ACADEMIC PRESS
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Journal of Algebra 258 (2002) 543–562
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Hook interpolations

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Received 10 September 2001

Communicated by G.D. James

Abstract

The hook components of $V^{\otimes n}$ interpolate between the symmetric power $\mathrm{Sym}^n(V)$ and the exterior power $\wedge^n(V)$. When V is the vector space of $k \times m$ matrices over $\boldsymbol{C}$, we decompose the hook components into irreducible $GL_k(\boldsymbol{C}) \times GL_m(\boldsymbol{C})$-modules. In particular, classical theorems are proved as boundary cases. For the algebra of square matrices over $\boldsymbol{C}$, a bivariate interpolation is presented and studied.

1. Introduction

The vector space $M_{k,m}$ of $k \times m$ matrices over $\boldsymbol{C}$ carries a (left) $GL_k(\boldsymbol{C})$-action and a (right) $GL_m(\boldsymbol{C})$-action. A classical theorem of Ehresmann [2] describes the decomposition of an exterior power of $M_{k,m}$ into irreducible bimodules. The symmetric analogue was given later (cf. [6]). See Section 2.3 below.

In this paper we present a natural interpolation between these theorems, in terms of hook components of the n-th tensor power of $M_{k,m}$. Duality and asymptotics of the decomposition of hook components follow.

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1 Research was supported in part by the Israel Science Foundation, founded by the Israel Academy of Sciences and Humanities, and by internal research grants from Bar-Ilan University.

0021-8693/02/$ – see front matter
PII: S0021-8693(02)00642-7

Similar methods are then applied to the diagonal two-sided $GL_k(\boldsymbol{C})$-action on the vector space of $k \times k$ matrices. Classical theorems of Thrall [18] and James [7] (for the symmetric powers of symmetric matrices), and of Helgason [4], Shimura [14] and Howe [5] (for the symmetric powers of anti-symmetric matrices) are extended, and a bivariate interpolation is presented.

Proofs are obtained using the representation theory of the symmetric and hyperoctahedral groups, together with Schur–Weyl duality; no use is made of highest-weight theory.

1.1. *Main results*

Let $M_{k,m}$ be the vector space of $k \times m$ matrices over $\boldsymbol{C}$. The tensor power $M_{k,m}^{\otimes n}$ carries a natural S_n-action by permuting the factors. This action decomposes the tensor power into irreducible S_n-modules. Let $M_{k,m}^{\otimes n}(i)$ be the isotypic component of $M_{k,m}^{\otimes n}$ corresponding to the irreducible S_n-representation indexed by the hook $(n-i, 1^i)$, where $0 \leqslant i \leqslant n-1$. This component still carries a $GL_k(\boldsymbol{C}) \times GL_m(\boldsymbol{C})$-action.

Theorem 1.1. *Let λ and μ be partitions of n, of lengths at most k and m, respectively. For every $0 \leqslant i \leqslant n$ the multiplicity of the irreducible $GL_k(\boldsymbol{C}) \times GL_m(\boldsymbol{C})$-module $V_k^\lambda \otimes V_m^\mu$ in $M_{k,m}^{\otimes n}(i-1) \oplus M_{k,m}^{\otimes n}(i)$ is*

$$\sum_{\alpha \vdash n-i,\ \beta \vdash i} c_{\alpha\beta}^{\lambda} c_{\alpha\beta'}^{\mu}$$

where $c_{\alpha\beta}^{\lambda}$ are Littlewood–Richardson coefficients, β' is the partition conjugate to β, and $M_{k,m}^{\otimes n}(-1) = M_{k,m}^{\otimes n}(n) = 0$.

See Theorem 3.3 below; for definitions and notation see Section 2 below. Theorem 1.1 interpolates between two well-known classical theorems (Theorems 2.4 and 2.5 below; see the remark following Theorem 3.3).

The following corollary generalizes the duality between Theorem 2.4 and Theorem 2.5.

Corollary 1.2. *Let $\mu \subseteq (m^m)$ and λ be partitions of n. For every $0 \leqslant i \leqslant n-1$ the multiplicity of $V_k^\lambda \otimes V_m^\mu$ in $M_{k,m}^{\otimes n}(i)$ is equal to the multiplicity of $V_k^\lambda \otimes V_m^{\mu'}$ in $M_{k,m}^{\otimes n}(n-1-i)$.*

See Corollary 3.4 below.

Let λ and μ be partitions of n. Define the *distance*

$$d(\lambda, \mu) := \frac{1}{2} \sum_i |\lambda_i - \mu_i|.$$

Theorem 1.3. *If $V_k^\lambda \otimes V_m^\mu$ appears as a factor in $M_{k,m}^{\otimes n}(t)$ (for some $0 \leqslant t \leqslant n-1$) then*

$$d(\lambda, \mu) < km.$$

See Theorem 4.3 below. This shows that, for $V_k^\lambda \otimes V_m^\mu$ to appear in a hook component, λ and μ must be very "close" to each other (for k and m fixed, n tending to infinity).

Consider now the vector space $M_{k,k}$ of $k \times k$ square matrices over $\boldsymbol{C}$. Let $M_{k,k}^{\otimes n}(i, j)$ be the component of $M_{k,k}^{\otimes n}(i)$ consisting of tensors with j skew symmetric and $n - j$ symmetric factors. $M_{k,k}^{\otimes n}(i, j)$ carries a $GL_k(\boldsymbol{C})$ two-sided diagonal action. The following theorem describes its decomposition as a $GL_k(\boldsymbol{C})$-module.

Theorem 1.4. *Let λ be a partition of $2n$ of length at most k. For every $0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant n$, the multiplicity of V_k^λ in $M_{k,k}^{\otimes n}(i, j) \oplus M_{k,k}^{\otimes n}(i-1, j)$ is*

$$\sum_{\substack{|\alpha|+|\beta|+|\gamma|+|\delta|=n,\\ |\beta|+|\delta|=j,\\ |\gamma|+|\delta|=i}} c^\lambda_{2\cdot\alpha,(2\cdot\beta)',2*\gamma,(2*\delta)'},$$

where the sum runs over all partitions $\alpha, \beta, \gamma, \delta$ with total size n such that β and δ have distinct parts and total size j, and γ and δ have total size i. The operations $*$ and $\cdot$ are defined in Section 2.1. Definition of the (extended) Littlewood–Richardson coefficients is given in Section 2.2.

