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Exact snapping loads of a buckled beam under a midpoint force

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ABSTRACT

A buckled beam possesses two stable equilibrium configurations and is a natural bistable device. This paper first derives the exact critical load Q^S_{cr} for a hinged buckled beam when it is subject to a concentrated force Q at the midpoint quasi-statically. In the case when the midpoint force is applied suddenly, the exact expression of a conservative dynamic critical load Q^D_{cr} is derived, which guarantees that snapping will not occur as long as Q is smaller than this value.

1. Introduction

An initially straight beam can be buckled into a curved shape by edge thrust. If both ends of the buckled beam are hinged in space, it becomes a natural bistable device. When the buckled beam is loaded laterally, the buckled beam may jump from one side to the other suddenly. This phenomenon is called snap-through buckling, which has wide applications in the design of bistable devices. The mathematical modeling of buckled beams may be divided into two categories. In the first category, exact geometry and elastica theory are used in the analysis of the deformation. Complicated numerical procedures are usually required to solve the nonlinear boundary value problem [1–3]. In the second category in which small deformation is assumed, mathematical analysis may be simplified significantly. In some cases, exact solutions of the snapping loads may be derived. This small-deformation analysis has attracted research interests recently, especially in MEMS applications. This paper deals with the snapping loads of a buckled beam within the small-deformation range.

Seide [4] studied the snapping loads of a hinged buckled beam under a uniformly distributed lateral load. By retaining only the first two terms in an infinite harmonic series, some closed-form expressions may be obtained for the critical loads. Vangbo [5] fabricated a clamped–clamped buckled beam with MEMS technology and analyzed the load–deflection relation when the buckled beam is subject to a midpoint force. Pinto and Goncalves [6] proposed a strategy for active control of a hinged buckled beam under a sinusoidally distributed load. Cazottes et al. [7] studied the deformation of a clamped–clamped buckled beam under a point force, which may be central or offset. In these previous research works, the prediction of snapping load is of great interest. Although the critical loads may be written in closed-form formulas by a two-term approximation [4], or obtained with numerical methods [5–7], simple exact formulas are always desirable. In this paper we present the exact closed-form expressions of the snapping loads, both static and dynamic, of a hinged buckled beam under a midpoint force.

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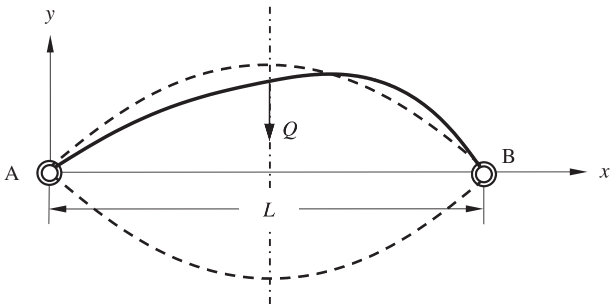

Fig. 1. A buckled beam under a point force at the midpoint. Dashed lines represent the two stable equilibrium configurations when $Q = 0$. The solid line represents an unsymmetric deformation.

2. Equations of motion

We assume that the original distance of the unstressed uniform beam is L_0. The beam is compressed by an axial force and buckles with the distance of the two ends A and B being shortened to L. The end shortening is defined as $e = L_0 - L$. There are two stable equilibrium positions, as shown by the dashed lines in Fig. 1. The origin of an xy-coordinate system is fixed at point A. It is noted that the shapes of the stable unloaded buckled beam are symmetric with respect to the central line $x = L/2$. We assume that the buckled beam is in the upper stable position and is under a point force Q downward at the midpoint. The deformed shape may be symmetric or unsymmetric. The solid line in Fig. 1 represents an unsymmetric deformation $y(x,t)$ under Q.

The equation of motion of the loaded buckled beam can be written as

$$\rho A y_{,tt} = -EIy_{,xxxx} - py_{,xx} - Q\delta\left(x - \frac{L}{2}\right). \tag{1}$$

The parameters E, ρ, A, and I are Young's modulus, mass density, area, and moment of inertia of the cross section of the beam. $\delta(\bullet)$ is the Dirac delta function. p is the axial force,

$$p(t) = \frac{AE}{L}\left[e - \frac{1}{2}\int_0^L (y_{,x})^2 dx\right] \tag{2}$$

It is noted that a positive p represents a compressive force in the beam.

