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On a min–max theorem on bipartite graphs

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Abstract

Frank et al. (Math. Programming Stud. 22 (1984) 99–112) proved that for any connected bipartite graft (G,T), the minimum size of a T-join is equal to the maximum value of a partition of A, where A is one of the two colour classes of G. Their proof consists of constructing a partition of A of value $|F|$, by using a minimum T-join F. That proof depends heavily on the properties of distances in graphs with conservative weightings. We follow the dual approach, that is starting from a partition of A of maximum value k, we construct a T-join of size k. Our proof relies only on Tutte's theorem on perfect matchings. It is known (J. Combin. Theory Ser. B 61 (2) (1994) 263–271) that the results of Lovász on 2-packing of T-cuts, of Seymour on packing of T-cuts in bipartite graphs and in grafts that cannot be T-contracted onto $(K_4, V(K_4))$, and of Sebő on packing of T-borders are implied by this theorem of Frank et al. The main contribution of the present paper is that all of these results can be derived from Tutte's theorem.

Keywords: Matchings; T-joins

1. Introduction

This paper concerns matchings and T-joins. Since T-joins are generalizations of matching, the minimum weight T-join problem contains the minimum weight perfect matching problem. On the other hand, Edmonds and Johnson [2] showed that the former problem can be reduced to the latter one. Thus, these problems are—in fact—equivalent.

In matching theory lots of min–max results are known. Concerning matchings, in fact, we shall consider Tutte's theorem [11] on the existence of perfect matchings in general graphs, and not the min–max version, the Tutte–Berge formula. Concerning

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T-joins, we mention the following min–max theorems: The results of Edmonds–Johnson [2], Lovász [7] on 2-packing of T-cuts, of Seymour [9,10] on packing of T-cuts in bipartite graphs and in grafts that cannot be T-contracted onto $(K_4, V(K_4))$, of Sebő [8] on packing of T-borders and a generalization of Seymour's theorem due to Frank et al. [4]. (For the definitions and the theorems see [3] or [5].) There are some easy known implications between these results, some others can be found in [5], where we showed that the result of Frank et al. [4] implies all of these results, including the Tutte theorem.

Our aim in this paper is to demonstrate a new (surprising) implication, namely, Tutte's theorem implies the result of Frank et al. [4], and consequently, all of these min–max results can be derived from Tutte's theorem.

2. Definitions, notation

In this paper $H=(V,E)$ denotes a graph where V is the set of vertices and E is the set of edges. $G=(A,B;E)$ denotes always a bipartite connected graph and $T\subseteq A\cup B$ a subset of vertices of even cardinality. The pair (G,T) is called a bipartite *graft*. An edge set $F\subseteq E$ is a *T-join* if $T=\{v\in A\cup B\colon d_F(v) \text{ is odd}\}$. The minimum size of a T-join is denoted by $\tau(G,T)$. We mention that a bipartite graft (G,T) contains always a T-join.

For a bipartite graft $(G=(A,B;E),T)$ let us introduce an auxiliary graph $G_A := (T,E_A)$ on the vertex set T, where for $u,v\in T$, $uv\in E_A$ if at least one of u and v belongs to A and there exists a path in G connecting u and v of length one or two.

Let K be a vertex set in G. Then $\delta(K)$ denotes the set of edges connecting K and $(A\cup B)-K$. $G[K]$ denotes the subgraph induced by K. b_K^T is defined to be 0 or 1 depending on the parity of $|T\cap K|$. K is called *T-odd* if $b_K^T=1$ and *T-even* if $b_K^T=0$. For a subgraph K of G, $\bar{K}=G[V(G)-V(K)]$.

We shall need the following operation applied for grafts. For a connected subgraph K of G, by *T-contracting* K we mean the graft (G',T') obtained from (G,T) where $G'=G/K$ (that is K is contracted into one vertex v_K) and $T'=T-V(K)$ if $b_K^T=0$ and $T'=T-V(K)+\{v_K\}$ if $b_K^T=1$.

In what follows a *component* of a graph means a connected component. For $X\subseteq V(G)$, $\mathscr{K}(G-X)$ denotes the set of components of $G-X$ and $\mathscr{K}_T(G-X)$ denotes the set of T-odd components of $G-X$. Let $q_T(G-X)=|\mathscr{K}_T(G-X)|$.