See Theorem 5.7 below. Theorem 1.4, for $i = 0$, interpolates between classical results, regarding symmetric powers of the spaces of symmetric and skew symmetric matrices (Theorems 2.6 and 2.7 below). Another boundary case, $i = n$, gives an interpolation between exterior powers of the same matrix spaces.

Corollary 1.5. *Let $\lambda \subseteq (k^k)$ be a partition of $2n$. For every $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant n$, the multiplicity of V_k^λ in $M_{k,k}^{\otimes n}(i, j)$ is equal to the multiplicity of $V_k^{\lambda'}$ in $M_{k,k}^{\otimes n}(i, n-j)$.*

See Corollary 5.8 below.

2. Background and notation

2.1. Partitions

Let n be a positive integer. A *partition* of n is a vector of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$ and $\lambda_1 + \cdots + \lambda_k = n$. We denote this by $\lambda \vdash n$. The *size* of a partition $\lambda \vdash n$, denoted $|\lambda|$, is n, and its *length*, $\ell(\lambda)$, is the number of parts. The empty partition $\emptyset$ has size and length zero: $|\emptyset| = \ell(\emptyset) = 0$. The set of all partitions of n with at most k parts is denoted by $\mathrm{Par}_k(n)$.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ define the *conjugate partition* $\lambda' = (\lambda'_1, \ldots, \lambda'_t)$ by letting λ'_i be the number of parts of λ that have size at least i.

A partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ may be viewed as the subset

$$\left\{(i, j) \,\middle|\, 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant \lambda_i\right\} \subset \boldsymbol{Z}^2,$$

the corresponding *Young diagram*. Using this interpretation, we may speak of the intersection $\lambda \cap \mu$ and the set difference $\lambda \backslash \mu$ of any two partitions. The set difference is called a *skew shape*; when $\mu \subseteq \lambda$ it is usually denoted λ/μ.

Let $(k^m) := (k, \ldots, k)$ (m equal parts). Thus, for example, $\lambda \subseteq (k^m)$ means $\lambda_1 \leqslant k$ and $\lambda'_1 \leqslant m$.

We shall also use the Frobenius notation for partitions, defined as follows: Let λ be a partition of n and set $d := \max\{i \mid \lambda_i - i \geqslant 0\}$ (i.e., the length of the main diagonal in the Young diagram of λ). Then the Frobenius notation for λ is $(\lambda_1 - 1, \ldots, \lambda_d - d \mid \lambda'_1 - 1, \ldots, \lambda'_d - d)$.

For any partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n define the following doubling operation:

$$2 \cdot \lambda := (2\lambda_1, \ldots, 2\lambda_k) \vdash 2n.$$

If all the parts of λ are distinct, define also

$$2 * \lambda := (\lambda_1, \ldots, \lambda_k \mid \lambda_1 - 1, \ldots, \lambda_k - 1) \vdash 2n,$$

in the Frobenius notation.

2.2. Representations

For any group G denote the trivial representation by 1_G. In this paper we shall denote the irreducible S_n-modules (Specht modules) by S^λ, and the irreducible $GL_k(\boldsymbol{C})$-modules (Weyl modules) by V_k^λ.

The Littlewood–Richardson coefficients describe the decomposition of tensor products of Weyl modules. Let $\mu \vdash t$ and $\nu \vdash n - t$. Then

$$V_k^\mu \otimes V_k^\nu \cong \bigoplus_{\lambda \vdash n} c_{\mu,\nu}^\lambda V_k^\lambda,$$

for $k \geqslant \max\{\ell(\lambda), \ell(\mu), \ell(\nu)\}$ (and the coefficients $c_{\mu,\nu}^\lambda$ are then independent of k). By Schur–Weyl duality they are also the coefficients of the outer product of Specht modules. Namely,

$$(S^\mu \otimes S^\nu)\uparrow_{S_t \times S_{n-t}}^{S_n} \cong \bigoplus_{\lambda \vdash n} c_{\mu,\nu}^\lambda S^\lambda.$$

The following identity is well known: for all triples of partitions λ, μ, ν,

$$c^{\lambda}_{\mu,\nu} = c^{\lambda'}_{\mu',\nu'}. \tag{2.1}$$

We shall also use the following notation for *Littelwood–Richardson coefficients*:

$$c^{\lambda}_{\alpha,\beta,\gamma,\delta} := \sum_{\mu,\nu} c^{\lambda}_{\alpha,\mu} c^{\mu}_{\beta,\nu} c^{\nu}_{\gamma,\delta};$$

so that

$$V_k^{\alpha} \otimes V_k^{\beta} \otimes V_k^{\gamma} \otimes V_k^{\delta} = \bigoplus_{\lambda} c^{\lambda}_{\alpha,\beta,\gamma,\delta} V_k^{\lambda}.$$

Let B_n be the Weyl group of type B and rank n, also known as the hyperoctahedral group or the group of signed permutations. A *bipartition* of n is an ordered pair (μ, ν) of partitions of total size $|\mu| + |\nu| = n$. The irreducible characters of B_n are indexed by bipartitions of n; denote by $\chi^{\mu,\nu}$ the character indexed by (μ, ν).

Consider the following natural embeddings of S_n into B_n and of B_n into S_{2n}: S_{2n} is the group of permutations on $\{-n, \ldots, -1, 1, \ldots, n\}$. B_n is embedded as the subgroup of all $\pi \in S_{2n}$ satisfying $\pi(-i) = -\pi(i)$ $(1 \leqslant i \leqslant n)$. S_n is embedded as the subgroup of all $\pi \in B_n$ satisfying also $\pi(i) > 0$ $(1 \leqslant i \leqslant n)$.

The following lemmas, used in Section 5, describe certain induced characters via the above embeddings. Lemma 2.1 is an immediate consequence of [11, Chapter I, Section 7, Example 4; Chapter I, Section 8, Examples 5–6; and Chapter VII, (2.4)]. See also [17].

Lemma 2.1.