We define the following dimensionless parameters (with asterisks),

$$y^* = \frac{y}{r}, \quad x^* = \frac{\pi x}{L}, \quad e^* = \frac{Le}{\pi^2 r^2}, \quad \delta^*(\cdot) = \frac{L}{\pi}\delta(\cdot),$$
$$t^* = \frac{\pi^2 t}{L^2}\sqrt{\frac{EI}{A\rho}}, \quad p^* = \frac{L^2 p}{\pi^2 EI}, \quad Q^* = \frac{2QL^3}{\pi^4 EIr}. \tag{3}$$

r is the radius of gyration of the cross section $\sqrt{\frac{I}{A}}$. $p^* = 1$ represents the first Euler buckling load. $Q^* = 1$ means that the point force is $\frac{\pi^2}{2}\frac{r}{L}$ times of the buckling load. After substituting relations (3) into Eqs. (1) and (2), and dropping all the superposed asterisks thereafter for simplicity, we obtain the dimensionless version of the equation of motion

$$y_{,tt} + y_{,xxxx} + py_{,xx} = -\frac{\pi}{2}Q\delta\left(x - \frac{\pi}{2}\right), \tag{4}$$

$$p(t) = e - \frac{1}{2\pi}\int_0^\pi (y_{,x})^2 dx. \tag{5}$$

The boundary conditions for y at $x = 0$ and π are

$$y(0) = y_{,xx}(0) = y(\pi) = y_{,xx}(\pi) = 0. \tag{6}$$

We expand y in Eqs. (4) and (5) as follows,

$$y(x,t) = \lim_{N\to\infty}\sum_{n=1}^{N}\alpha_n(t)\sin nx. \tag{7}$$

N denotes the number of modes used in the expansions. After substituting Eq. (7) into (4) and (5) we obtain the equations governing α_n,

$$\ddot{\alpha}_n + n^4\alpha_n - n^2 p\alpha_n + q_n = 0, \quad n = 1, 2, 3, \ldots \tag{8}$$

where

$$p = e - \frac{1}{4}\sum_{k=1}^{\infty} k^2\alpha_k^2, \tag{9}$$

$$q_n = Q\sin\frac{n\pi}{2}, \quad n = 1, 2, 3, \ldots \tag{10}$$

The overhead dot in Eq. (8) represents derivative with respect to time.

3. Equilibrium configurations

For quasi-static loading, Eq. (8) ($\ddot{\alpha}_n$ neglected) represents an infinite number of coupled nonlinear algebraic equations for the infinite number of coordinates α_n. The first question arises is how many equilibrium configurations are possible for a specified e when $Q = 0$. This question can be readily answered from Eqs. (8) and (9). First of all, there always exists a trivial solution with all $\alpha_n = 0$. This is a straight configuration, denoted configuration P_0. The other equilibrium solutions involve only one harmonic mode $\pm\sin nx$, whose amplitude α_n depends on the end shortening e as,

$$\alpha_n = \pm\frac{2\sqrt{e - n^2}}{n}. \tag{11}$$

These one-mode solutions are denoted configurations P_n^+ and P_n^-, or collectively, $P_n^\pm$. The configuration $P_n^\pm$ is symmetric when n is an odd number, and is anti-symmetric when n is an even number.

It is noted that α_n in Eq. (11) is real only when $n^2 \leqslant e$. In the case when $e < 1$, P_0 is the only static solution. When $4 > e \geqslant 1$, there are only three static solutions; they are P_0, P_1^+ and P_1^-. The number of static solutions increases as e increases. Among the many static solutions only P_1^+ and P_1^- are stable, as depicted by the dashed lines in Fig. 1. In the following, we assume that the buckled beam is in position P_1^+ before external load Q is applied. In this position the initial height at the midpoint is, from Eq. (11), $h = 2\sqrt{e-1}$.

