(\|\mu\|),$$
$$\bar{n}_2^{2,4} = (\omega_2^{44})^{-1}s_2^{\rho_2/\lambda_1}\bar{v}_2^0 + M_2^4\mu + o(\|\mu\|),$$
$$F_3 = P_1^1 \circ P_2^0 : S_2^0 \to S_1^0,$$
$$n_1^{2,1} = (\omega_1^{12})^{-1}s_3^{\rho_1/\lambda_1}\delta - (\omega_1^{12})^{-1}\omega_1^{42}(\omega_1^{44})^{-1}s_3^{\rho_2/\lambda_1}v_2^0 + M_1^1\mu + o(\|\mu\|),$$
$$n_1^{2,3} = u_1^1 - \delta_1^u - \omega_1^{13}(\omega_1^{12})^{-1}s_3^{\rho_1/\lambda_1}\delta + [\omega_1^{13}(\omega_1^{12})^{-1}\omega_1^{42} - \omega_1^{43}](\omega_1^{44})^{-1}s_3^{\rho_2/\lambda_1}v_2^0 + M_1^3\mu + o(\|\mu\|),$$
$$n_1^{2,4} = (\omega_1^{44})^{-1}s_3^{\rho_2/\lambda_1}v_2^0 + M_1^4\mu + o(\|\mu\|).$$

Now, the successor function

$$G(s_1, s_2, s_3, u_1^1, \bar{u}_2^1, u_2^1, v_1^0, \bar{v}_2^0, v_2^0) = (G_1^1, G_1^3, G_1^4, G_2^1, G_2^3, G_2^4, G_3^1, G_3^3, G_3^4) = (F_1(q_1^0) - \bar{q}_2^0, \bar{F}_2(\bar{q}_2^0) - q_2^0, F_3(q_2^0) - q_1^0)$$

is given by the following:

$$G_1^1 = -(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1}\delta - (\omega_2^{12})^{-1}\omega_2^{42}(\omega_2^{44})^{-1}s_1^{\rho_2/\lambda_1}v_1^0 - s_2\delta + \omega_2^{31}(\omega_2^{33})^{-1}s_2^{\lambda_2/\lambda_1}\bar{u}_2^1 + M_2^1\mu + o(\|\mu\|),$$
$$G_1^3 = u_2^1 - \delta_2^u + \omega_2^{13}(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1}\delta + [\omega_2^{13}(\omega_2^{12})^{-1}\omega_2^{42} - \omega_2^{43}](\omega_2^{44})^{-1}s_1^{\rho_2/\lambda_1}v_1^0 - (\omega_2^{33})^{-1}s_2^{\lambda_2/\lambda_1}\bar{u}_2^1 + M_2^3\mu + o(\|\mu\|),$$
$$G_1^4 = -\bar{v}_2^0 + \delta_2^v + (\omega_2^{44})^{-1}s_1^{\rho_2/\lambda_1}v_1^0 + \bar{\omega}_2^{14}\delta s_2 - [\bar{\omega}_2^{14}\omega_2^{31} - \omega_2^{34}](\omega_2^{33})^{-1}s_2^{\lambda_2/\lambda_1}\bar{u}_2^1 + M_2^4\mu + o(\|\mu\|),$$
$$G_2^1 = (\omega_2^{12})^{-1}s_2^{\rho_1/\lambda_1}\delta - (\omega_2^{12})^{-1}\omega_2^{42}(\omega_2^{44})^{-1}s_2^{\rho_2/\lambda_1}\bar{v}_2^0 + s_3\delta + \omega_2^{31}(\omega_2^{33})^{-1}s_3^{\lambda_2/\lambda_1}u_1^1 + M_2^1\mu + o(\|\mu\|),$$
$$G_2^3 = \bar{u}_2^1 - \delta_2^u - \omega_2^{13}(\omega_2^{12})^{-1}s_2^{\rho_1/\lambda_1}\delta + [\omega_2^{13}(\omega_2^{12})^{-1}\omega_2^{42} - \omega_2^{43}](\omega_2^{44})^{-1}s_2^{\rho_2/\lambda_1}\bar{v}_2^0 - (\omega_2^{33})^{-1}s_3^{\lambda_2/\lambda_1}u_1^1 + M_2^3\mu + o(\|\mu\|),$$
$$G_2^4 = -v_2^0 + \delta_2^v + (\omega_2^{44})^{-1}s_2^{\rho_2/\lambda_1}\bar{v}_2^0 - \bar{\omega}_2^{14}\delta s_3 - [\bar{\omega}_2^{14}\omega_2^{31} - \omega_2^{34}](\omega_2^{33})^{-1}s_3^{\lambda_2/\lambda_1}u_1^1 + M_2^4\mu + o(\|\mu\|),$$
$$G_3^1 = (\omega_1^{12})^{-1}s_3^{\rho_1/\lambda_1}\delta - (\omega_1^{12})^{-1}\omega_1^{42}(\omega_1^{44})^{-1}s_3^{\rho_2/\lambda_1}v_2^0 - s_1\delta + \omega_1^{31}(\omega_1^{33})^{-1}s_1^{\lambda_2/\lambda_1}u_2^1 + M_1^1\mu + o(\|\mu\|),$$
$$G_3^3 = u_1^1 - \delta_1^u - \omega_1^{13}(\omega_1^{12})^{-1}s_3^{\rho_1/\lambda_1}\delta + [\omega_1^{13}(\omega_1^{12})^{-1}\omega_1^{42} - \omega_1^{43}](\omega_1^{44})^{-1}s_3^{\rho_2/\lambda_1}v_2^0 - (\omega_1^{33})^{-1}s_1^{\lambda_2/\lambda_1}u_2^1 + M_1^3\mu + o(\|\mu\|),$$
$$G_3^4 = -v_1^0 + \delta_1^v + (\omega_1^{44})^{-1}s_3^{\rho_2/\lambda_1}v_2^0 + \bar{\omega}_1^{14}\delta s_1 - [\bar{\omega}_1^{14}\omega_1^{31} - \omega_1^{34}](\omega_1^{33})^{-1}s_1^{\lambda_2/\lambda_1}u_2^1 + M_1^4\mu + o(\|\mu\|).$$

Therefore, there is a correspondence between the solution $Q = (s_1, s_2, s_3, u_1^1, \bar{u}_2^1, u_2^1, v_1^0, \bar{v}_2^0, v_2^0)$ of

$$(G_1^1, G_1^3, G_1^4, G_2^1, G_2^3, G_2^4, G_3^1, G_3^3, G_3^4) = 0$$

with $s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$, and the existence of 1–1 double homoclinic loops, 2–1 double homoclinic loops, 2–1 right homoclinic loop, 1–1 large homoclinic loop, 2–1 large homoclinic loop and 2–1 large period orbit of system (1).

Solving $(u_2^1, \bar{v}_2^0, \bar{u}_2^1, v_2^0, u_1^1, v_1^0)$ from $(G_1^3, G_1^4, G_2^3, G_2^4, G_3^3, G_3^4) = 0$ and substituting it into the equations $(G_1^1, G_2^1, G_3^1) = 0$, we obtain the following bifurcation equations

$$\begin{aligned} &-(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1} - s_2 + \delta^{-1}M_2^1\mu + h.o.t. = 0,\\ &(\omega_2^{12})^{-1}s_2^{\rho_1/\lambda_1} + s_3 + \delta^{-1}M_2^1\mu + h.o.t. = 0,\\ &(\omega_1^{12})^{-1}s_3^{\rho_1/\lambda_1} - s_1 + \delta^{-1}M_1^1\mu + h.o.t. = 0. \end{aligned} \tag{8}$$

4. Bifurcation results with single twisted orbit

In this section, we study the existence, uniqueness and non-coexistence problem of p–q double homoclinic loops, p–q right homoclinic loop, p–q left homoclinic loop together with p–q large homoclinic loop and p–q large period orbit for a nontwisted orbit Γ_1 and a twisted Γ_2. Similarly, we can consider the corresponding problem for twisted Γ_1 and nontwisted Γ_2.

Firstly, we have the following result concerning the uniqueness and the incoexistence.