(a) $1_{B_n}\uparrow^{S_{2n}}_{B_n} = \chi^{(n),\emptyset}\uparrow^{S_{2n}}_{B_n} = \sum_{\lambda\vdash n} \chi^{2\cdot\lambda}$;

(b) $\chi^{\emptyset,(n)}\uparrow^{S_{2n}}_{B_n} = \sum_{\lambda\vdash n} \chi^{(2\cdot\lambda)'}$;

(c) $\chi^{(1^n),\emptyset}\uparrow^{S_{2n}}_{B_n} = \sum_{\lambda\vdash n} \chi^{2*\lambda}$;

(d) $\chi^{\emptyset,(1^n)}\uparrow^{S_{2n}}_{B_n} = \sum_{\lambda\vdash n} \chi^{(2*\lambda)'}$;

where the last two sums are over partitions with distinct parts.

Lemma 2.2.

(a) $\chi^{(n)}\uparrow^{B_n}_{S_n} = \sum_{i=0}^{n} \chi^{(i),(n-i)}$.

(b) $\chi^{(1^n)}\uparrow^{B_n}_{S_n} = \sum_{i=0}^{n} \chi^{(1^i),(1^{n-i})}$.

For a proof, see Appendix A.1.

The following lemma is a special case of the Littlewood–Richardson rule for B_n; cf. [16, Lemma 7.1].

Lemma 2.3. $\chi^{(i),(n-i)} = (\chi^{(i),\emptyset} \otimes \chi^{\emptyset,(n-i)})\uparrow_{B_i \times B_{n-i}}^{B_n}$.

2.3. Symmetric and exterior powers of matrix spaces

In this subsection we cite well-known classical theorems, concerning the decomposition into irreducibles of symmetric and exterior powers of matrix spaces, which are to be generalized in this paper.

Let $M_{k,m}$ be the vector space of $k \times m$ matrices over $\boldsymbol{C}$. Then $M_{k,m}$ carries a (left) $GL_k(\boldsymbol{C})$-action and a (right) $GL_m(\boldsymbol{C})$-action. A classical theorem of Ehresmann [2] (see also [10]) describes the decomposition of an exterior power of $M_{k,m}$ into irreducible $GL_k(\boldsymbol{C}) \times GL_m(\boldsymbol{C})$-modules.

Theorem 2.4. *The n-th exterior power of $M_{k,m}$ is isomorphic, as a $GL_k(\boldsymbol{C}) \times GL_m(\boldsymbol{C})$-module, to*

$$\wedge^n(M_{k,m}) \cong \bigoplus_{\substack{\lambda \vdash n \\ \lambda \subseteq (m^k)}} V_k^{\lambda} \otimes V_m^{\lambda'},$$

where λ' is the partition conjugate to λ.

The following three results on symmetric powers were proved several times independently; these results may be found in [3,6].

The symmetric analogue of Theorem 2.4 was studied, for example, in [6, (11.1.1)] and [3, Theorem 5.2.7].

Theorem 2.5. *The n-th symmetric power of $M_{k,m}$ is isomorphic, as a $GL_k(\boldsymbol{C}) \times GL_m(\boldsymbol{C})$-module, to*

$$\mathrm{Sym}^n(M_{k,m}) \cong \bigoplus_{\substack{\lambda \vdash n \\ \ell(\lambda) \leqslant \min(k,m)}} V_k^{\lambda} \otimes V_m^{\lambda}.$$

Let $M_{k,k}^+$ be the vector space of symmetric $k \times k$ matrices over $\boldsymbol{C}$. This space carries a natural two sided $GL_k(\boldsymbol{C})$-action. The following theorem describes the decomposition of its symmetric powers into irreducible $GL_k(\boldsymbol{C})$-modules.

Theorem 2.6. *The n-th symmetric power of $M_{k,k}^+$ is isomorphic, as a $GL_k(\boldsymbol{C})$-module, to*

$$\mathrm{Sym}^n\left(M_{k,k}^+\right) \cong \bigoplus_{\lambda \in \mathrm{Par}_k(n)} V_k^{2\cdot\lambda}.$$

This theorem was proved by A.T. James [7], but had already appeared in an early work of Thrall [18]. See also [5,14], [6, (11.2.2)] and [3, Theorem 5.2.9] for further proofs and references.

Let $M^-_{k,k}$ be the vector space of skew symmetric $k \times k$ matrices over $\boldsymbol{C}$.

Theorem 2.7. *The n-th symmetric power of $M^-_{k,k}$ is isomorphic, as a $GL_k(\boldsymbol{C})$-module, to*

$$\mathrm{Sym}^n\left(M^-_{k,k}\right) \cong \bigoplus_{(2\cdot\lambda)' \in \mathrm{Par}_k(2n)} V_k^{(2\cdot\lambda)'}.$$

This theorem is proved in [4,5,14]. See also [6, (11.3.2)] and [3, Theorem 5.2.11].

3. Hook components of $M^{\otimes n}_{k,m}$

Consider $M = M_{k,m} = \boldsymbol{C}^{k\times m}$, the vector space of $k \times m$ matrices over $\boldsymbol{C}$. Then $M \cong V \otimes W$, where $V \cong \boldsymbol{C}^k$ and $W \cong \boldsymbol{C}^m$. Thus M carries a (left) $GL(V)$-action and a (right) $GL(W)$-action, which commute. Its tensor power $M^{\otimes n} \cong V^{\otimes n} \otimes W^{\otimes n}$ thus carries a $GL(V) \times S_n \times S_n \times GL(W)$ linear representation; one copy of the symmetric group S_n permutes the factors in $V^{\otimes n}$, and the other copy of S_n permutes the factors in $W^{\otimes n}$. The actions of all four groups clearly commute. We are interested in the $GL(V) \times S_n \times GL(W)$-action on $M^{\otimes n}$ obtained through the diagonal embedding $S_n \hookrightarrow S_n \times S_n$, $\pi \mapsto (\pi, \pi)$.

Lemma 3.1.

$$M^{\otimes n} \cong \bigoplus_{\substack{\lambda\in\mathrm{Par}_k(n)\\ \nu\in\mathrm{Par}(n)\\ \mu\in\mathrm{Par}_m(n)}} \alpha_{\lambda\mu\nu} V_k^\lambda \otimes S^\nu \otimes V_m^\mu,$$

where $\alpha_{\lambda\mu\nu} := \langle \chi^\lambda \chi^\mu \chi^\nu, 1_{S_n}\rangle$.