We denote by $\mathscr{P}_A := \{u\colon u\in A\}$ the partition of A where the elements of $\mathscr{P}_A$ are the vertices in A as singletons. The value of a (sub)partition $\mathscr{P}=\{A_1,\dots,A_k\}$ of A is defined to be

$$\operatorname{val}(\mathscr{P})=\sum\{q_T(G-A_i)\colon A_i\in\mathscr{P}\}, \tag{1}$$

in other words,

$$\operatorname{val}(\mathscr{P})=\sum\left\{b_K^T\colon K\in\bigcup_{A_i\in\mathscr{P}}\mathscr{K}(G-A_i)\right\}. \tag{2}$$

The theorem of Frank et al. [4] that generalizes all the min–max results mentioned in the Introduction is as follows.

Theorem 1 (Frank, Sebő, Tardos). *If* (G,T) *is a bipartite graft with* $G=(A,B;E)$, *then*

$$\tau(G,T)=\max\{\mathrm{val}(\mathscr{P}):\ \mathscr{P}\ \textit{is a partition of}\ A\}. \tag{3}$$

In order to be able to prove Theorem 1 by induction we will have to prove a slightly stronger result than Theorem 1. To present it we need some definitions. An edge set C of a connected graph G is called *bicut* if $G-C$ has exactly two connected components. Note that each edge of a tree is a bicut. Let $\mathscr{P}=\{A_1,\dots,A_k\}$ be a partition of A and let $\mathscr{Q}=\{B_1,\dots,B_l\}$ be a partition of B. Then $\mathscr{P}\cup\mathscr{Q}$ is called a *bi-partition* of $A\cup B$ in G. Let us denote by $G/(\mathscr{P}\cup\mathscr{Q})$ the bipartite graph obtained from G by identifying the vertices in R for every member $R\in\mathscr{P}\cup\mathscr{Q}$ and by taking the underlying simple graph. A bi-partition $\mathscr{P}\cup\mathscr{Q}$ of $A\cup B$ is called *admissible* if

(i) $F:=G/(\mathscr{P}\cup\mathscr{Q})$ is a tree, and
(ii) for each edge e of F, the edge set of G that corresponds to e forms a bicut of G.

By Claim 4, for any bipartite graft there exists an admissible bi-partition.

Theorem 2. *If* (G,T) *is a bipartite graft with* $G=(A,B;E)$, *then*

$$\tau(G,T)=\max\{\mathrm{val}(\mathscr{P}):\ \mathscr{P}\cup\mathscr{Q}\ \textit{is an admissible bi-partition of}\ A\cup B\}. \tag{4}$$

The proof of Frank et al. [4] for Theorem 1 consists of constructing a partition of A of value $|F|$, by using a minimum T-join F. That proof depends heavily on the properties of distances in graphs with conservative weightings. We follow the dual approach, that is starting from a bi-partition of $A\cup B$ of maximum value k, we construct a T-join of size k. Our proof applies induction. Taking a special optimal admissible bi-partition either we can use induction for some contracted graphs (and here we need admissibility of the bi-partition) or we can apply Tutte's theorem on perfect matchings, namely a graph H has a perfect matching if and only if $q_V(H-X)\leqslant|X|$ for every vertex set X of $V(H)$.

We must mention two papers on this topic. Kostochka [6] and Ageev and Kostochka [1] proved results similar to Theorem 2. Their proof technique is different from the present one.

3. Preliminary results

Claim 3. *Let* $(G=(A,B;E),T)$ *be a bipartite graft.*

(a) *Then the bi-partition* $\mathscr{P}\cup\mathscr{Q}$ *of* $A\cup B$ *satisfies* (i) *where* $\mathscr{P}:=\{a:\ a\in A\}$ *and* $\mathscr{Q}:=\{B\}$.

(b) *If* $X \subseteq A$, *then the bi-partition* $\mathscr{P} \cup \mathscr{Q}$ *of* $A \cup B$ *satisfies* (i) *where* $\mathscr{P} := \{a\colon a \in A - X\} \cup \{X\}$ *and* $\mathscr{Q} := \{K \cap B\colon K \in \mathscr{K}(G - X)\}$.

The following claim (whose proof is left for the reader) shows that for any bipartite graft there exists an admissible bi-partition.