Theorem 4.1. *Assume that hypotheses* (H_1)–(H_5) *hold. Then, for* $\|\mu\|$ *sufficiently small, system* (1) *has at most one* 1–1 *double homoclinic loop, or one* 2–1 *double homoclinic loop, or one* 2–1 *right homoclinic loop, or one* 1–1 *large homoclinic loop, or one* 2–1 *large homoclinic loop or one* 2–1 *large period orbit in the small neighbourhood of* Γ. *Moreover these orbits do not coexist.*

Proof. Let $Q = (s_1, s_2, s_3, u_1^1, \bar{u}_2^1, u_2^1, v_1^0, \bar{v}_2^0, v_2^0)$ then

$$W = \left.\frac{\partial(G_3^1, G_1^1, G_2^1, G_3^3, G_1^3, G_2^3, G_3^4, G_1^4, G_2^4)}{\partial Q}\right|_{Q=0,\mu=0} = \begin{pmatrix} W_{11} & 0 & 0 \\ 0 & I_{3m} & 0 \\ W_{13} & 0 & I_{3n} \end{pmatrix}$$

where I_k denotes the identity matrix of order k,

$$W_{11} = \mathrm{diag}(-\delta, -\delta, \delta), \qquad W_{13} = \mathrm{diag}(\bar{\omega}_1^{14}\delta, \bar{\omega}_2^{14}\delta, -\bar{\omega}_2^{14}\delta)$$

are diagonal matrices. Notice that $\det W = -\delta^3 \neq 0$. According to the implicit function theorem, in the neighborhood of $(Q, \mu) = (0, 0)$, there exists a unique solution $s_i = s_i(\mu), u_i^1 = u_i^1(\mu), v_i^0 = v_i^0(\mu), \bar{u}_2^1 = \bar{u}_2^1(\mu), \bar{v}_2^0 = \bar{v}_2^0(\mu)$ satisfying $s_i(0) = 0, u_i^1(0) = 0, v_i^0(0) = 0, \bar{u}_2^1(0) = 0, \bar{v}_2^0(0) = 0$, for $i = 1, 2$.

Then, depending on the solutions s_i, one may have the following possibilities which have relations with the bifurcation problem.

If $s_1 = s_2 = 0$, then necessarily $s_3 = 0$. By the uniqueness, we see that the double homoclinic loop is persistent and it is impossible to appear two different homoclinic loops near Γ_2 forming bellows configuration.

If $s_2 = s_3 = 0$ and $s_1 > 0$, then Γ_2 is persistent, and meanwhile system (1) has a unique 1–1 large homoclinic loop.

If $s_1 = s_3 = 0$ and $s_2 > 0$, then system (1) has a unique 2–1 double homoclinic loop.

If $s_1 = 0, s_2 > 0, s_3 > 0$ or $s_3 = 0, s_1 > 0, s_2 > 0$, then system (1) has a unique 2–1 large homoclinic loop.

If $s_2 = 0, s_1 > 0$ and $s_3 > 0$, then system (1) has a unique 2–1 right homoclinic loop.

If $s_1 > 0, s_2 > 0, s_3 > 0$, system (1) has a unique 2–1 large period orbit.

Clearly, the uniqueness guarantees that all these kinds of orbits do not coexist. □

Remark 4.1. If there exists any $p - q$ large homoclinic (or periodic) orbit for large p and q, then from (H_5), $p = 2q$ must be satisfied. However, due to the uniqueness of solution, it is impossible for the $2q - q$ $(q > 1)$ large homoclinic orbit to exist, and all the $2q - q$ $(q > 1)$ large periodic orbits are in fact the 2–1 large periodic orbit.

Remark 4.2. If $s_1 = s_2 = s_3 = 0$ is the solution of Eq. (8), then $G_1^j = G_2^j$, for $j = 1, 3, 4$, thus the first two equations of (8) are the same.

In the forthcoming section, we study the different bifurcation manifolds and their existence regions for the single twisted orbit case.

By substituting $s_1 = s_2 = s_3 = 0$ into the first two equations we obtain $M_2^1\mu + h.o.t. = 0$. If $M_2^1 \neq 0$, then this equation defines a manifold L_2 of codimension 1 with a normal vector M_2^1 at $\mu = 0$. One concludes that the first two equations of (8) have solution $s_1 = s_2 = s_3 = 0$ when $\mu \in L_2$ and $\|\mu\| \ll 1$, which means that Γ_2 is persistent.

Similarly, there is a codimension 1 manifold L_1 defined by $M_1^1\mu + h.o.t. = 0$ with normal vector M_1^1 at $\mu = 0$ such that the third equation of (8) has solution $s_1 = s_2 = s_3 = 0$ as $\mu \in L_1$ and $\|\mu\| \ll 1$. Therefore Γ_1 is persistent. Suppose $\mathrm{rank}(M_1^1, M_2^1) = 2$, then $L_{12} = L_1 \cap L_2$ is a codimension 2 manifold with normal plane $\mathrm{Span}\{M_1^1, M_2^1\}$ such that the double homoclinic orbit $\Gamma = \Gamma_1 \cup \Gamma_2$ is persistent for $\mu \in L_{12}$.

Substituting $s_2 = s_3 = 0$ into Eq. (8), we obtain

$$\begin{aligned}
&-(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1} + \delta^{-1}M_2^1\mu + h.o.t. = 0,\\
&\delta^{-1}M_2^1\mu + h.o.t. = 0,\\
&-s_1 + \delta^{-1}M_1^1\mu + h.o.t. = 0.
\end{aligned}$$

Therefore we get $s_1 = \delta^{-1}M_1^1\mu + h.o.t.$ If $M_1^1\mu > 0$ then we have $s_1 > 0$. Substituting it into the first two equations, we obtain the codimension 2 bifurcation set H_{23}^1 such that a 1–1 large homoclinic loop bifurcates and Γ_2 persists. We have

$$\begin{aligned}
H_{23}^1 : &-(\omega_2^{12})^{-1}(\delta^{-1}M_1^1\mu)^{\rho_1/\lambda_1} + \delta^{-1}M_2^1\mu + h.o.t. = 0,\\
&\delta^{-1}M_2^1\mu + h.o.t. = 0,
\end{aligned}$$

which is well defined at least in the region $\{\mu : M_1^1\mu > 0,\ M_2^1\mu < 0, M_1^1\mu = o(|M_2^1\mu|^{\lambda_1/\rho_1})\}$.

Similarly, if Eq. (8) has $s_1 = s_3 = 0, s_2 > 0$ as its solution, we need to have

$$\begin{aligned}
&-s_2 + \delta^{-1}M_2^1\mu + h.o.t. = 0,\\
&(\omega_2^{12})^{-1}s_2^{\rho_1/\lambda_1} + \delta^{-1}M_2^1\mu + h.o.t. = 0,\\
&\delta^{-1}M_1^1\mu + h.o.t. = 0.
\end{aligned}$$

As the first equation induces $s_2 = \delta^{-1}M_2^1\mu + h.o.t.$, so we can get the bifurcation manifold for a 2–1 double homoclinic loop:

$$H_{13}^2 : \delta^{-1}M_2^1\mu + h.o.t. = 0, \qquad \delta^{-1}M_1^1\mu + h.o.t. = 0.$$

Accordingly, for $\mathrm{rank}\{M_1^1, M_2^1\} = 2$, $\dim\mu = \ell > 2$, and $0 < M_2^1\mu << 1$, we have $H_{13}^2 \cap \{\mu \mid s_2(\mu) > 0\} \neq \emptyset$, so there do exist 2–1 double homoclinic orbits with these conditions. If not, there exist no 2–1 double homoclinic orbits.

Proposition 4.1. *There exists no p–q large homoclinic loop for any $p \geq 2, q \geq 1$.*

Proof. If Eq. (8) has a solution with $s_1 = 0, s_2 > 0, s_3 > 0$. One has

$$-s_2 + \delta^{-1}M_2^1\mu + h.o.t. = 0,$$
$$(\omega_2^{12})^{-1}s_2^{\rho_1/\lambda_1} + s_3 + \delta^{-1}M_2^1\mu + h.o.t. = 0,$$
$$(\omega_1^{12})^{-1}s_3^{\rho_1/\lambda_1} + \delta^{-1}M_1^1\mu + h.o.t. = 0,$$

which implies that

$$s_2 = \delta^{-1}M_2^1\mu + h.o.t., \qquad s_3 = -\delta^{-1}M_2^1\mu + h.o.t.$$

Hence, it is impossible to have a 2–1 large homoclinic loop bifurcation for system (1). Moreover, reminding ourselves of the Remark 4.1, our proof is completed. □

If $s_2 = 0, s_1 > 0, s_3 > 0$ is the solution of (8), we obtain

$$-(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1} + \delta^{-1}M_2^1\mu + h.o.t. = 0,$$
$$s_3 + \delta^{-1}M_2^1\mu + h.o.t. = 0,$$
$$(\omega_1^{12})^{-1}s_3^{\rho_1/\lambda_1} - s_1 + \delta^{-1}M_1^1\mu + h.o.t. = 0.$$

Thus,

$$s_1 = (\omega_2^{12}\delta^{-1}M_2^1\mu)^{\lambda_1/\rho_1} + h.o.t., \qquad s_3 = -\delta^{-1}M_2^1\mu + h.o.t.$$

So the codimension 1 bifurcation manifold for the 2–1 right homoclinic loop is

$$H_2^{13} = \{\mu \mid -(\omega_2^{12}\delta^{-1}M_2^1\mu)^{\lambda_1/\rho_1} + \delta^{-1}M_1^1\mu + h.o.t. = 0\}$$

which is well defined at least in the region $\{\mu : M_1^1\mu > 0, M_2^1\mu < 0\}$ and has normal vector M_2^1 at $\mu = 0$.