Proof. By Schur–Weyl duality (the double commutant theorem) [3, Theorem 9.1.2],

$$V^{\otimes n} \cong \bigoplus_{\lambda\in\mathrm{Par}_k(n)} V_k^\lambda \otimes S^\lambda$$

as $GL(V) \times S_n$-modules. Similarly,

$$W^{\otimes n} \cong \bigoplus_{\lambda\in\mathrm{Par}_m(n)} V_m^\lambda \otimes S^\lambda$$

as $GL(W) \times S_n$-modules. Therefore

$$M^{\otimes n} \cong V^{\otimes n} \otimes W^{\otimes n} \cong \bigoplus_{\substack{\lambda \in \mathrm{Par}_k(n) \\ \mu \in \mathrm{Par}_m(n)}} V_k^\lambda \otimes S^\lambda \otimes S^\mu \otimes V_m^\mu$$

as $GL(V) \times S_n \times S_n \times GL(W)$-modules.

Using the diagonal embedding $S_n \hookrightarrow S_n \times S_n$,

$$M^{\otimes n} \cong \bigoplus_{\substack{\lambda \in \mathrm{Par}_k(n) \\ \mu \in \mathrm{Par}_m(n)}} V_k^\lambda \otimes \left(S^\lambda \otimes S^\mu\right)\downarrow_{S_n}^{S_n \times S_n} \otimes V_m^\mu$$

as $GL(V) \times S_n \times GL(W)$-modules.

Note that the S_n-character of $(S^\lambda \otimes S^\mu)\downarrow_{S_n}^{S_n \times S_n}$ is the standard inner tensor product (sometimes called Kronecker product) of the S_n-characters χ^λ and χ^μ. Hence, by elementary representation theory,

$$\left(S^\lambda \otimes S^\mu\right)\downarrow_{S_n}^{S_n \times S_n} \cong \bigoplus_{\nu \vdash n} \alpha_{\lambda\mu\nu} S^\nu, \quad \text{where}$$

$$\alpha_{\lambda\mu\nu} = \langle \chi^\lambda \chi^\mu, \chi^\nu \rangle = \frac{1}{n!} \sum_{\pi \in S_n} \chi^\lambda(\pi)\chi^\mu(\pi)\chi^\nu(\pi) = \langle \chi^\lambda \chi^\mu \chi^\nu, 1_{S_n} \rangle. \qquad \square$$

In particular, Lemma 3.1 gives Theorems 2.4 and 2.5.

Corollary 3.2.

(1) $\mathrm{Sym}^n(M) \cong \bigoplus_{\substack{\lambda \vdash n \\ \ell(\lambda) \leqslant \min(k,m)}} V_k^\lambda \otimes V_m^\lambda.$

(2) $\wedge^n(M) \cong \bigoplus_{\substack{\lambda \vdash n \\ \lambda \subseteq (m^k)}} V_k^\lambda \otimes V_m^{\lambda'}.$

Proof. $\mathrm{Sym}^n(M)$ is the isotypic component of $M^{\otimes n}$ corresponding to the trivial character $\chi^{(n)}$ of the symmetric group. Thus, by Lemma 3.1,

$$\mathrm{Sym}^n(M) \cong \bigoplus_{\substack{\lambda \in \mathrm{Par}_k(n) \\ \mu \in \mathrm{Par}_m(n)}} \alpha_{\lambda,\mu,(n)} V_k^\lambda \otimes S^{(n)} \otimes V_m^\mu.$$

But by the orthonormality of irreducible characters,

$$\alpha_{\lambda,\mu,(n)} = \langle \chi^\lambda \chi^\mu, \chi^{(n)} \rangle = \langle \chi^\lambda, \chi^\mu \chi^{(n)} \rangle = \langle \chi^\lambda, \chi^\mu \rangle = \delta_{\lambda\mu}.$$

This proves (1), namely Theorem 2.5.

The n-th exterior power is the isotypic component of $M^{\otimes n}$ corresponding to the sign character $\chi^{(1^n)}$ of the symmetric group. Recall that for any partition $\mu \vdash n$, $\chi^{\mu}\chi^{(1^n)} = \chi^{\mu'}$. Thus

$$\alpha_{\lambda,\mu,(1^n)} = \langle \chi^{\lambda}\chi^{\mu}, \chi^{(1^n)} \rangle = \langle \chi^{\lambda}, \chi^{\mu}\chi^{(1^n)} \rangle = \langle \chi^{\lambda}, \chi^{\mu'} \rangle = \delta_{\lambda\mu'}.$$

This proves (2), namely, Theorem 2.4. □

Let M be the vector space of $k \times m$ matrices as before. The tensor power $M^{\otimes n}$ carries a natural S_n-action by permuting the factors. This action decomposes into irreducible S_n-representations. Let $M^{\otimes n}(t)$ be the component of $M^{\otimes n}$, corresponding to the irreducible hook representation $(n-t, 1^t)$, $0 \leqslant t \leqslant n-1$. This component carries a $GL_k(\boldsymbol{C}) \times GL_m(\boldsymbol{C})$-action.

Theorem 3.3. *Let $\lambda \in \mathrm{Par}_k(n)$ and $\mu \in \mathrm{Par}_m(n)$. For every $0 \leqslant t \leqslant n$, the multiplicity of the irreducible $GL_k(\boldsymbol{C}) \times GL_m(\boldsymbol{C})$-module $V_k^{\lambda} \otimes V_m^{\mu}$ in $M^{\otimes n}(t-1) \oplus M^{\otimes n}(t)$ is*

$$\sum_{\substack{\alpha \vdash n-t \\ \beta \vdash t}} c^{\lambda}_{\alpha\beta} c^{\mu}_{\alpha\beta'}$$

where $c^{\lambda}_{\alpha\beta}$ are Littlewood–Richardson coefficients, β' is the partition conjugate to β, and $M^{\otimes n}(-1) = M^{\otimes n}(n) = 0$.