Claim 4. *Let* $(G = (A, B; E), T)$ *be a bipartite graft.*

(a) *If there is no cut vertex in* A *then* $\mathscr{P} \cup \mathscr{Q}$ *is an admissible bi-partition of* $A \cup B$, *where* $\mathscr{P} := \{a\colon a \in A\}$ *and* $\mathscr{Q} := \{B\}$.
(b) *If there is a cut vertex* $v \in A$, *that is* G *can be decomposed into two connected bipartite subgraphs* $G_1 = (A_1, B_1; E_1)$ *and* $G_2 = (A_2, B_2; E_2)$ *with exactly one vertex in common, namely* v, *then let us denote by* (G_1, T_1) *and* (G_2, T_2) *the two grafts obtained from* (G, T) *by* T*-contracting* $V(G_2)$ *and* $V(G_1)$. *If for* $i = 1, 2$, $\mathscr{P}_i \cup \mathscr{Q}_i$ *is an admissible bi-partition of* $A_i \cup B_i$ *and* $v \in A'_i$ *then* $\mathscr{P} \cup \mathscr{Q}$ *is an admissible bi-partition of* $A \cup B$, *where* $\mathscr{P} := (\mathscr{P}_1 - A'_1) \cup (\mathscr{P}_2 - A'_2) \cup \{A'_1 \cup A'_2\}$ *and* $\mathscr{Q} := \mathscr{Q}_1 \cup \mathscr{Q}_2$.

The definition of an admissible bi-partition implies at once the following claim.

Claim 5. *Let* $\mathscr{P} \cup \mathscr{Q}$ *be an admissible bi-partition of* $A \cup B$.

(a) $K \in \mathscr{K}_T(G - A_i)$ *for some* $A_i \in \mathscr{P}$ *if and only if* $\bar{K} \in \mathscr{K}_T(G - B_j)$ *for some* $B_j \in \mathscr{Q}$.
(b) $\mathrm{val}(\mathscr{P}) = \mathrm{val}(\mathscr{Q})$.

Claim 6. *Let* $\mathscr{P}$ *be a partition of* A *and* F *a* T*-join in a bipartite graft* $(G = (A, B; E), T)$.

(a) *Then* $\mathrm{val}(\mathscr{P}) \leqslant |F|$.
(b) *Moreover, if* $\mathrm{val}(\mathscr{P}) = |F|$, *then for every component* K *of* $G - A_i$ *for any* $A_i \in \mathscr{P}$, $|\delta(K) \cap F| = b_K^T$.

Proof. Let $\mathscr{R} := \bigcup_{A_i \in \mathscr{P}} \mathscr{K}(G - A_i)$. By parity, for each $K \in \mathscr{R}$,

$$b_K^T \leqslant |\delta(K) \cap F|.$$

Since for $K_1, K_2 \in \mathscr{R}, \delta(K_1) \cap \delta(K_2) = \emptyset$, we have

$$\mathrm{val}(\mathscr{P}) = \sum_{K \in \mathscr{R}} b_K^T \leqslant \sum_{K \in \mathscr{R}} |\delta(K) \cap F| \leqslant |F|. \qquad \square$$

Claim 7. *For every partition* $\mathscr{P}$ *of* A *in a bipartite graft* $(G = (A, B; E), T)$,

$$\mathrm{val}(\mathscr{P}) \equiv |T \cap A| \pmod 2.$$

Proof. Since $|T|$ is even, for each $A_i \in \mathscr{P}, q_T(G - A_i) \equiv |T \cap A_i| \pmod 2$. Thus

$$\mathrm{val}(\mathscr{P}) = \sum_{A_i \in \mathscr{P}} q_T(G - A_i) \equiv \sum_{A_i \in \mathscr{P}} |T \cap A_i| = |T \cap A|. \qquad \square$$

We shall deal with some bi-partitions along the proofs. The admissibility of these bi-partitions can always be easily verified. The following easy fact may be useful.