Let μ be situated in the neighborhood of H_2^{13}, differentiating Eq. (8), take values at H_2^{13}, and denoting by $s_{i\mu}$ the gradient of $s_i(\mu)$ with respect to μ, we get

$$-(\omega_2^{12})^{-1}\rho_1(\omega_2^{12}\delta^{-1}M_2^1\mu)^{(\rho_1-\lambda_1)/\rho_1}s_{1\mu} - \lambda_1 s_{2\mu} + \lambda_1\delta^{-1}M_2^1 + h.o.t. = 0,$$
$$s_{3\mu} + \delta^{-1}M_2^1 + h.o.t. = 0,$$
$$(\omega_1^{12})^{-1}\rho_1(-\delta^{-1}M_2^1\mu)^{(\rho_1-\lambda_1)/\lambda_1}s_{3\mu} - \lambda_1 s_{1\mu} + \lambda_1\delta^{-1}M_1^1 + h.o.t. = 0.$$

Accordingly, we have $s_{2\mu} = \delta^{-1}M_2^1 + O(|\omega_2^{12}\delta^{-1}M_2^1\mu|^{(\rho_1-\lambda_1)/\rho_1})$. Therefore, $s_2 = s_2(\mu)$ increases along the direction M_2^1 in a small neighborhood of H_2^{13}.

Suppose $s_3 = 0, s_1 > 0, s_2 > 0$ is the solution of (8), then one has

$$-(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1} - s_2 + \delta^{-1}M_2^1\mu + h.o.t. = 0,$$
$$(\omega_2^{12})^{-1}s_2^{\rho_1/\lambda_1} + \delta^{-1}M_2^1\mu + h.o.t. = 0,$$
$$-s_1 + \delta^{-1}M_1^1\mu + h.o.t. = 0.$$

Hereafter,

$$s_1 = \delta^{-1}M_1^1\mu + h.o.t., \qquad s_2 = (-\omega_2^{12}\delta^{-1}M_2^1\mu)^{\lambda_1/\rho_1} + h.o.t.,$$

and the codimension of one 2–1 large homoclinic loop bifurcation manifold is

$$H_3^{12} = \{\mu \mid -(\omega_2^{12})^{-1}(\delta^{-1}M_1^1\mu)^{\rho_1/\lambda_1} - (-\omega_2^{12}\delta^{-1}M_2^1\mu)^{\lambda_1/\rho_1} + h.o.t. = 0\}$$

with normal vector $M_2^1(resp.\ M_1^1)$ at $\mu = 0$ as $M_2^1 \neq 0(resp.\ M_2^1 = 0)$, which is well defined at least in the region $\{\mu \mid M_1^1\mu > 0, M_2^1\mu > 0\}$.

When $\mu \in H_3^{12}$, based on (8) we get

$$-(\omega_2^{12})^{-1}\rho_1(\delta^{-1}M_1^1\mu)^{(\rho_1-\lambda_1)/\lambda_1}s_{1\mu} - \lambda_1 s_{2\mu} + \lambda_1\delta^{-1}M_2^1 + h.o.t. = 0,$$
$$(\omega_2^{12})^{-1}\rho_1[(-\omega_2^{12}\delta^{-1}M_2^1\mu)]^{(\rho_1-\lambda_1)/\rho_1}s_{2\mu} + \lambda_1 s_{3\mu} + \lambda_1\delta^{-1}M_2^1 + h.o.t. = 0,$$
$$-s_{1\mu} + \delta^{-1}M_1^1 + h.o.t. = 0.$$

Then we have $s_{3\mu} = -\delta^{-1}M_2^1 + O(|\omega_2^{12}\delta^{-1}M_2^1\mu|^{(\rho_1-\lambda_1)/\rho_1})$ such that $s_3 = s_3(\mu)$ increases along the direction of the gradient $-M_2^1$ in a small neighborhood of H_3^{12}.

Now, we study the 2–1 large period orbit bifurcation and its existence regions.

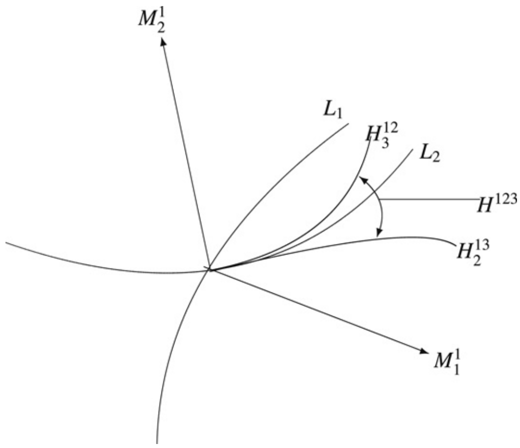

Fig. 3. Bifurcation diagram in single twisted case as $\mathrm{rank}(M_1^1, M_2^1) = 2$.

Due to $(8)_1$– $(8)_2$, we get $s_2 + s_3 = -(\omega_2^{12})^{-1} s_1^{\rho_1/\lambda_1} + h.o.t.$, so $s_3 = o(s_1)$. Because of this and owing to $(8)_3$, we have $s_1 = \delta^{-1} M_1^1 \mu + h.o.t.$ Meanwhile $(8)_1$, $(8)_2$ produce

$$s_2 = \delta^{-1} M_2^1 \mu - (\omega_2^{12})^{-1} (\delta^{-1} M_1^1 \mu)^{\rho_1/\lambda_1} + h.o.t.,$$
$$s_3 = -\delta^{-1} M_2^1 \mu - (\omega_2^{12})^{-1} (\delta^{-1} M_2^1 \mu - (\omega_2^{12})^{-1} (\delta^{-1} M_1^1 \mu)^{\rho_1/\lambda_1})^{\rho_1/\lambda_1} + h.o.t.$$

From the former lines we deduce that for μ sitting on the set H^{123} defined by

$$\{\mu \mid M_1^1 \mu > 0 \text{and (9) is verified}\},$$

a 2–1 large period orbit persists, where

$$(\omega_2^{12})^{-1} (\delta^{-1} M_1^1 \mu)^{\rho_1/\lambda_1} < \delta^{-1} M_2^1 \mu < -(\omega_2^{12})^{-1} (\delta^{-1} M_2^1 \mu - (\omega_2^{12})^{-1} (\delta^{-1} M_1^1 \mu)^{\rho_1/\lambda_1})^{\rho_1/\lambda_1}, \tag{9}$$

and it is nonempty when rank $\{M_1^1, M_2^1\} = 2$.