Remark. Theorem 3.3 may be considered as an interpolation between Theorems 2.4 and 2.5. $M^{\otimes n}(0) \cong \mathrm{Sym}^n(M)$ and $M^{\otimes n}(-1) = 0$. Substituting $t = 0$ forces $\beta = \emptyset$. Hence $\lambda = \alpha = \mu$. So, the multiplicity is $\delta_{\lambda\mu}$. This gives Theorem 2.5.

Similarly, $M^{\otimes n}(n-1) \cong \wedge^n(M)$ and $M^{\otimes n}(n) = 0$. Substituting $t = n$ forces $\alpha = \emptyset$. Hence $\lambda = \beta = \mu'$. So, the multiplicity is $\delta_{\lambda\mu'}$. This gives Theorem 2.4.

Proof. By Lemma 3.1,

$$M^{\otimes n}(t) \cong \bigoplus_{\substack{\lambda \in \mathrm{Par}_k(n) \\ \mu \in \mathrm{Par}_m(n)}} \alpha_{\lambda,\mu,(n-t,1^t)} V_k^{\lambda} \otimes S^{(n-t,1^t)} \otimes V_m^{\mu}$$

is the decomposition of this component into irreducibles.

Denote by 1_t and ε_t the trivial and sign characters, respectively, of S_t. By the combinatorial interpretation of the Littlewood–Richardson rule (cf. [8, Theorem 2.8.13]), for every $0 \leqslant t \leqslant n$

$$(1_{n-t} \otimes \varepsilon_t)\uparrow^{S_n}_{S_{n-t} \times S_t} = \chi^{(n-t,1^t)} + \chi^{(n-t+1,1^{t-1})}. \tag{3.1}$$

Hence, by Frobenius reciprocity,

$$\begin{aligned}\alpha_{\lambda,\mu,(n-t,1^t)} + \alpha_{\lambda,\mu,(n-t+1,1^{t-1})} &= \langle \chi^\lambda \chi^\mu, \chi^{(n-t,1^t)} + \chi^{(n-t+1,1^{t-1})} \rangle \\ &= \langle \chi^\lambda \chi^\mu, (1_{n-t} \otimes \varepsilon_t)\uparrow_{S_{n-t}\times S_t}^{S_n} \rangle \\ &= \langle (\chi^\lambda \chi^\mu)\downarrow_{S_{n-t}\times S_t}^{S_n}, 1_{n-t} \otimes \varepsilon_t \rangle \\ &= \langle \chi^\lambda \downarrow_{S_{n-t}\times S_t}^{S_n}, \chi^\mu \downarrow_{S_{n-t}\times S_t}^{S_n} \cdot (1_{n-t} \otimes \varepsilon_t) \rangle.\end{aligned}$$

By the Littlewood–Richardson rule the last expression is equal to

$$\begin{aligned}&\Big\langle \sum_{\alpha\vdash n-t,\ \beta\vdash t} c^\lambda_{\alpha\beta} \chi^\alpha \otimes \chi^\beta, \sum_{\alpha\vdash n-t,\ \beta\vdash t} c^\mu_{\alpha\beta} \chi^\alpha \otimes \chi^\beta \cdot (1_{n-t} \otimes \varepsilon_t) \Big\rangle \\ &\quad = \Big\langle \sum_{\alpha\vdash n-t,\ \beta\vdash t} c^\lambda_{\alpha\beta} \chi^\alpha \otimes \chi^\beta, \sum_{\alpha\vdash n-t,\ \beta\vdash t} c^\mu_{\alpha\beta} \chi^\alpha \otimes \chi^{\beta'} \Big\rangle \\ &\quad = \sum_{\alpha\vdash n-t,\ \beta\vdash t} c^\lambda_{\alpha\beta} c^\mu_{\alpha\beta'}. \qquad \square\end{aligned}$$

The following corollary generalizes the "duality" of Theorems 2.4 and 2.5.

Corollary 3.4. *Let $\lambda \in \mathrm{Par}_k(n)$, and let $\mu, \mu' \in \mathrm{Par}_m(n)$ be conjugate partitions. Then, for every $0 \leqslant t \leqslant n-1$, the multiplicity of $V_k^\lambda \otimes V_m^\mu$ in $M^{\otimes n}(t)$ is equal to the multiplicity of $V_k^\lambda \otimes V_m^{\mu'}$ in $M^{\otimes n}(n-1-t)$.*

Proof. It suffices to show that the multiplicity of $V_k^\lambda \otimes V_m^\mu$ in $M^{\otimes n}(t-1) \oplus M^{\otimes n}(t)$ is equal to the multiplicity of $V_k^\lambda \otimes V_m^{\mu'}$ in $M^{\otimes n}(n-t) \oplus M^{\otimes n}(n-t-1)$. By Theorem 3.3, this is equivalent to the identity

$$\sum_{\alpha\vdash n-t,\ \beta\vdash t} c^\lambda_{\alpha\beta} c^\mu_{\alpha\beta'} = \sum_{\alpha\vdash t,\ \beta\vdash n-t} c^\lambda_{\alpha\beta} c^{\mu'}_{\alpha\beta'}.$$

But this follows from (2.1). $\square$

Examples. Let $\lambda \in \mathrm{Par}_k(n)$, $\mu, \mu' \in \mathrm{Par}_m(n)$. The multiplicities of $V_k^\lambda \otimes V_m^\mu$ in $M^{\otimes n}(t)$ for $t = 0$ and $t = n-1$ are given by Theorems 2.5 and 2.4. Consider two other pairs of t-values.

- $t = 1$. For $\lambda = \mu$ the multiplicity is the number of (inner) corners in λ, minus 1. For $\lambda \neq \mu$ this is 1 if $|\lambda \setminus \mu| = 1$, and zero otherwise.
- $t = n-2$. For $\lambda = \mu'$ the multiplicity is the number of (inner) corners in λ, minus 1. For $\lambda \neq \mu'$ this is 1 if $|\lambda \setminus \mu'| = 1$, and zero otherwise.
- $t = 2$ ($n > 2$). The multiplicity is nonzero iff there is a partition α of $n-2$ such that λ/α is a horizontal strip and μ/α is a vertical strip, or vice versa.
- $t = n-3$ ($n > 2$). The multiplicity is nonzero iff there is a partition α of $n-2$ such that λ/α is a horizontal strip and μ'/α is a vertical strip, or vice versa.