As Q increases from zero, in addition to the dominant mode $\pm\sin nx$, the static solutions consist of also an infinite number of odd-number modes. As a consequence, all symmetric (unsymmetric) solutions remain symmetric (unsymmetric). Although the solutions involve an infinite number of harmonic components when $Q \neq 0$, for simplicity we still use the notations $P_n^\pm$ to name the solutions even if they are no longer one-mode solutions. The symmetric and unsymmetric equilibrium configurations can be analyzed analytically as follows.

3.1. Symmetric solutions ($P_0, P_1^\pm, P_3^\pm, P_5^\pm$, etc.)

Symmetric solutions include P_0 and the solutions involving only odd number of n in Eq. (7). The equations with even number of n in Eq. (8) are satisfied automatically because of Eq. (10). The remaining coordinates α_{2i+1} can be related to α_1 from Eq. (8) as

$$\alpha_{2i+1} = \frac{(-1)^{i+1}\alpha_1 Q}{(2i+1)^2[4i(i+1)\alpha_1 - Q]}, \quad i = 1, 2, 3, \ldots \tag{12}$$

After substituting Eq. (12) into Eq. (9), and substituting the resulting p into Eq. (8) for $n = 1$, we obtain the following equation for α_1,

$$\alpha_1\left(1 - e + \frac{\alpha_1^2}{4}\right) + Q + \frac{1}{4}\sum_{i=1}^{\infty}\left\{\frac{Q^2\alpha_1^3}{(2i+1)^2[4i(i+1)\alpha_1 - Q]^2}\right\} = 0. \tag{13}$$

After solving α_1 from Eq. (13), the other coordinates α_{2i+1} can be obtained from Eq. (12). It is noted that Eq. (13) includes the α_1 corresponding to position P_0.

3.2. Unsymmetric solutions ($P_2^\pm, P_4^\pm, P_6^\pm$, etc.)

This type of solutions involves one dominant even-number ($n = 2j$) harmonic component and an infinite number of odd-number components. For this type of solution we can solve for p from the $2j$th equation of Eq. (8) as

$$p = 4j^2. \tag{14}$$

After substituting Eq. (14) into the $(2i+1)$th equation in Eq. (8) we can solve for α_{2i+1} exactly as

$$\alpha_{2i+1} = \frac{q_{2i+1}}{(2i+1)^2[4j^2 - (2i+1)^2]}. \tag{15}$$

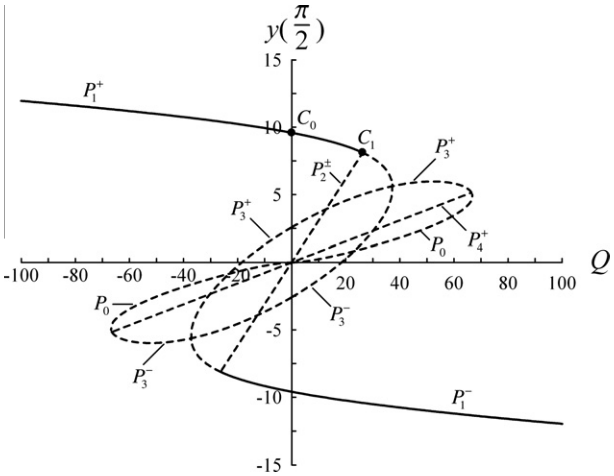

Fig. 2. Relation between the height of the midpoint $y(\frac{\pi}{2})$ and the magnitude Q of the point force for the case when $e = 24$. Solid and dashed lines represent stable and unstable deformations, respectively. Unsymmetrical snapping occurs at point C_1.

After substituting Eqs. (14) and (15) into Eq. (9) we can solve for α_{2j} as

$$\alpha_{2j} = \pm\sqrt{\frac{e}{j^2} - 4 - \frac{1}{4j^2}\sum_{i=1,3,5,\dots}^{\infty}\frac{q_i^2}{i^2[4j^2 - i^2]^2}}. \tag{16}$$

It is noted that the coordinates α_i of an unsymmetric solution can be written in closed forms, while α_i of a symmetric solution cannot.