With the above analysis, we state the following result:

Theorem 4.2. *Suppose that* (H_1)–(H_5) *are fulfilled, then we have the following.*

1. *If $M_1^1 \neq 0$, there exists a unique manifold L_1 with codimension* 1 *and normal vector M_1^1 at $\mu = 0$, such that system* (1) *has a homoclinic loop near Γ_1 if and only if $\mu \in L_1$ and $\|\mu\| \ll 1$.*
   If $M_2^1 \neq 0$, there exists a unique manifold L_2 with codimension 1 *and normal vector M_2^1 at $\mu = 0$, such that system* (1) *has a homoclinic loop near Γ_2.*
   If $\mathrm{rank}(M_1^1, M_2^1) = 2$, *then $L_{12} = L_1 \cap L_2$ is a codimension* 2 *manifold and $0 \in L_{12}$ such that system* (1) *has an* 1–1 *double homoclinic loop near Γ as $\mu \in L_{12}$ and $\|\mu\| \ll 1$, namely, Γ is persistent.*
2. *In the region defined by $\{\mu : M_1^1 \mu > 0, M_2^1 \mu < 0, M_1^1 \mu = o(|M_2^1 \mu|^{\lambda_1/\rho_1})\}$, there exists a unique bifurcation set H_{23}^1 which is tangent to L_2 such that system* (1) *has one* 1–1 *large homoclinic loop and Γ_2 persists as $\mu \in H_{23}^1$.*
   In the region defined by $\{\mu : 0 < M_2^1 \mu \ll 1\}$, there does exist a unique codimension 2 *bifurcation manifold H_{13}^2 which is tangent to $L_1 \cup L_2$ at $\mu = 0$ with the normal plane span$\{M_1^1, M_2^1\}$ when* $\mathrm{rank}\{M_1^1, M_2^1\} = 2$, *and for $\mu \in H_{13}^2$, system* (1) *has a unique* 2–1 *double homoclinic loop near Γ.*
   In the region defined by $\{\mu : M_1^1 \mu > 0, M_2^1 \mu < 0\}$, there exists a unique codimension 1 *bifurcation set H_2^{13} with normal vector M_2^1 (resp. M_1^1) at $\mu = 0$ as $M_2^1 \neq 0$ (resp. $M_2^1 = 0, M_1^1 \neq 0$) such that for $\mu \in H_2^{13}$, system* (1) *has a unique* 2–1 *right homoclinic loop near Γ.*
   In the region defined by $\{\mu : M_1^1 \mu > 0, M_2^1 \mu > 0\}$, there exists a unique 2–1 *large homoclinic loop bifurcation manifold H_3^{12} of codimension* 1 *with normal vector M_2^1 (resp. M_1^1) at $\mu = 0$ as $M_2^1 \neq 0$ (resp. $M_2^1 = 0, M_1^1 \neq 0$) such that for $\mu \in H_3^{12}$, system* (1) *has a unique* 2–1 *large homoclinic loop near Γ.*
3. *When μ belongs to the region* $H^{123} = \{\mu \mid M_1^1 \mu > 0$ and (9) is verified$\}$ *which is bounded by H_2^{13} and H_3^{12}, system* (1) *has a unique* 2–1 *large period orbit, and when μ is situated in the region* $\{\mu \mid M_1^1 \mu \leq 0\} \bigcup \{\mu \mid M_2^1 \mu \leq (\omega_2^{12})^{-1} (\delta^{-1} M_1^1 \mu)^{\rho_1/\lambda_1}\} \bigcup \{\mu \mid M_2^1 \mu \geq -(\omega_2^{12})^{-1} (\delta^{-1} M_2^1 \mu - (\omega_2^{12})^{-1} (\delta^{-1} M_1^1 \mu)^{\rho_1/\lambda_1})^{\rho_1/\lambda_1}\}$, *system* (1) *has no large period orbit. (See Fig. 3)*

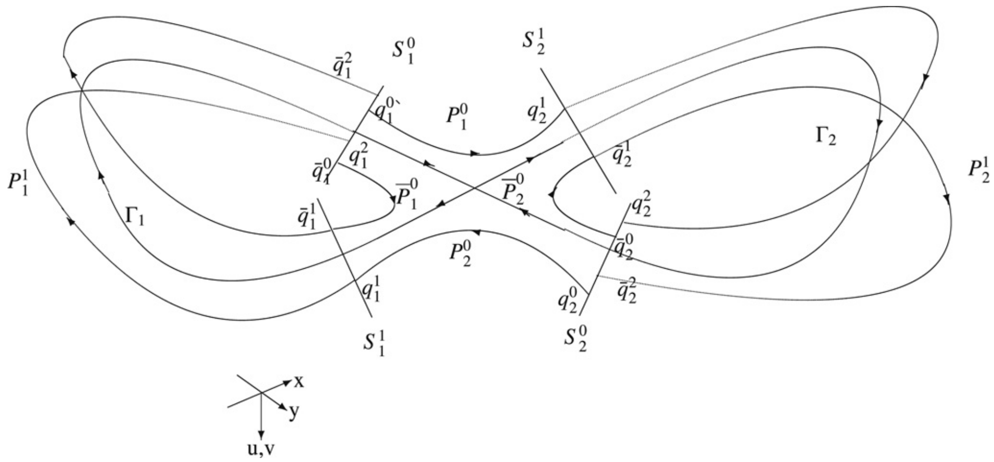

Fig. 4. Poincaré map with double twisted orbits.

5. Bifurcation with double twisted orbits

5.1. Bifurcation equations with double twisted orbits

We now study the bifurcation problem of double twisted orbits, which means that the following hypothesis is verified.

(H_6) Suppose that both Γ_1 and Γ_2 are twisted, that is, $\omega_1^{12} < 0$ and $\omega_2^{12} < 0$.

Let $P_1^0, \overline{P}_2^0, P_2^0$ be the same as in Section 3 and let $\overline{P}_1^0 : S_1^0 \to S_1^1$ (see Fig. 4) be given by

$$\overline{P}_1^0 : \bar{q}_1^0(\bar{x}_1^0, \bar{y}_1^0, \bar{u}_1^0, \bar{v}_1^0) \to \bar{q}_1^1(\bar{x}_1^1, \bar{y}_1^1, \bar{u}_1^1, \bar{v}_1^1),$$
$$\bar{x}_1^0 = s_4\bar{x}_1^1, \qquad \bar{y}_1^1 = s_4^{\rho_1/\lambda_1}\bar{y}_1^0, \qquad \bar{u}_1^0 = s_4^{\lambda_2/\lambda_1}\bar{u}_1^1, \qquad \bar{v}_1^1 = s_4^{\rho_2/\lambda_1}\bar{v}_1^0,$$

where $s_4 = e^{-\lambda_1\tau_4}$ and τ_4 is the flying time from $\bar{q}_1^0$ to $\bar{q}_1^1$, $\bar{x}_1^1 \approx -\delta,\ \bar{y}_1^0 \approx -\delta$. Like before, we have

$$\bar{n}_1^{2j,1} = \bar{x}_1^{2j} - \omega_1^{31}(\omega_1^{33})^{-1}\bar{u}_1^{2j},$$
$$\bar{n}_1^{2j,3} = (\omega_1^{33})^{-1}\bar{u}_1^{2j},$$
$$\bar{n}_1^{2j,4} = \bar{v}_1^{2j} - \delta_1^v - \bar{\omega}_1^{14}\bar{x}_1^{2j} + (\bar{\omega}_1^{14}\omega_1^{31} - \omega_1^{34})(\omega_1^{33})^{-1}\bar{u}_1^{2j},$$
$$\bar{n}_1^{2j+1,1} = (\omega_1^{12})^{-1}\bar{y}_1^{2j+1} - (\omega_1^{12})^{-1}\omega_1^{42}(\omega_1^{44})^{-1}\bar{v}_1^{2j+1},$$
$$\bar{n}_1^{2j+1,3} = \bar{u}_1^{2j+1} - \delta_1^u - \omega_1^{13}(\omega_1^{12})^{-1}\bar{y}_1^{2j+1} + [\omega_1^{13}(\omega_1^{12})^{-1}\omega_1^{42} - \omega_1^{43}](\omega_1^{44})^{-1}\bar{v}_1^{2j+1},$$
$$\bar{n}_1^{2j+1,4} = (\omega_1^{44})^{-1}\bar{v}_1^{2j+1},$$

and

$$F_4 = P_1^1 \circ \overline{P}_1^0 : S_1^0 \to S_1^0,$$
$$\bar{n}_1^{2,1} = -(\omega_1^{12})^{-1}s_4^{\rho_1/\lambda_1}\delta - (\omega_1^{12})^{-1}\omega_1^{42}(\omega_1^{44})^{-1}s_4^{\rho_2/\lambda_1}\bar{v}_1^0 + M_1^1\mu + o(\|\mu\|),$$
$$\bar{n}_1^{2,3} = \bar{u}_1^1 - \delta_1^u + \omega_1^{13}(\omega_1^{12})^{-1}s_4^{\rho_1/\lambda_1}\delta + [\omega_1^{13}(\omega_1^{12})^{-1}\omega_1^{42} - \omega_1^{43}](\omega_1^{44})^{-1}s_4^{\rho_2/\lambda_1}\bar{v}_1^0 + M_1^3\mu + o(\|\mu\|),$$
$$\bar{n}_1^{2,4} = (\omega_1^{44})^{-1}s_4^{\rho_2/\lambda_1}\bar{v}_1^0 + M_1^4\mu + o(\|\mu\|).$$