4. Asymptotics

Let λ and μ be partitions of the same number n. Recalling the definition of the set difference $\lambda\backslash\mu$ from Section 2.1, define the *distance*

$$d(\lambda,\mu) := |\lambda \setminus \mu| \quad \left(= \frac{1}{2}\sum_i |\lambda_i - \mu_i|\right).$$

Lemma 4.1. *If $V_k^\lambda \otimes V_m^\mu$ appears as a factor in $M^{\otimes n}(t)$ (for some $0 \leqslant t \leqslant n-1$) then $d(\lambda,\mu) \leqslant t$ and $d(\lambda,\mu') \leqslant n-1-t$.*

Proof. By assumption, $V_k^\lambda \otimes V_m^\mu$ appears as a factor in both $M^{\otimes n}(t-1) \oplus M^{\otimes n}(t)$ and $M^{\otimes n}(t) \oplus M^{\otimes n}(t+1)$.

By Theorem 3.3, if $V_k^\lambda \otimes V_m^\mu$ appears as a factor in $M^{\otimes n}(t-1) \oplus M^{\otimes n}(t)$ then there exists a pair of partitions, $\alpha \vdash n-t$ and $\beta \vdash t$ such that, $c^\lambda_{\alpha\beta} c^\mu_{\alpha\beta'} \neq 0$. $c^\lambda_{\alpha\beta} \neq 0 \Rightarrow \alpha \subseteq \lambda$ and $c^\mu_{\alpha\beta'} \neq 0 \Rightarrow \alpha \subseteq \mu$. Hence $|\lambda\backslash\mu| \leqslant |\lambda\backslash\alpha| = t$. Also, $c^\lambda_{\alpha\beta} \neq 0 \Rightarrow \beta \subseteq \lambda$ and $c^\mu_{\alpha\beta'} \neq 0 \Rightarrow \beta \subseteq \mu'$. Hence $|\lambda\backslash\mu'| \leqslant |\lambda\backslash\beta| = n-t$.

Similarly, if $V_k^\lambda \otimes V_m^\mu$ appears as a factor in $M^{\otimes n}(t) \oplus M^{\otimes n}(t+1)$ then $|\lambda\backslash\mu| \leqslant t+1$ and $|\lambda\backslash\mu'| \leqslant n-t-1$.

Altogether, we get the desired claim. □

Let ψ be an S_n-character (not necessarily irreducible). Define the *height* of ψ by

$$\text{height}(\psi) := \max\bigl\{\ell(\nu) \bigm| \nu \vdash n,\ \langle\psi,\chi^\nu\rangle \neq 0\bigr\}.$$

The following result was proved by Regev.

Lemma 4.2 [8, Theorem 12]. *For any $\lambda,\mu \vdash n$,*

$$\text{height}\bigl(\chi^\lambda\chi^\mu\bigr) \leqslant \ell(\lambda)\cdot\ell(\mu).$$

Theorem 4.3. *If $V_k^\lambda \otimes V_m^\mu$ appears as a factor in $M^{\otimes n}(t)$ (for some $0 \leqslant t \leqslant n-1$) then*

$$d(\lambda,\mu) \leqslant km.$$

Proof.

$$d(\lambda,\mu) \overset{(1)}{\leqslant} t \overset{(2)}{\leqslant} \text{height}\bigl(\chi^\lambda\chi^\mu\bigr) - 1 \overset{(3)}{\leqslant} \ell(\lambda)\cdot\ell(\mu) - 1 \leqslant km - 1.$$

Inequalities (1), (2), and (3) follow from Lemmas 4.1, 3.1 (for $\nu = (n-t, 1^t)$), and 4.2, respectively. □

Let ψ be an S_n-character (not necessarily irreducible). Define the *height* of ψ by

$$\text{width}(\psi) := \max\bigl\{\mu_1 \bigm| \nu \vdash n,\ \langle \psi, \chi^\nu \rangle \neq 0\bigr\}.$$

The following result of Dvir strengthens Lemma 4.2.

Lemma 4.4 [2, Theorem 1.6]. *For any $\lambda, \mu \vdash n$,*

$$\begin{aligned} &(1) \quad \text{width}\bigl(\chi^\lambda \chi^\mu\bigr) = |\lambda \cap \mu| \quad \textit{and} \\ &(2) \quad \text{height}\bigl(\chi^\lambda \chi^\mu\bigr) = |\lambda \cap \mu'|. \end{aligned}$$

This result gives another way of proving Theorem 4.3.

Second proof of Theorem 4.3.

$$\begin{aligned} d(\lambda, \mu) &= |\lambda \backslash \mu| = n - |\lambda \cap \mu| \overset{(1)}{\leqslant} t \overset{(2)}{\leqslant} \text{height}\bigl(\chi^\lambda \chi^\mu\bigr) - 1 \overset{(3)}{=} |\lambda \cap \mu'| - 1 \\ &\leqslant km - 1. \end{aligned}$$

Inequality (1) follows from Lemma 4.4(1), since $n - t \leqslant \text{width}(\chi^\lambda \chi^\mu)$. Inequality (2) follows from Lemma 3.1. Equality (3) is Lemma 4.4(2). □

Note. For any two partitions λ, μ of n with $\ell(\lambda) \leqslant k$ and $\ell(\mu) \leqslant m$, $V_k^\lambda \otimes V_m^\mu$ appears as a factor in $M_{k,m}^{\otimes n}$. Theorem 4.3 shows that, in order to appear in a hook component, λ and μ must be very "close" to each other (for fixed k and m and n tending to infinity).

5. Square matrices

Consider now the vector space $M_k = M_{k,k}$ of $k \times k$ matrices over $\boldsymbol{C}$. This space carries a diagonal (left and right) $GL_k(\boldsymbol{C})$-action, defined by

$$g(m) := g \cdot m \cdot g^t \quad (\forall g \in GL_k(\boldsymbol{C}),\ \forall m \in M_k).$$

5.1. Symmetric powers

Recall from Section 2.1 the definition of $2 \cdot \lambda$, for a partition λ.