Claim 8. *Let X be a subset of vertices of a connected graph H. Let K be a component of $H-X$. If X is contained in one of the components of $H-K$, then $H-K$ is connected.*

Claim 9. *Let H be a connected graph with $|V(H)|$ even. If X is a minimal vertex set with $q_V(H-X)>|X|$, then for every component K of $H-X$, $H-K$ is connected.*

Proof. By assumption, using the usual parity argument, $q_V(H-X)\geqslant |X|+2$. Let K be any component of $H-X$. Then at least one component N of $H-K$ contains more odd components of $H-X$ than vertices in X, that is $q_V(H-(N\cap X))>|N\cap X|$. Then, by the minimality of X, $N\cap X=X$, that is, by Claim 8, $H-K$ is connected. □

Claim 10. *Let $(G=(A,B;E),T)$ be a bipartite graft. If the auxiliary graph G_A has a perfect matching M then G contains a T-join of cardinality $|T\cap A|$.*

Proof. For every edge $uv\in M$ there exists a (u,v)-path in G of length at most two. Since M is a matching these paths are edge disjoint. The union F of these paths is a T-join of G because M covers all the vertices of T. By construction, $|F|=|T\cap A|$. □

4. The proof of Theorem 2

Let (G,T) be a counterexample with minimum number of vertices in G. By Claim 6(a), for any admissible bi-partition $\mathscr{P}\cup\mathscr{Q}$ of $A\cup B$, $\mathrm{val}(\mathscr{P})\leqslant\tau(G,T)$, so $\mathrm{val}(\mathscr{P})<\tau(G,T)$.

Lemma 11. *G is 2-connected.*

Proof. Suppose that G contains a cut vertex v, by symmetry we may suppose that $v\in A$. We use the notation of Claim 4. For $i=1,2$, (G_i,T_i) is a bipartite graft and $|A_i\cup B_i|<|A\cup B|$ so there exists an admissible bi-partition $\mathscr{P}_i\cup\mathscr{Q}_i$ of $A_i\cup B_i$ with

$$\tau(G_i,T_i)=\mathrm{val}(\mathscr{P}_i). \quad (5)$$

Clearly,

$$\tau(G,T)=\tau(G_1,T_1)+\tau(G_2,T_2). \quad (6)$$

Let $\mathscr{P}\cup\mathscr{Q}$ be the admissible bi-partition of $A\cup B$ defined in Claim 4(b). Note that

$$\mathrm{val}(\mathscr{P})=\mathrm{val}(\mathscr{P}_1)+\mathrm{val}(\mathscr{P}_2). \quad (7)$$

Then, by (6), (5) and (7), $\tau(G,T)=\mathrm{val}(\mathscr{P})$ showing that (G,T) is not a counterexample. □

Let us denote by MAX the maximum value of an admissible bi-partition of $A \cup B$. Observe that MAX $\geqslant |T \cap A|$ and MAX $\geqslant |T \cap B|$. The first comes from the admissible bi-partition $\mathscr{P} = \{v\colon v \in A\}, \mathscr{Q} = \{B\}$, the other one from $\mathscr{P} = \{A\}, \mathscr{Q} = \{v\colon v \in B\}$. These bi-partitions are admissible by Claim 4(a).

Case 1: First suppose that MAX $= |T \cap A|$ (or MAX $= |T \cap B|$).

Lemma 12. *If the auxiliary graph G_A has no perfect matching then there exists an admissible bi-partition $\mathscr{P} \cup \mathscr{Q}$ of $A \cup B$ with* $\mathrm{val}(\mathscr{P}) > |T \cap A|$.

Proof. By Tutte's Theorem, there exists a set $X \subset T$ so that $q_T(G_A - X) > |X|$. Let us take a minimal such set.

We claim that $X \cap B = \emptyset$. Suppose that $a \in X \cap B$. Suppose that a is connected to two odd components K_1 and K_2 of $G_A - X$. Then, by the definition of G_A, there is an edge between K_1 and K_2, that is they cannot be different components of $G_A - X$. Thus a is connected to at most one odd component of $G_A - X$. Hence, $q_T(G_A - (X - a)) \geqslant q_T(G_A - X) - 1 \geqslant |X| > |X - a|$, contradicting the minimality of X.

Let us denote by B_1 the set of vertices in $B - T$ that has at least one neighbour in $A \cap T$ and let $B_2 := B - T - B_1$. Let $G_1 := G[T \cup B_1]$ and $G_2 := G[(A - T) \cup B_2]$. Note that by the definition of G_A there is a bijection between the components of $G_A - X$ and the components of $G_1 - X$ different from isolated vertices in B_1. Moreover, the T parity of the corresponding components are the same. Let $\mathscr{R} = \mathscr{K}(G_2)$. Note that if $R \in \mathscr{R}$ then there is no edge between $R \cap B_2$ and $A \cap T$. We distinguish two cases.