Up to now, the successor function is given by

$$\begin{aligned} G(s_1, s_2, s_3, s_4, u_1^1, \bar{u}_2^1, u_2^1, \bar{u}_1^1, v_1^0, \bar{v}_2^0, v_2^0, \bar{v}_1^0) &= (G_1^1, G_1^3, G_1^4, G_2^1, G_2^3, G_2^4, G_3^1, G_3^3, G_3^4, G_4^1, G_4^3, G_4^4) \\ &= (F_1(q_1^0) - \bar{q}_2^0, F_2(\bar{q}_2^0) - q_2^0, F_3(q_2^0) - \bar{q}_1^0, F_4(\bar{q}_1^0) - q_1^0) \end{aligned}$$

where

$$G_1^1 = -(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1}\delta - (\omega_2^{12})^{-1}\omega_2^{42}(\omega_2^{44})^{-1}s_1^{\rho_2/\lambda_1}v_1^0 - s_2\delta + \omega_2^{31}(\omega_2^{33})^{-1}s_2^{\lambda_2/\lambda_1}\bar{u}_2^1 + M_2^1\mu + o(\|\mu\|),$$
$$G_1^3 = u_2^1 - \delta_2^u + \omega_2^{13}(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1}\delta + [\omega_2^{13}(\omega_2^{12})^{-1}\omega_2^{42} - \omega_2^{43}](\omega_2^{44})^{-1}s_1^{\rho_2/\lambda_1}v_1^0 - (\omega_2^{33})^{-1}s_2^{\lambda_2/\lambda_1}\bar{u}_2^1 + M_2^3\mu + o(\|\mu\|),$$
$$G_1^4 = -\bar{v}_2^0 + \delta_2^v + (\omega_2^{44})^{-1}s_1^{\rho_2/\lambda_1}v_1^0 + \bar{\omega}_2^{14}\delta s_2 - [\bar{\omega}_1^{14}\omega_2^{31} - \omega_2^{34}](\omega_2^{33})^{-1}s_2^{\lambda_2/\lambda_1}\bar{u}_2^1 + M_2^4\mu + o(\|\mu\|),$$
$$G_2^1 = (\omega_2^{12})^{-1}s_2^{\rho_1/\lambda_1}\delta - (\omega_2^{12})^{-1}\omega_2^{42}(\omega_2^{44})^{-1}s_2^{\rho_2/\lambda_1}\bar{v}_2^0 + s_3\delta + \omega_2^{31}(\omega_2^{33})^{-1}s_3^{\lambda_2/\lambda_1}u_1^1 + M_2^1\mu + o(\|\mu\|),$$

$$G_2^3 = \bar{u}_2^1 - \delta_2^u - \omega_2^{13}(\omega_2^{12})^{-1}s_2^{\rho_1/\lambda_1}\delta + [\omega_2^{13}(\omega_2^{12})^{-1}\omega_2^{42} - \omega_2^{43}](\omega_2^{44})^{-1}s_2^{\rho_2/\lambda_1}\bar{v}_2^0 - (\omega_2^{33})^{-1}s_3^{\lambda_2/\lambda_1}u_1^1 + M_2^3\mu + o(\|\mu\|),$$
$$G_2^4 = -v_2^0 + \delta_2^v + (\omega_2^{44})^{-1}s_2^{\rho_2/\lambda_1}v_2^- \bar{\omega}_2^{14}\delta s_3 - [\bar{\omega}_1^{14}\omega_2^{31} - \omega_2^{34}](\omega_2^{33})^{-1}s_3^{\lambda_2/\lambda_1}u_1^1 + M_2^4\mu + o(\|\mu\|),$$
$$G_3^1 = (\omega_1^{12})^{-1}s_3^{\rho_1/\lambda_1}\delta - (\omega_1^{12})^{-1}\omega_1^{42}(\omega_1^{44})^{-1}s_3^{\rho_2/\lambda_1}v_2^0 + s_4\delta + \omega_1^{31}(\omega_1^{33})^{-1}s_4^{\lambda_2/\lambda_1}\bar{u}_1^1 + M_1^1\mu + o(\|\mu\|),$$
$$G_3^3 = u_1^1 - \delta_1^u + \omega_1^{13}(\omega_1^{12})^{-1}s_3^{\rho_1/\lambda_1}\delta + [\omega_1^{13}(\omega_1^{12})^{-1}\omega_1^{42} - \omega_1^{43}](\omega_1^{44})^{-1}s_3^{\rho_2/\lambda_1}v_2^0 - (\omega_1^{33})^{-1}s_4^{\lambda_2/\lambda_1}\bar{u}_1^1 + M_1^3\mu + o(\|\mu\|),$$
$$G_3^4 = -\bar{v}_1^0 + \delta_1^v + (\omega_1^{44})^{-1}s_3^{\rho_2/\lambda_1}v_2^0 - \bar{\omega}_1^{14}\delta s_4 - [\bar{\omega}_1^{14}\omega_1^{31} - \omega_1^{34}](\omega_1^{33})^{-1}s_4^{\lambda_2/\lambda_1}\bar{u}_1^1 + M_1^4\mu + o(\|\mu\|),$$
$$G_4^1 = -(\omega_1^{12})^{-1}s_4^{\rho_1/\lambda_1}\delta - (\omega_1^{12})^{-1}\omega_1^{42}(\omega_1^{44})^{-1}s_4^{\rho_2/\lambda_1}\bar{v}_1^0 - s_1\delta + \omega_1^{31}(\omega_1 x^{33})^{-1}s_1^{\lambda_2/\lambda_1}u_2^1 + M_1^1\mu + o(\|\mu\|),$$
$$G_4^3 = \bar{u}_1^1 - \delta_1^u + \omega_1^{13}(\omega_1^{12})^{-1}s_4^{\rho_1/\lambda_1}\delta + [\omega_1^{13}(\omega_1^{12})^{-1}\omega_1^{42} - \omega_1^{43}](\omega_1^{44})^{-1}s_4^{\rho_2/\lambda_1}\bar{v}_1^0 - (\omega_1^{33})^{-1}s_1^{\lambda_2/\lambda_1}u_2^1 + M_1^3\mu + o(\|\mu\|),$$
$$G_4^4 = -v_1^0 + \delta_1^v + (\omega_1^{44})^{-1}s_4^{\rho_2/\lambda_1}\bar{v}_1^0 + \bar{\omega}_2^{14}\delta s_1 - [\bar{\omega}_1^{14}\omega_1^{31} - \omega_1^{34}](\omega_1^{33})^{-1}s_1^{\lambda_2/\lambda_1}u_2^1 + M_1^4\mu + o(\|\mu\|).$$

Thereafter, there is a correspondence between the solution $Q = (s_1, s_2, s_3, s_4, u_1^1, \bar{u}_2^1, u_2^1, \bar{u}_1^1, v_1^0, \bar{v}_2^0, v_2^0, \bar{v}_1^0)$ of

$$(G_1^1, G_1^3, G_1^4, G_2^1, G_2^3, G_2^4, G_3^1, G_3^3, G_3^4, G_4^1, G_4^3, G_4^4) = 0$$

with $s_1 \ge 0, s_2 \ge 0, s_3 \ge 0, s_4 \ge 0$, and the existence of 1–1 double homoclinic loop, 1–2 double homoclinic loop, 2–1 double homoclinic loop, 2–2 double homoclinic loop, 2–1 large homoclinic loop, 1–2 large homoclinic loop, 2–2 large homoclinic loop, 2–2 right homoclinic loop, 2–2 large homoclinic loop, 2–2 left homoclinic loop and 2–2 large period orbit of system (1).

From equation $(G_1^3, G_1^4, G_2^3, G_2^4, G_3^3, G_3^4, G_4^3, G_4^4) = 0$, we can solve $(u_2^1, \bar{v}_2^0, \bar{u}_2^1, v_2^0, u_1^1, \bar{v}_1^0, \bar{u}_1^1, v_1^0)$ as in the former section. Substituting it into $(G_1^1, G_2^1, G_3^1, G_4^1) = 0$, we obtain the bifurcation equations

$$\begin{aligned}
&-(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1} - s_2 + \delta^{-1}M_2^1\mu + h.o.t. = 0,\\
&(\omega_2^{12})^{-1}s_2^{\rho_1/\lambda_1} + s_3 + \delta^{-1}M_2^1\mu + h.o.t. = 0,\\
&(\omega_1^{12})^{-1}s_3^{\rho_1/\lambda_1} + s_4 + \delta^{-1}M_1^1\mu + h.o.t. = 0,\\
&-(\omega_1^{12})^{-1}s_4^{\rho_1/\lambda_1} - s_1 + \delta^{-1}M_1^1\mu + h.o.t. = 0.
\end{aligned} \tag{10}$$

5.2. 2–2 bifurcations results with double twisted orbits

In this section, we study the existence, uniqueness and incoexistence problem of the *p*–*q* double homoclinic loops, *p*–*q* large homoclinic loop, *p*–*q* left (right) homoclinic loop, *p*–*q* large period orbit for the double twisted homoclinic orbits Γ.