Fig. 2 shows the relation between the height of the midpoint $y(\frac{\pi}{2})$ and the magnitude Q of the point force for the case when $e = 24$. For the case when $Q = 0$ in Fig. 2, there are 9 equilibrium solutions; they are P_0, and $P_n^\pm$, where $n = 1,2,3,4$. The two straight lines are the loci of the four unsymmetric solutions $P_2^\pm$ and $P_4^\pm$, whose coordinates α_n are obtained exactly from Eqs. (15) and (16). It is noted that P_2^+ and P_2^- are two different solutions, but share the same locus in Fig. 2. The five curved loci in Fig. 2 represent the symmetric solutions P_0, $P_1^\pm$ and $P_3^\pm$, whose coordinate α_1 can be obtained by solving the nonlinear Eq. (13) with a root finder in Mathematica. In the numerical work, instead of $N = \infty$, it is found that a finite number of modes $N = 7$ in Eq. (7) is sufficient to guarantee the accuracy and convergence of the series.

4. Exact static critical load Q_{cr}^S

The stability of the equilibrium configurations can be determined by either a vibration method or an energy method [8]. In Fig. 2, solid and dashed lines represent stable and unstable deformations, respectively. If the magnitude of the midpoint force Q is increased slowly from zero, the midpoint position starts from point C_0 to C_1, at which the P_1^+ locus meets the $P_2^\pm$ locus. The buckled beam snaps unsymmetrically at point C_1, and the corresponding Q is called the static critical load.

Fig. 3 shows another case when $e = 4.5$, in which the P_1^+ locus meets the P_0 locus first, instead of $P_2^\pm$, and forms a limit point at C_1. In this case the buckled beam snaps symmetrically. Apparently, there exists a special e between the ones in Figs. 2 and 3, denoted as $\bar{e}$, at which P_1^+ locus meets both P_0 and $P_2^\pm$ loci simultaneously. This situation occurs when Eq. (13) admits a double root, which requires the derivative of Eq. (13) with respect to α_1 to vanish,

$$4 - 4e + 3\alpha_1^2 + \sum_{i=1}^{\infty}\left\{\frac{Q^2\alpha_1^2}{(2i+1)^2[4i(i+1)\alpha_1 - Q]^2}\left[3 - \frac{8i(i+1)\alpha_1}{4i(i+1)\alpha_1 - Q}\right]\right\} = 0. \tag{17}$$

From Eq. (15) $\alpha_1(P_2^\pm)$ can be written explicitly as

$$\alpha_1(P_2^\pm) = \frac{Q}{3}. \tag{18}$$

After substituting Eq. (18) into Eqs. (13) and (17), both equations can be rearranged further into the forms

$$1024(e-4) - 3\pi^2Q^2 = 0, \tag{19}$$

$$8192(1-e) + 69\pi^2Q^2 = 0. \tag{20}$$

After eliminating Q from Eqs. (19) and (20), $\bar{e}$ is found to be 5.6. Therefore, for a buckled beam with $e \geqslant \bar{e}$, static snap-through from position P_1^+ to P_1^- will occur unsymmetrically. For this case the static critical loads can be solved exactly from Eq. (19) as

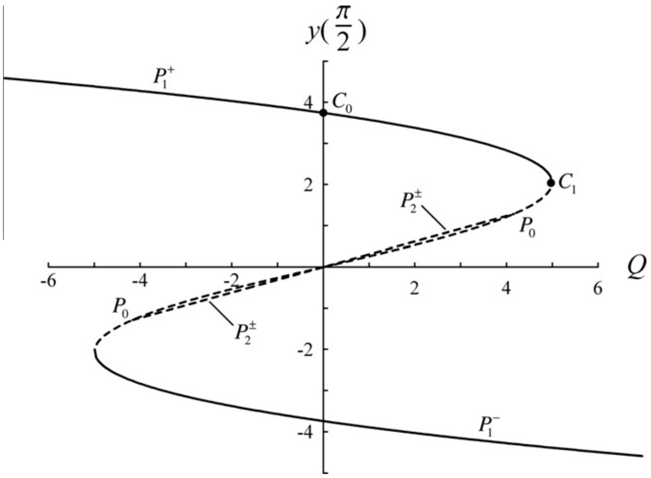

Fig. 3. Relation between the height of the midpoint $y(\frac{\pi}{2})$ and the magnitude Q of the point force for the case when $e = 4.5$. Solid and dashed lines represent stable and unstable deformations, respectively. Symmetrical snapping occurs at point C_1.

$$Q^S_{cr}(e) = \frac{32}{\pi}\sqrt{\frac{e-4}{3}} \quad \text{for } e \geqslant 5.6. \tag{21}$$

On the other hand, for $1 < e < \bar{e}$, the buckled beam will snap symmetrically, whose critical load cannot be expressed in closed form, because the relation between $\alpha_1(P_1^+)$ and Q cannot be written in an explicit manner.