Case I: First, suppose that $X = \emptyset$ that is $q_T(G_1) \geqslant 1$, in other words $q_T(G - (A - T)) \geqslant 1$. Let $\mathscr{R}_1 \subseteq \mathscr{R}$ be a minimal subset of $\mathscr{R}$ so that $q_T(G - A') \geqslant 1$, where $A' := \bigcup\{R \cap A\colon R \in \mathscr{R}_1\}$. Let $\mathscr{P} = \{u\colon u \in A - A'\} \cup \{A'\}$ and let $\mathscr{Q} = \{R \cap B\colon R \in \mathscr{K}(G - A')\}$. By Claim 3(b), $\mathscr{P} \cup \mathscr{Q}$ satisfies (i). Since $A' \subseteq A - T$, $|(V(G) - A') \cap T|$ is even so $q_T(G - A') \geqslant 2$ and, by the minimality of $\mathscr{R}_1$, each such component has at least one neighbour in every $R \in \mathscr{R}_1$. Since G is 2-connected and for every $R \in \mathscr{R}_1, G[R]$ is connected, it follows that for every $D \in \mathscr{K}(G - A')$, $G - D$ is connected, that is (ii) is also satisfied, so $\mathscr{P} \cup \mathscr{Q}$ is an admissible bi-partition and

$$\mathrm{val}(\mathscr{P}) = \sum_{A_i \in \mathscr{P}} q_T(G - A_i) \geqslant \sum_{t \in A - A'} b_t^T + q_T(G - A') \geqslant |T \cap A| + 2.$$

Case II: Secondly, suppose that $X \neq \emptyset$. By the minimality of X, $X \subset V(G')$ where $G' \in \mathscr{K}(G_1)$. Let $\mathscr{R}_1 \subseteq \mathscr{R}$ be a minimal subset of $\mathscr{R}$ so that all the components of $G' - X$ rest in different components of $G - A'' - X$, where $A'' := \bigcup\{R \cap A\colon R \in \mathscr{R}_1\}$. Let $\mathscr{P} := \{X \cup A''\} \cup \{u\colon u \in A - (X \cup A'')\}$ and let $\mathscr{Q} = \{R \cap B\colon R \in \mathscr{K}(G - X - A'')\}$. By Claim 3(b), $\mathscr{P} \cup \mathscr{Q}$ satisfies (i). For each $R \in \mathscr{R}_1$, $G[R]$ is connected and, by the minimality of $\mathscr{R}_1$, R has neighbours in at least two different components of $G - X - A''$. Moreover, by Claim 9, for each $K \in \mathscr{K}(G' - X)$, $G' - K$ is connected, hence $(G - \bigcup\{R\colon R \in \mathscr{R}_1\}) - K'$ is connected, where $K' \in \mathscr{K}(G - X - A'')$ that contains K. It follows that $X \cup A''$ is contained in one of the components of $G - K'$. Thus, by Claim 8 and by 2-connectivity,

$\mathscr{P} \cup \mathscr{Q}$ is an admissible bi-partition of $A \cup B$ and

$$\mathrm{val}(\mathscr{P}) = \sum_{A_i \in \mathscr{P}} q_T(G - A_i) = \sum_{t \in A - X - A''} b_t^T + q_T(G - (X \cup A''))$$

$$= |A \cap T| - |X| + q_T(G_A - X) > |T \cap A|. \qquad \square$$

By Lemma 12, G_A (G_B, resp.) has a perfect matching and thus, by Claim 10, G contains a T-join of cardinality $|T \cap A|$ ($|T \cap B|$, resp.). By Claim 6(a), the proof of the theorem is complete.