Theorem 5.1. *For $\lambda \in \text{Par}_k(2n)$, the multiplicity of V_k^λ in $\text{Sym}^n(M_k)$ is*

$$\sum_{|\mu|+|\nu|=n} c^\lambda_{2\cdot\mu,(2\cdot\nu)'}.$$

Corollary 5.2. *Let* $\lambda \in \mathrm{Par}(2n)$, $\lambda \subseteq (k^k)$ *(i.e.,* $\lambda, \lambda' \in \mathrm{Par}_k(2n)$*). Then the multiplicities of* V_k^λ *and of* $V_k^{\lambda'}$ *in* $\mathrm{Sym}^n(M_k)$ *are equal.*

Proof. This is an immediate consequence of Theorem 5.1, applying identity (2.1). □

Proof of Theorem 5.1. Let $V \cong \boldsymbol{C}^k$. Then $V \otimes V$ carries a diagonal (left) GL_k-action, and

$$M_k \cong V \otimes V$$

as GL_k-modules. Thus

$$M_k^{\otimes n} \cong V^{\otimes 2n}$$

as GL_k-modules. Moreover, $V^{\otimes 2n}$ carries an $S_{2n} \times GL_k$-action: S_{2n} permutes the $2n$ factors in the tensor product, and GL_k acts on all of them simultaneously (on the left). The S_{2n}- and GL_k-actions satisfy Schur–Weyl duality (the double commutant theorem), so that

$$V^{\otimes 2n} \cong \bigoplus_{\lambda \in \mathrm{Par}_k(2n)} V_k^\lambda \otimes S^\lambda,$$

as $GL_k \times S_{2n}$-modules.

Now, $\mathrm{Sym}^n(M_k)$ is the part of $M_k^{\otimes n}$ which is invariant under the action of $S_n \hookrightarrow S_{2n}$, where the embedding $S_n \hookrightarrow S_n \times S_n \subseteq S_{2n}$ is diagonal: $\pi \longmapsto (\pi, \pi)$. It follows that the multiplicity of V_k^λ in $\mathrm{Sym}^n(M_k)$ is equal to the multiplicity of the trivial character 1_{S_n} in the restriction $\chi^\lambda \downarrow_{S_n}^{S_{2n}}$, where S_n is diagonally embedded.

By Frobenius reciprocity,

$$\langle 1_{S_n}, \chi^\lambda \downarrow_{S_n}^{S_{2n}} \rangle = \langle 1_{S_n} \uparrow_{S_n}^{S_{2n}}, \chi^\lambda \rangle.$$

We conclude that, for $\lambda \in \mathrm{Par}_k(2n)$, the multiplicity of V_k^λ, in $\mathrm{Sym}^n(M_k)$ is

$$\langle 1_{S_n} \uparrow_{S_n}^{S_{2n}}, \chi^\lambda \rangle.$$

We shall compute these multiplicities in several steps.

First, we induce in two steps:

$$1_{S_n} \uparrow_{S_n}^{S_{2n}} = \left(1_{S_n} \uparrow_{S_n}^{B_n}\right) \uparrow_{B_n}^{S_{2n}}.$$

By Lemmas 2.2(a) and 2.3,

$$\begin{aligned} \left(\chi^{(n)} \uparrow_{S_n}^{B_n}\right) \uparrow_{B_n}^{S_{2n}} &= \sum_{i=0}^{n} \chi^{(i),(n-i)} \uparrow_{B_n}^{S_{2n}} \\ &= \sum_{i=0}^{n} \left(\chi^{(i),\emptyset} \otimes \chi^{\emptyset,(n-i)}\right) \uparrow_{B_i \times B_{n-i}}^{B_n} \uparrow_{B_n}^{S_{2n}} \end{aligned}$$

$$= \sum_{i=0}^{n} \left(\chi^{(i),\emptyset} \otimes \chi^{\emptyset,(n-i)}\right)\uparrow_{B_i \times B_{n-i}}^{S_{2n}}.$$

Again, let us induce in two steps:

$$\begin{aligned}\left(\chi^{(i),\emptyset} \otimes \chi^{\emptyset,(n-i)}\right)\uparrow_{B_i \times B_{n-i}}^{S_{2n}} &= \left(\left(\chi^{(i),\emptyset} \otimes \chi^{\emptyset,(n-i)}\right)\uparrow_{B_i \times B_{n-i}}^{S_{2i} \times S_{2n-2i}}\right)\uparrow_{S_{2i} \times S_{2n-2i}}^{S_{2n}} \\ &= \left(\chi^{(i),\emptyset}\uparrow_{B_i}^{S_{2i}} \otimes \chi^{\emptyset,(n-i)}\uparrow_{B_{n-i}}^{S_{2n-2i}}\right)\uparrow_{S_{2i} \times S_{2n-2i}}^{S_{2n}}.\end{aligned}$$

By Lemma 2.1, (a) and (b), the right-hand side is equal to

$$\left(\sum_{\mu \vdash i} \chi^{2\cdot\mu} \otimes \sum_{\nu \vdash n-i} \chi^{(2\cdot\nu)'}\right)\uparrow_{S_{2i} \times S_{2n-2i}}^{S_{2n}}.$$

We conclude that

$$1_{S_n}\uparrow_{S_n}^{S_{2n}} = \sum_{i=0}^{n} \sum_{\mu \vdash i,\ \nu \vdash n-i} \left(\chi^{2\cdot\mu} \otimes \chi^{(2\cdot\nu)'}\right)\uparrow_{S_{2i} \times S_{2n-2i}}^{S_{2n}}.$$

Applying the Littlewood–Richardson rule completes the proof. □

5.2. A graded refinement of symmetric powers

The space $M_k^{\otimes n}$ carries not only an S_n-action but also a B_n-action, where the signed permutation $(i, -i)$ $(1 \leqslant i \leqslant n)$ acts by transposing the i-th factor in the tensor product of n square matrices. $M_k = M_k^+ \oplus M_k^-$, where M_k^+ (M_k^-) is the vector space of symmetric (skew symmetric) matrices of order $k \times k$. Consequently, $M_k^{\otimes n}$ is graded by the number of skew symmetric factors. The component of $M_k^{\otimes n}$ with i skew symmetric factors, denoted $M_k^{\otimes n}(i)$, is invariant under the B_n-action, as well as under the diagonal two-sided GL_k-action.