It is noted that Eq. (19) can also be obtained from Eq. (16) by requiring the value in the square root to vanish for $2j = 2$. Therefore, Eq. (19) can also be interpreted as the minimum end shortening, denoted e_2, for positions $P_2^\pm$ to exist when the buckled beam is under point force Q. $e_2(Q)$ can be solved from Eq. (19) as

$$e_2(Q) = 4 + \frac{3\pi^2}{1024}Q^2. \tag{22}$$

Similarly, there exists a minimum e, denoted $e_1(Q)$, for positions P_1^- of the loaded buckled beam to exist. The relation $e_1(Q)$ can only be obtained numerically from Eq. (13). In the special case when $Q = 0$, $e_1 = 1$. In the case when $e < e_1(Q)$, positions P_1^- does not exist, and there will be no snapping at all. $e_1(Q)$ and $e_2(Q)$ are related to dynamic snapping of the buckled beam, as discussed in the next section.

5. Exact dynamic critical load Q^D_{cr}

In the case when the point force is applied at time $t = 0$ suddenly instead of quasi-statically, the buckled beam may snap from P_1^+ to P_1^- dynamically. While it is in general difficult to determine the necessary and sufficient condition for dynamic snap-through to occur, it is possible to establish a sufficient condition against dynamic snap-through in terms of the total potential U of the loaded buckled beam. Based on this concept, we can define a critical value Q^D_{cr}. If the magnitude Q of the concentrated force is smaller than Q^D_{cr}, it is guaranteed that the buckled beam will return to P_1^+ position after the transient vibration is settled by damping. On the other hand, there is no guarantee whether the buckled beam will snap or not when Q is greater than Q^D_{cr}. This conservative critical load Q^D_{cr} can be determined from the condition that the energy barrier between the two stable positions equals the total potential of the original stable position before external load is applied [9].

The dimensionless total potential U of an equilibrium configuration is defined as

$$U = U_s + U_Q, \tag{23}$$

where U_s is the strain energy of the deformed buckled beam and U_Q is the potential corresponding to the point load,

$$U_s = 2p^2 + \frac{2}{\pi}\int_0^\pi (y_{,xx})^2 dx = 2p^2 + \sum_{n=1}^{\infty}[n^4\alpha_n^2], \tag{24}$$

$$U_Q = 2Q\left[y\left(\frac{\pi}{2}\right) - y_0\left(\frac{\pi}{2}\right)\right] = 2Q\left[\sum_{n=1,3,5,\ldots}^{\infty}(-1)^{\frac{n-1}{2}}\alpha_n - 2\sqrt{e-1}\right]. \tag{25}$$

The initial total potential when $Q = 0$ is denoted U_0.

The total potentials of the equilibrium configurations P_j^+ and P_j^- are equal for $Q \neq 0$ when j is an even number, denoted $U(P_j^\pm)$. $U(P_j^+)$ and $U(P_j^-)$ are not equal when j is an odd number. In the case when the loaded buckled beam possesses solutions $P_2^{\pm}$ (when $e \geqslant e_2$), the energy barrier between the two stable configurations P_1^+ and P_1^- can be proved to be $U(P_2^{\pm})$. On the other hand, when $P_2^{\pm}$ do not exist but P_1^- does ($e_1 < e < e_2$), the energy barrier will be $U(P_0)$.