Case 2: Secondly, suppose that MAX $> |T \cap A|$ and MAX $> |T \cap B|$. Then, by Lemma 11, every optimal admissible bi-partition contains a set A_i with $1 < |A_i| < |A|$. Let us choose an optimal admissible bi-partition $\mathscr{P} \cup \mathscr{Q}$ of $A \cup B$ so that such a set A_i of $\mathscr{P}$ is as large as possible. Let $K \in \mathscr{K}(G - A_i)$ so that $|V(K)| \geqslant 2$. (Since $|A_i| < |A|$ such a set exists.) Then, by Claim 5, $\bar{K} \in \mathscr{K}(G - B_j)$ for some $B_j \in \mathscr{Q}$ and $|V(\bar{K})| \geqslant 2$. Let us denote by (G_1, T_1) and (G_2, T_2) the two bipartite grafts obtained from (G, T) by T-contracting the connected subgraphs K and $\bar{K}$, respectively. The colour classes of G_r will be denoted by A^r and B^r, while the contracted vertex of G_r is denoted by v_r for $r = 1, 2$. Let $\mathscr{P}_1 := \{A_k \in \mathscr{P}\colon A_k \subseteq A^1\}$ and $\mathscr{Q}_1 := \{B_l \in \mathscr{Q}\colon B_l \subseteq B^1\} \cup \{v_1\}$. Let $\mathscr{P}_2 := \{A_k \in \mathscr{P}\colon A_k \subseteq A^2\} \cup \{v_2\}$ and $\mathscr{Q}_2 := \{B_l \in \mathscr{Q}\colon B_l \subseteq B^2\}$. The admissibility of the bi-partition $\mathscr{P} \cup \mathscr{Q}$ implies the following Claim.

Claim 13.

(a) *$\mathscr{P}_r \cup \mathscr{Q}_r$ is an admissible bi-partition of $A^r \cup B^r$ in $G_r, r = 1, 2$.*
(b) $\mathrm{val}_{(G,T)}(\mathscr{P}) = \mathrm{val}_{(G_1,T_1)}(\mathscr{P}_1) - b_{v_1}^{T_1} + \mathrm{val}_{(G_2,T_2)}(\mathscr{P}_2)$.

Lemma 14. *For $r = 1, 2$, $\mathscr{P}_r \cup \mathscr{Q}_r$ is an optimal admissible bi-partition of $A^r \cup B^r$ in (G_r, T_r).*

Proof. By Claim 13(a), only the optimality must be verified. By symmetry, it is enough to prove it for $r = 2$. Suppose that $\mathscr{P}' \cup \mathscr{Q}'$ is an admissible bi-partition of $A^2 \cup B^2$ in G_2 with $\mathrm{val}_{(G_2,T_2)}(\mathscr{P}') > \mathrm{val}_{(G_2,T_2)}(\mathscr{P}_2)$. Let us denote by X that member of $\mathscr{P}'$ that contains v_2. Since $\mathscr{P}_1 \cup \mathscr{Q}_1$ and $\mathscr{P}' \cup \mathscr{Q}'$ are admissible bi-partitions and $\bar{K}$ is connected, $\mathscr{P}'' := (\mathscr{P}_1 - A_i) \cup (\mathscr{P}' - X) \cup \{(X - v_2) \cup A_i\}$, $\mathscr{Q}'' = (\mathscr{Q}_1 - \{v_1\}) \cup \mathscr{Q}'$ is an admissible bi-partition of $A \cup B$ in G. By Claim 13(b),

$$\mathrm{val}_{(G,T)}(\mathscr{P}'') = \mathrm{val}_{(G_1,T_1)}(\mathscr{P}_1) - b_{v_1}^{T_1} + \mathrm{val}_{(G_2,T_2)}(\mathscr{P}')$$

$$> \mathrm{val}_{(G_1,T_1)}(\mathscr{P}_1) - b_{v_1}^{T_1} + \mathrm{val}_{(G_2,T_2)}(\mathscr{P}_2) = \mathrm{val}_{(G,T)}(\mathscr{P}),$$

a contradiction. $\square$

Lemma 15. *If K is T-odd, then for every edge v_2u of G_2, $\mathscr{P}_2 \cup \mathscr{Q}_2$ is an optimal admissible bi-partition of $A^2 \cup B^2$ in (G_2, T_2') of value $\mathrm{val}_{(G_2,T_2)}(\mathscr{P}_2) - 1$, where $T_2' := T_2 \oplus \{v_2, u\}$.*