First, let us give the following result concerning the uniqueness and the incoexistence.

Theorem 5.1. *Assume that* (H_1)–(H_4) *and* (H_6) *hold. Then, for* $\|\mu\|$ *sufficient small, system* (1) *has at most one* 1–1 *double homoclinic loops, one* 1–2 *double homoclinic loops, one* 2–1 *double homoclinic loops, one* 2–2 *double homoclinic loops, one* 2–1 *large homoclinic loop, one* 1–2 *large homoclinic loop, one* 2–2 *large homoclinic loop, one* 2–2 *right homoclinic loop, one* 2–2 *large homoclinic loop, one* 2–2 *left homoclinic loop or one* 2–2 *large period orbit in the small neighborhood of* Γ*. Moreover these orbits do not coexist.*

Proof. Let $Q = (s_1, s_2, s_3, s_4, u_1^1, \bar{u}_2^1, u_2^1, \bar{u}_1^1, v_1^0, \bar{v}_2^0, v_2^0, \bar{v}_1^0)$ and

$$W = \frac{\partial(G_4^1, G_1^1, G_2^1, G_3^1, G_4^3, G_1^3, G_2^3, G_3^3, G_4^4, G_1^4, G_2^4, G_3^4)}{\partial(s_1, s_2, s_3, s_4, u_1^1, \bar{u}_2^1, u_2^1, \bar{u}_1^1, v_1^0, \bar{v}_2^0, v_2^0, \bar{v}_1^0)}\Big|_{Q=0,\ \mu=0},$$

then $\det W = \delta^4 \ne 0$. Due to the implicit function theorem, in the neighborhood of $(Q, \mu) = (0, 0)$, there exists a unique solution $s_i = s_i(\mu), u_i^1 = u_i^1(\mu),\ v_i^0 = v_i^0(\mu),\ \bar{u}_i^1 = \bar{u}_i^1(\mu),\ \bar{v}_i^0 = \bar{v}_i^0(\mu)$ satisfying $s_i(0) = 0, u_i^1(0) = 0,\ v_i^0(0) = 0, \bar{u}_i^1(0) = 0, \bar{v}_i^0(0) = 0, i = 1, 2$.

It indicates that, if $s_1 = s_2 = s_3 = s_4 = 0$, system (1) has a unique 1–1 double homoclinic loops, that is to say, the double homoclinic loop Γ persists.

If $s_1 = s_2 = s_3 = 0, s_4 > 0$, then there exists a unique 1–2 double homoclinic loops, i.e. Γ_1 becomes a 2-homoclinic orbit and Γ_2 persists.

If $s_1 = s_3 = s_4 = 0, s_2 > 0$, then there exists a unique 2–1 double homoclinic loops, namely, Γ_2 becomes a 2-homoclinic orbit and Γ_1 persists.

If $s_1 = s_3 = 0, s_2 > 0, s_4 > 0$, system (1) has a unique 2–2 double homoclinic loop.

If $s_1 = s_4 = 0, s_2 > 0, s_3 > 0$, then Γ_1 is persistent, and meanwhile system (1) has a unique 2–1 large homoclinic loop.

If $s_2 = s_3 = 0, s_1 > 0, s_4 > 0$, then Γ_2 is persistent, and meanwhile system (1) has a unique 2–1 large homoclinic loop.

If $s_1 = 0, s_2 > 0, s_3 > 0, s_4 > 0$, there exists a unique 2–2 large homoclinic loop.

If $s_2 = 0, s_1 > 0, s_3 > 0, s_4 > 0$, system (1) has a unique 2–2 right homoclinic loop.

If $s_3 = 0, s_1 > 0, s_2 > 0, s_4 > 0$, there exists a unique 2–2 large homoclinic loop.

If $s_4=0, s_1>0, s_2>0, s_3>0$, system (1) has a unique 2–2 left homoclinic loop.

If $s_1>0, s_2>0, s_3>0, s_4>0$, system (1) has a unique one 2–2 large period orbit.

Clearly, the uniqueness guarantees that all these kinds of orbits do not coexist. And all other cases are impossible based on the definition of the Poincaré map. □

We now study the bifurcation problem for the double twisted orbits case. It can be remarked that if $s_1=s_2=s_3=0$ ($s_1=s_3=s_4=0$) is the solution of Eq. (10), then $G_1^j=G_2^j$ ($G_3^j=G_4^j$) for $j=1,3,4$, thus the first (or last) two equations of (10) are the same one.

By the same reason as in Section 4, if $s_1=s_2=s_3=s_4=0$ is the solution of the first (or second) equation of (10), then we have $M_2^1\mu+h.o.t.=0$. In the case of $M_2^1\neq 0$, there exists a codimension 1 manifold L_2 with a normal vector M_2^1 at $\mu=0$, such that the first two equations of (10) have solution $s_1=s_2=s_3=s_4=0$ as $\mu\in L_2$ and $\|\mu\|\ll 1$, that is, Γ_2 is persistent. Similarly, there is a codimension 1 manifold L_1 defined by $M_1^1\mu+h.o.t.=0$ with normal vector M_1^1 at $\mu=0$ when $M_1^1\neq 0$ such that the third and the fourth equations of (10) have solution $s_1=s_2=s_3=s_4=0$ as $\mu\in L_1$ and $\|\mu\|\ll 1$, which indicates that Γ_1 is persistent. Suppose $\operatorname{rank}(M_1^1, M_2^1)=2$, then $L_{12}=L_1\cap L_2$ is a codimension 2 manifold with normal plane $span\{M_1^1, M_2^1\}$ such that (10) has solution $s_1=s_2=s_3=s_4=0$ as $\mu\in L_{12}$ and $\|\mu\|\ll 1$, namely, the double homoclinic orbit $\Gamma=\Gamma_1\cup\Gamma_2$ is persistent.

Suppose $s_1=s_2=s_3=0, s_4>0$ is the solution of (10). We have $s_4=-\delta^{-1}M_1^1\mu+h.o.t.$ for $M_1^1\mu<0$. Substituting it into the last equation, we obtain the codimension 2 bifurcation set

$$H_{123}^4: M_2^1\mu+h.o.t.=0,\qquad M_1^1\mu+h.o.t.=0,$$

which is well defined at least in the region $\{\mu: M_1^1\mu<0\}$ with normal plane $span\{M_1^1, M_2^1\}$ at $\mu=0$ when $\operatorname{rank}\{M_1^1, M_2^1\}=2$ such that a unique 1–2 double homoclinic loop bifurcates from Γ for $\mu\in H_{123}^4$. That is, Γ_2 persists, while Γ_1 becomes a 2-homoclinic orbit.

Similarly, we get the bifurcation set

$$H_{134}^2: M_2^1\mu+h.o.t.=0,\qquad M_1^1\mu+h.o.t.=0,$$

such that (10) has solution $s_1=s_3=s_4=0, s_2>0$ as $\mu\in H_{134}^2$, that is, system (1) has a 2–1 double homoclinic loop near Γ. Clearly, H_{134}^2 which is well defined at least in the region $\{\mu: M_2^1\mu>0\}$ when $\operatorname{rank}\{M_1^1, M_2^1\}=2$, has codimension 2 and a normal plane $span\{M_1^1, M_2^1\}$ at $\mu=0$.

If (10) has $s_1=s_2=s_4=0, s_3>0$ as its solution, then $s_3=-\delta^{-1}M_2^1\mu+h.o.t.$. Hence, the bifurcation set

$$\begin{aligned}H_{124}^3:\ & M_2^1\mu+h.o.t.=0,\qquad M_1^1\mu+h.o.t.=0,\\ & (\omega_1^{12})^{-1}(-\delta^{-1}M_2^1\mu)^{\rho_1/\lambda_1}+\delta^{-1}M_1^1\mu+h.o.t.=0,\end{aligned}$$

where Γ persists and a 1–1 large homoclinic orbit bifurcates near Γ, is well defined at least in the region $\{\mu: M_1^1\mu>0, M_2^1\mu<0\}$. When $\operatorname{rank}\{M_1^1, M_2^1\}=2$, it has a codimension no less than 2.