Exact expression of the conservative critical load Q_{cr}^D for the case when $e \geqslant e_2$ is possible because the coordinates α_2 and α_{2i+1} of $P_2^{\pm}$ can be written in closed forms. From Eqs. (23)–(25), $U(P_2^{\pm})$ can be written as

$$U(P_2^{\pm}) = 32 - 4Q\sqrt{e-1} + 16\alpha_2^2 + \sum_{n=1,3,5,\ldots}^{\infty}\left[n^4\alpha_n^2 + 2(-1)^{\frac{n-1}{2}}Q\alpha_n\right]. \tag{26}$$

After substituting α_2 from Eq. (16) and α_{2i+1} from Eq. (15) into Eq. (26), the total potential $U(P_2^{\pm})$ can be simplified to the following form,

$$U(P_2^{\pm}) = \frac{\pi^2}{32}Q^2 - 4Q\sqrt{e-1} + 16(e-2). \tag{27}$$

The total potential of the original stable position before external load is applied is $U_0(P_1^{\pm}) = 4e-2$. From the condition that $U(P_2^{\pm})$ in Eq. (27) equals $U_0(P_1^{\pm})$, we can derive the exact expression of Q_{cr}^D as

$$Q_{cr}^D(e) = \frac{8}{\pi^2}\left[8\sqrt{e-1} - \sqrt{64(e-1)+\pi^2(15-6e)}\right] \quad \text{for } e \geqslant 4.38. \tag{28}$$

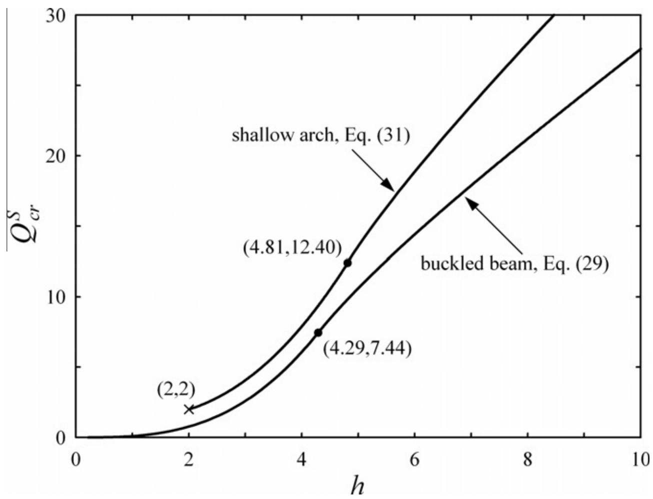

Fig. 4. Relation between Q_{cr}^S and h for a buckled beam and a shallow arch. Exact formulas can be found for the curves beyond the black dots.

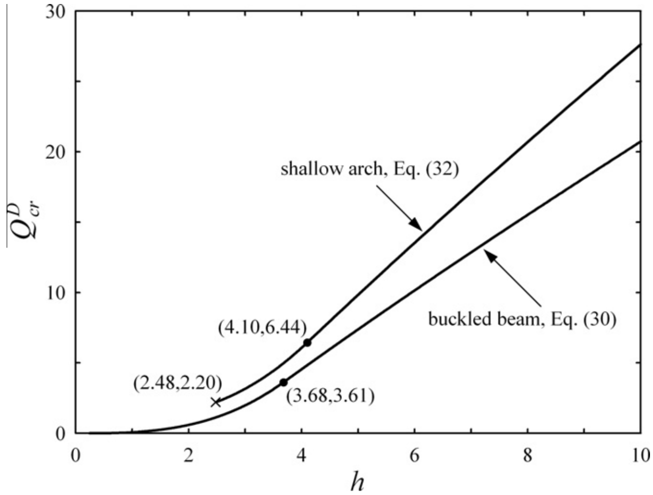

Fig. 5. Relation between Q_{cr}^D and h for a buckled beam and a shallow arch. Exact formulas can be found for the curves beyond the black dots.

Eq. (28) is valid when $P_2^{\pm}$ exists, i.e., $e \geqslant e_2$. From Eqs. (22) and (28) we can solve for the minimum e above which Eq. (28) is valid as $e = 4.38$.

6. Comparison to a shallow arch

It is noted that a buckled beam may look like a shallow arch in appearance before the lateral load is applied. A shallow arch is a curved beam which is stress free before external forces are applied. A buckled beam, on the other hand, is stress free when it is straight. The exact static and dynamic critical loads of a half-sine shallow arch have been derived by Chen et al. [10] with a similar method. It is of interest to compare the critical loads of a shallow arch and a buckled beam.