Proof. By Claim 13(a), only the optimality must be verified. $\mathrm{val}_{(G_2,T_2')}(\mathscr{P}_2)=\mathrm{val}_{(G_2,T_2)}(\mathscr{P}_2)-1$ because for a component L of G_2-R with $R\in\mathscr{P}_2-\{v_2\}$, $|L\cap T_2|\equiv|L\cap T_2'|$ (mod 2) and the unique component K of G_2-v_2 becomes T_2'-even. Suppose that $\mathscr{P}'\cup\mathscr{Q}'$ is an admissible bi-partition of $A^2\cup B^2$ in (G_2,T_2') with $\mathrm{val}_{(G_2,T_2')}(\mathscr{P}')>\mathrm{val}_{(G_2,T_2)}(\mathscr{P}_2)-1$. By Claim 7, $\mathrm{val}_{(G_2,T_2')}(\mathscr{P}')\geqslant\mathrm{val}_{(G_2,T_2)}(\mathscr{P}_2)+1$. Note that since K is T-odd, $b_{v_1}^{T_1}=1$. Let us denote by X that member of $\mathscr{P}'$ that contains v_2. Since K and $\bar{K}$ are connected, $\mathscr{P}'':=(\mathscr{P}_1-A_i)\cup(\mathscr{P}'-X)\cup\{(X-v_2)\cup A_i\}$, $\mathscr{Q}''=(\mathscr{Q}_1-\{v_1\})\cup\mathscr{Q}'$ is an admissible bi-partition of $A\cup B$ in G.

If $X=v_2$ then, by Claim 13(b),

$$\mathrm{val}_{(G,T)}(\mathscr{P}'')=\mathrm{val}_{(G_1,T_1)}(\mathscr{P}_1)+\mathrm{val}_{(G_2,T_2')}(\mathscr{P}')$$
$$\geqslant\mathrm{val}_{(G_1,T_1)}(\mathscr{P}_1)+\mathrm{val}_{(G_2,T_2)}(\mathscr{P}_2)+1>\mathrm{val}_{(G,T)}(\mathscr{P}),$$

a contradiction.

If $X\neq v_2$, then, by Claim 13(b),

$$\mathrm{val}_{(G,T)}(\mathscr{P}'')\geqslant(\mathrm{val}_{(G_1,T_1)}(\mathscr{P}_1)-1)+(\mathrm{val}_{(G_2,T_2')}(\mathscr{P}')-1)$$
$$\geqslant\mathrm{val}_{(G_1,T_1)}(\mathscr{P}_1)-1+\mathrm{val}_{(G_2,T_2)}(\mathscr{P}_2)=\mathrm{val}_{(G,T)}(\mathscr{P}),$$

that is $\mathscr{P}''\cup\mathscr{Q}''$ is an optimal admissible bi-partition of $A\cup B$ in G, but $|(X-v_2)\cup A_i|>|A_i|$, contradicting the maximality of A_i. □

By induction ($|V(G_1)|<|V(G)|$ because $|V(K)|>2$) and by Lemma 14, there exists a T_1-join F_1 in G_1 with $|F_1|=\mathrm{val}(\mathscr{P}_1)$.

First suppose that K is a T-even component of $G-A_i$. By induction ($|V(G_2)|<|V(G)|$ because $|V(\bar{K})|\geqslant 2$) and by Lemma 14, there exists a T_2-join F_2 in G_2 with $|F_2|=\mathrm{val}(\mathscr{P}_2)$. Then, by Claim 6(b), $|F_1\cap\delta(K)|=0=|F_2\cap\delta(K)|$, hence $F:=F_1\cup F_2$ is a T-join and, by Claim 13(b), $|F|=|F_1|+|F_2|=\mathrm{val}(\mathscr{P}_1)+\mathrm{val}(\mathscr{P}_2)=\mathrm{val}(\mathscr{P})$. By Claim 6(a), we are done.

Now suppose that K is a T-odd component of $G-A_i$. Then, by Claim 6(b), $|F_1\cap\delta(K)|=1$. This edge corresponds to an edge v_2u in G_2. By induction ($|V(G_2)|<|V(G)|$ because $|V(\bar{K})|\geqslant 2$) and by Lemma 15 with edge v_2u, there exists a T_2'-join F_2 in G_2 with $|F_2|=\mathrm{val}(\mathscr{P}_2)-1$. Then $F:=F_1\cup F_2$ is a T-join and, by Claim 13(b), $|F|=|F_1|+|F_2|=\mathrm{val}(\mathscr{P}_1)+\mathrm{val}(\mathscr{P}_2)-1=\mathrm{val}(\mathscr{P})$. By Claim 6(a), we are done. □

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