Similarly, another bifurcation set

$$\begin{aligned}H_{234}^1:\ & M_2^1\mu+h.o.t.=0,\qquad M_1^1\mu+h.o.t.=0,\\ & -(\omega_2^{12})^{-1}(\delta^{-1}M_1^1\mu)^{\rho_1/\lambda_1}+\delta^{-1}M_2^1\mu+h.o.t.=0,\end{aligned}$$

such that Γ persists and a 1–1 large homoclinic orbit bifurcates near Γ for $\mu\in H_{2,3,4}^1$ is well defined at least in the region $\{\mu: M_1^1\mu>0, M_2^1\mu<0\}$. It has a codimension no less than 2 as $\operatorname{rank}\{M_1^1, M_2^1\}=2$.

Suppose $s_1=s_3=0, s_2>0, s_4>0$ is the solution of (10). Consequently, we have $s_2=\delta^{-1}M_2^1\mu+h.o.t., s_4=-\delta^{-1}M_1^1\mu+h.o.t.$. Substituting it into the second and fourth equation, the 2–2 double homoclinic loop bifurcation set $H_{13}^{24}: M_1^1\mu+h.o.t.=0,\ M_2^1\mu+h.o.t.=0$ is obtained, which is well defined at least in the region $\{\mu: M_1^1\mu<0, M_2^1\mu>0\}$ as $\operatorname{rank}\{M_1^1, M_2^1\}=2$. It is of codimension 2 and has normal plane $span\{M_1^1, M_2^1\}$ at $\mu=0$.

When $\mu\in H_{13}^{24}$, system (1) has unique 2–2 double homoclinic loops near Γ.

Using the same reasoning, we can obtain the bifurcation set

$$\begin{aligned}H_{24}^{13}:\ & -(\omega_2^{12})^{-1}(\delta^{-1}M_1^1\mu)^{\rho_1/\lambda_1}+\delta^{-1}M_2^1\mu+h.o.t.=0,\\ & (\omega_1^{12})^{-1}(-\delta^{-1}M_2^1\mu)^{\rho_1/\lambda_1}+\delta^{-1}M_1^1\mu+h.o.t.=0,\end{aligned}$$

which is situated in the region $\{\mu: M_1^1\mu>0, M_2^1\mu<0\}$ such that (10) has a solution $s_2=s_4=0, s_1>0,\ s_3>0$ as $\mu\in H_{24}^{13}$ and the corresponding system (1) has two 1–1 large homoclinic orbits near Γ.

Ulteriorly, as the similar analysis tells us that it is impossible for (10) to have a solution (s_1, s_2, s_3, s_4) with $s_i=0$ and for $j\neq i, s_j>0$ or $s_i>0$ for $i, j=1,2,3,4$. So there exists no 2–2 large period orbit.

Thanks to the above analysis, we have

Theorem 5.2. *Suppose that* (H_1)–(H_4), (H_6) *are valid, then*

1. *If* $M_i^1 \neq 0$, *there exists a unique manifold* L_i *with codimension* 1 *and normal vector* M_i^1 *at* $\mu = 0$, *such that system* (1) *has a homoclinic loop near* Γ_i *if and only if* $\mu \in L_i$ *and* $\|\mu\| \ll 1, i = 1, 2$.
   If $\mathrm{rank}(M_1^1, M_2^1) = 2$, *then* $L_{12} = L_1 \cap L_2$ *is a codimension* 2 *manifold and* $0 \in L_{12}$ *such that system* (1) *has an* 1–1 *double homoclinic loop near* Γ *as* $\mu \in L_{12}$ *and* $\|\mu\| \ll 1, i = 1, 2$ *namely,* Γ *is persistent.*
2. *In the region defined by* $\{\mu : M_1^1\mu < 0\}$, *there exists a unique codimension* 2 *bifurcation set* H_{123}^4 *such that system* (1) *has one* 1–2 *double homoclinic loop and* Γ_2 *persists.*
   In the region defined by $\{\mu : M_2^1\mu > 0\}$, *there exists a unique codimension* 2 *bifurcation set* H_{134}^2 *such that system* (1) *has one* 2–1 *double homoclinic loop and* Γ_1 *persists.*
   In the region defined by $\{\mu : M_1^1\mu < 0, M_2^1\mu > 0\}$, *there exists a unique* 2–2 *double homoclinic loop bifurcation set* H_{13}^{24} *of codimension* 2 . *For* $\mu \in H_{13}^{24}$, *system* (1) *has a unique* 2–2 *double homoclinic loop near* Γ.
   In the region defined by $\{\mu : M_1^1\mu > 0, M_2^1\mu < 0\}$, *there exists a codimension* 2 *bifurcation set* H_{24}^{13} *such that system* (1) *has* 1–1 *large homoclinic orbits near* Γ *for* $\mu \in H_{24}^{13}$.
   In the region defined by $\{\mu : M_1^1\mu > 0, M_2^1\mu < 0\}$, *there exist two bifurcation sets* H_{124}^3 *and* H_{234}^1 *with a codimension no less than* 2 , *where* Γ *persists and an additional* 1–1 *large homoclinic orbit bifurcates near* Γ *for* $\mu \in H_{124}^3 \cup H_{234}^1$ *and* $\|\mu\| \ll 1$.
   There exists no 2–2 *large period orbit,* 2–2 *large homoclinic loop,* 2–2 *left homoclinic loop and* 2–2 *right homoclinic loop near* Γ.

5.3. 1–1 bifurcations results with double twisted orbits

In the sequel, we give a further study of the 1–1 large homoclinic orbit and 1–1 large period orbit bifurcation for the case of double twisted orbits.

Consider the following Poincaré maps:

$$F_1 = P_2^1 \circ P_1^0 : S_1^0 \to S_2^0, \qquad F_3 = P_1^1 \circ P_2^0 : S_2^0 \to S_1^0$$

and the successor function

$$\begin{aligned} G(s_1, s_3, u_1^1, u_2^1, v_1^0, v_2^0) &= (G_1^1, G_1^3, G_1^4, G_2^1, G_2^3, G_2^4) \\ &= (F_1(q_1^0) - q_2^0, F_3(q_2^0) - q_1^0). \end{aligned}$$

Using the same procedure as in Section 4, we have:

$$\begin{aligned}
G_1^1 &= -(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1}\delta - (\omega_2^{12})^{-1}\omega_2^{42}(\omega_2^{44})^{-1}s_1^{\rho_2/\lambda_1}v_1^0 + s_3\delta + \omega_2^{31}(\omega_2^{33})^{-1}s_3^{\lambda_2/\lambda_1}u_1^1 + M_2^1\mu + o(\|\mu\|),\\
G_1^3 &= u_2^1 - \delta_2^u + \omega_2^{13}(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1}\delta + [\omega_2^{13}(\omega_2^{12})^{-1}\omega_2^{42} - \omega_2^{43}](\omega_2^{44})^{-1}s_1^{\rho_2/\lambda_1}v_1^0 - (\omega_2^{33})^{-1}s_3^{\lambda_2/\lambda_1}u_1^1 + M_2^3\mu + o(\|\mu\|),\\
G_1^4 &= -v_2^0 + \delta_2^v + (\omega_2^{44})^{-1}s_1^{\rho_2/\lambda_1}v_1^0 - \bar{\omega}_2^{14}\delta s_3 - [\bar{\omega}_2^{14}\omega_2^{31} - \omega_2^{34}](\omega_2^{33})^{-1}s_3^{\lambda_2/\lambda_1}u_1^1 + M_2^4\mu + o(\|\mu\|),\\
G_2^1 &= (\omega_1^{12})^{-1}s_3^{\rho_1/\lambda_1}\delta - (\omega_1^{12})^{-1}\omega_1^{42}(\omega_1^{44})^{-1}s_3^{\rho_2/\lambda_1}v_2^0 - s_1\delta + \omega_1^{31}(\omega_1^{33})^{-1}s_1^{\lambda_2/\lambda_1}u_2^1 + M_1^1\mu + o(\|\mu\|),\\
G_2^3 &= u_1^1 - \delta_1^u - \omega_1^{13}(\omega_1^{12})^{-1}s_3^{\rho_1/\lambda_1}\delta + [\omega_1^{13}(\omega_1^{12})^{-1}\omega_1^{42} - \omega_1^{43}](\omega_1^{44})^{-1}s_3^{\rho_2/\lambda_1}v_2^0 - (\omega_1^{33})^{-1}s_1^{\lambda_2/\lambda_1}u_2^1 + M_1^3\mu + o(\|\mu\|),\\
G_2^4 &= -v_1^0 + \delta_1^v + (\omega_1^{44})^{-1}s_3^{\rho_2/\lambda_1}v_2^0 + \bar{\omega}_1^{14}\delta s_1 - [\bar{\omega}_1^{14}\omega_1^{31} - \omega_1^{34}](\omega_1^{33})^{-1}s_1^{\lambda_2/\lambda_1}u_2^1 + M_1^4\mu + o(\|\mu\|).
\end{aligned}$$

Therefore, there is a correspondence between the solutions $Q = (s_1, s_3, u_1^1, u_2^1, v_1^0, v_2^0)$ of

$$(G_1^1, G_1^3, G_1^4, G_2^1, G_2^3, G_2^4) = 0$$

with $s_1 \geq 0, s_3 \geq 0$, and the existence of 1–1 large homoclinic loops, and a 1–1 large period orbit of system (1).