$Q_{cr}^{S}(e)$ in Eq. (21) and $Q_{cr}^{D}(e)$ in Eq. (28) can be rewritten in terms of the initial midpoint height h by using the relation $h = 2\sqrt{e-1}$ as

$$Q_{cr}^{S}(h) = \frac{16}{\pi}\sqrt{\frac{h^2}{3} - 4} \quad \text{for } h \geqslant 4.29, \tag{29}$$

$$Q_{cr}^{D}(h) = \frac{4}{\pi^2}\left[8h - \sqrt{36\pi^2 + h^2(64 - 6\pi^2)}\right] \quad \text{for } h \geqslant 3.68. \tag{30}$$

The critical loads of a hinged shallow arch are

$$Q_{cr}^{S\pm}(h) = \frac{h \pm \sqrt{h^2 - (1+\kappa)(144 - 8h^2)}}{1+\kappa} \quad \text{for } h \geqslant 4.81, \tag{31}$$

$$Q_{cr}^{D}(h) = \frac{16}{3\pi^2}\left[8h - \sqrt{36\pi^2 + h^2(64 - 6\pi^2)}\right] \quad \text{for } h \geqslant 2\sqrt{3}. \tag{32}$$

The constant κ in Eq. (31) is

$$\kappa = \frac{27\pi^2 - 256}{256}. \tag{33}$$

It is noted that the dynamic critical load of a buckled beam is exactly three quarters of that of a shallow arch. Figs. 4 and 5 show the static and dynamic critical loads, respectively, as functions of h for a shallow arch and a buckled beam. For a buckled beam, bistable states always exist as long as h is nonzero. For a shallow arch, on the other hand, static critical load exists only when $h \geqslant 2$, and dynamic critical load exists only when $h \geqslant 2.48$.

7. Conclusions

This paper presents the critical conditions for snap-through buckling of a hinged buckled beam under a concentrated force at the midpoint. In the case when the concentrated force is applied quasi-statically, the buckled beam will snap unsymmetrically when the end shortening e is greater than 5.6. The static critical load Q_{cr}^{S} can be found in an exact form (Eq. (21)). In the case when the concentrated force is applied suddenly, the exact form of a conservative dynamic critical load Q_{cr}^{D} (Eq. (28)) can be derived when e is greater than 4.38. The critical loads of a buckled beam are approximately three quarters of the ones of a shallow arch with the same initial shape.

References

[1] J.V. Huddleston, Behavior of a steep pressured arch made from a buckled strut, ASME J. Appl. Mech. 37 (1970) 984–994.
[2] A.B. Pippard, The elastic arch and its modes of instability, Eur. J. Phys. 11 (1990) 359–365.
[3] P. Patricio, M. Adda-Bedia, M. Ben Amar, An elastica problem: instabilities of an elastic arch, Physica D 124 (1998) 285–295.
[4] P. Seide, Dynamic stability of laterally loaded buckled beams, J. Eng. Mech. 110 (1984) 1556–1569.
[5] M. Vangbo, An analytical analysis of a compressed bistable buckled beam, Sensors Actuators A 69 (1998) 212–216.
[6] O.C. Pinto, P.B. Goncalves, Nonlinear control of buckled beams under step loading, Mech. Syst. Signal Process. 14 (2000) 967–985.
[7] P. Cazottes, A. Fernandes, J. Pouget, M. Hafez, Bistable buckled beam: modeling of actuating force and experimental validations, ASME J. Mech. Des. 131 (2009) 101001.
[8] N.J. Hoff, V.G. Bruce, Dynamic analysis of the buckling of laterally loaded flat arches, J. Math. Phys. 32 (1954) 276–288.
[9] G.J. Simitses, Dynamic Stability of Suddenly Loaded Structures, Springer-Verlag, New York, 1990.
[10] J.-S. Chen, W.-C. Ro, J.-S. Lin, Exact static and dynamic critical loads of a shallow arch under a point force at the midpoint, Int. J. Non-Linear Mech. 44 (2009) 66–70.