Solve $(u_2^1, v_2^0, u_1^1, v_1^0)$ from $(G_1^3, G_1^4, G_2^3, G_2^4) = 0$ and then substitute it into $(G_1^1, G_2^1) = 0$, we obtain the bifurcation equation

$$\begin{aligned} &-(\omega_2^{12})^{-1}s_1^{\rho_1/\lambda_1} + s_3 + \delta^{-1}M_2^1\mu + h.o.t. = 0,\\ &(\omega_1^{12})^{-1}s_3^{\rho_1/\lambda_1} - s_1 + \delta^{-1}M_1^1\mu + h.o.t. = 0. \end{aligned} \tag{11}$$

Similarly as in the former sections, we state the following results.

Theorem 5.3. *Assume that* (H_1)–(H_4), (H_6) *hold. Then, for a sufficiently small* $\|\mu\|$, *the system* (1) *has at most one* 1–1 *large homoclinic loop or one* 1–1 *large period orbit in the small neighborhood of* Γ. *Moreover these orbits do not coexist.*

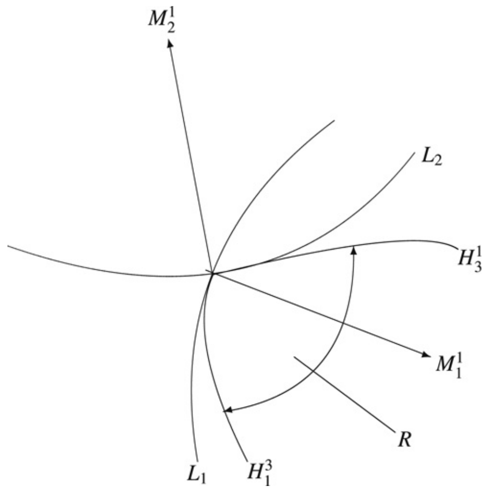

Fig. 5. 1–1 bifurcation diagram in double twisted case as $\text{rank}(M_1^1, M_2^1) = 2$.

Proof. Let $Q = (s_1, s_3, u_1^1, u_2^1, v_1^0, v_2^0)$ and $W = \left.\frac{\partial(G_2^1, G_1^1, G_2^3, G_1^3, G_2^4, G_1^4)}{\partial(s_1, s_3, u_1^1, u_2^1, v_1^0, v_2^0)}\right|_{Q=0,\mu=0}$. Then $\det W = -\delta^2 \neq 0$, According to the implicit function theorem, in the neighborhood of $(Q, \mu) = (0, 0)$, there exists a unique solution $s_i = s_i(\mu), u_i^1 = u_i^1(\mu), v_i^0 = v_i^0(\mu)$, satisfying $s_i(0) = 0, u_i^1(0) = 0, v_i^0(0) = 0, i = 1, 2$. Then if $s_1 = s_3 = 0$, by the uniqueness, we can see that the double homoclinic loop is persistent; if $s_1 = 0, s_3 > 0$ or $s_3 = 0, s_1 > 0$, then system (1) has a unique 1–1 large homoclinic loop; if $s_1 > 0, s_3 > 0$, system (1) has a unique one 1–1 large period orbit.

Clearly, the uniqueness guarantees that all these kinds of orbits do not coexist. □

If (11) has $s_1 = s_3 = 0$ as its solution, then $M_i^1\mu + h.o.t. = 0, i = 1, 2$. In the case of $M_2^1 \neq 0$, there exists a codimension 1 manifold L_2 with normal vector M_2^1 at $\mu = 0$ such that the first equation of (11) has the solution $s_1 = s_3 = 0$ as $\mu \in L_2$ and $\|\mu\| \ll 1$, that is, Γ_2 persists. Similarly, there is a codimension 1 manifold L_1 defined by $M_1^1\mu + h.o.t. = 0$ with normal vector M_1^1 at $\mu = 0$ such that the second equation of (11) has solution $s_1 = s_3 = 0$ as $\mu \in L_1$ and $\|\mu\| \ll 1$, that is, Γ_1 persists. Suppose $\text{rank}(M_1^1, M_2^1) = 2$, then $L_{12} = L_1 \cap L_2$ is a codimension 2 manifold with normal plane *span*$\{M_1^1, M_2^1\}$ such that the double homoclinic orbit $\Gamma = \Gamma_1 \cup \Gamma_2$ is persistent.

If (11) has solution $s_1 = 0, s_3 > 0$, then $s_3 = -\delta^{-1}M_2^1\mu + h.o.t.$ for $M_2^1\mu < 0$. Substituting it into the second equation, we obtain the bifurcation set

$$H_1^3 : (\omega_1^{12})^{-1}(-\delta^{-1}M_2^1\mu)^{\rho_1/\lambda_1} + \delta^{-1}M_1^1\mu + h.o.t. = 0,$$

which is well defined in the region $\{\mu : M_1^1\mu > 0, M_2^1\mu < 0\}$, such that system (1) has a unique 1–1 large homoclinic orbit for $\mu \in H_1^3$ and $\|\mu\| \ll 1$.

When $\mu \in H_1^3$, from (11) we have

$$s_{3\mu} + \delta^{-1}M_2^1 + h.o.t. = 0,$$
$$(\omega_1^{12})^{-1}\rho_1(-\delta^{-1}M_2^1\mu)^{(\rho_1-\lambda_1)/\lambda_1}s_{3\mu} - \lambda_1 s_{1\mu} + \lambda_1\delta^{-1}M_1^1 + h.o.t. = 0.$$

As $s_{1\mu} = \delta^{-1}M_1^1 + O(|M_2^1\mu|^{(\rho_1-\lambda_1)/\lambda_1})$, so s_1 increases along the direction of M_1^1 for $\mu \in H_1^3$ and $\|\mu\| \ll 1$.

Similarly, we get the bifurcation set

$$H_3^1 : -(\omega_2^{12})^{-1}(\delta^{-1}M_1^1\mu)^{\rho_1/\lambda_1} + \delta^{-1}M_2^1\mu + h.o.t. = 0,$$

which is well defined in the region $\{\mu : M_1^1\mu > 0,\ M_2^1\mu < 0\}$, such that (11) has solution $s_1 > 0, s_3 = 0$ as $\mu \in H_3^1$, that is, system (1) has a unique 1–1 large homoclinic orbit near Γ for $\mu \in H_3^1$ and $\|\mu\| \ll 1$. And s_3 increases along the direction $-M_2^1$.

Thus we have proved the following statement.

Theorem 5.4. *Assume that* (H_1)–(H_4), (H_6) *hold. Then*

1. *If* $M_i^1 \neq 0$, *then there exists codimension* 1 *manifold* L_i *with normal vector* M_i^1 *at* $\mu = 0$ *such that* Γ_i *persists for* $\mu \in L_i$ *and* $\|\mu\| \ll 1, i = 1, 2$.

   If $\text{rank}(M_1^1, M_2^1) = 2$, *then* $L_{12} = L_1 \cap L_2$ *is a codimension* 2 *manifold with normal plane span*$\{M_1^1, M_2^1\}$ *such that the double homoclinic orbit* $\Gamma = \Gamma_1 \cup \Gamma_2$ *is persistent as* $\mu \in L_{12}$ *and* $\|\mu\| \ll 1$.

2. *In the region defined by* $\{\mu : M_1^1\mu > 0,\ M_2^1\mu < 0\}$, *there exists a unique codimension* 1 *bifurcation set* H_1^3 *(resp.* H_3^1*) such that system* (1) *has a unique* 1–1 *large homoclinic orbit for* $\mu \in H_1^3$ *(resp.* H_3^1*) and* $\|\mu\| \ll 1$.
3. *There is a sector R bounded by* H_1^3 *and* H_3^1 *such that system* (1) *has a unique* 1–1 *large period orbit for* $\mu \in R$ *and* $\|\mu\| \ll 1$. *(See Fig. 5)*

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