Lemma 5.3. *If the irreducible B_n-character $\chi^{\mu,\nu}$ appears in the decomposition of the B_n-action on $M_k^{\otimes n}(i)$, then $|\nu| = i$.*

For a proof see Appendix A.2.

Since the components $M_k^{\otimes n}(i)$ are invariant under the S_n-action, the S_n-invariant subspace $\mathrm{Sym}^n(M_k)$ inherits the grading by the number of skew symmetric factors. Let $\mathrm{Sym}_i^n(M_k)$ denote the component with i skew symmetric factors. The following theorem refines Theorem 5.1.

Theorem 5.4. *For $\lambda \in \mathrm{Par}_k(2n)$, the multiplicity of V_k^λ in $\mathrm{Sym}_i^n(M_k)$ is*

$$\sum_{\mu \vdash n-i,\ \nu \vdash i} c_{2\cdot\mu,(2\cdot\nu)'}^{\lambda}.$$

Note. Theorem 5.4 interpolates between two classical results, Theorems 2.6 and 2.7. Indeed, $\mathrm{Sym}_0^n(M_k) = \mathrm{Sym}^n(M_k^+)$ is the n-th symmetric power of the vector space of symmetric matrices. In this case $i = 0$, so $\nu = \emptyset$. Hence

$$\sum_{\mu\vdash n} c^{\lambda}_{2\cdot\mu,\emptyset} = \begin{cases} 1, & \text{if } \lambda = 2\cdot\mu \text{ for some } \mu \vdash n; \\ 0, & \text{otherwise.} \end{cases}$$

This gives Theorem 2.6. Similarly, $\mathrm{Sym}_n^n(M_k) = \mathrm{Sym}^n(M_k^-)$. In this case $i = n$, $\mu = \emptyset$, and a similar computation gives Theorem 2.7.

An analogue of Corollary 3.4 follows.

Corollary 5.5. *Let $\lambda, \lambda' \in \mathrm{Par}_k(2n)$ be conjugate partitions. Then, for every $0 \leqslant i \leqslant n$, the multiplicity of V_k^{λ} in $\mathrm{Sym}_i^n(M_k)$ is equal to the multiplicity of $V_k^{\lambda'}$ in $\mathrm{Sym}_{n-i}^n(M_k)$.*

Proof. Combine Theorem 5.4 with identity (2.1). □

Proof of Theorem 5.4. This is a refinement of the proof of Theorem 5.1. In this refinement the group B_n appears in an essential way, whereas in the proof of Theorem 5.1 it was used only as a technical tool.

$M_k^{\otimes n}$ is a B_n-module, and $\mathrm{Sym}^n(M_k)$ is its submodule, for which the B_n-action, when restricted to S_n, is trivial. Hence, if the irreducible B_n-character $\chi^{\mu,\nu}$ appears in $\mathrm{Sym}^n(M_k)$, then

$$\left\langle \chi^{\mu,\nu}\downarrow^{B_n}_{S_n}, 1_{S_n}\right\rangle \neq 0.$$

By Lemma 2.2(a),

$$\left\langle \chi^{\mu,\nu}\downarrow^{B_n}_{S_n}, 1_{S_n}\right\rangle = \left\langle \chi^{\mu,\nu}, 1_{S_n}\uparrow^{B_n}_{S_n}\right\rangle = \left\langle \chi^{\mu,\nu}, \sum_{j=0}^{n} \chi^{(n-j),(j)}\right\rangle,$$

and this is nonzero (and equal to 1) if and only if $\mu = (n-j)$ and $\nu = (j)$ for some $1 \leqslant j \leqslant n$.

Combining this with Lemma 5.3 we conclude that $\chi^{(n-i),(i)}$ is the unique irreducible B_n-character in $\mathrm{Sym}_i^n(M_k)$.

Now, as in the proof of Theorem 5.1, the multiplicity of V_k^{λ} in $\mathrm{Sym}_i^n(M_k)$ is

$$\left\langle \chi^{\lambda}\downarrow^{S_{2n}}_{B_n}, \chi^{(n-i),(i)}\right\rangle = \left\langle \chi^{\lambda}, \chi^{(n-i),(i)}\uparrow^{S_{2n}}_{B_n}\right\rangle.$$

By Lemmas 2.3 and 2.1,

$$\begin{aligned} \chi^{(n-i),(i)}\uparrow^{S_{2n}}_{B_n} &= \left(\chi^{(n-i),\emptyset} \otimes \chi^{\emptyset,(i)}\right)\uparrow^{S_{2n}}_{B_{n-i}\times B_i} \\ &= \left(\sum_{\mu\vdash n-i} \chi^{2\cdot\mu} \otimes \sum_{\nu\vdash i} \chi^{(2\cdot\nu)'}\right)\uparrow^{S_{2n}}_{S_{2n-2i}\times S_{2i}}. \end{aligned}$$

The Littlewood–Richardson rule completes the proof of Theorem 5.4. □

5.3. *Hook components of tensor powers*

In this subsection we generalize the results of the previous sections to obtain a bivariate interpolation between symmetric and exterior powers of symmetric and skew symmetric matrices.

As before, the n-th tensor power $M_k^{\otimes n}$ carries an S_n-action. The symmetric power $\mathrm{Sym}^n(M_k)$ is the S_n-invariant part, i.e., corresponds to the trivial character $\chi^{(n)}$. The exterior power corresponds to the sign character $\chi^{(1^n)}$. We shall denote the factor corresponding to the hook character $\chi^{(n-t,1^t)}$ $(0 \leqslant t \leqslant n-1)$ by $M_k^{\otimes n}(t)$ with the convention $M_k^{\otimes n}(-1) = M_k^{\otimes n}(n) = 0$.

Theorem 5.6. *For every $0 \leqslant t \leqslant n-1$ and $\lambda \in \mathrm{Par}_k(2n)$, the multiplicity of V_k^λ in $M_k^{\otimes n}(t) \oplus M_k^{\otimes n}(t-1)$ is*

$$\sum_{\substack{|\alpha|+|\beta|=n-t,\\ |\gamma|+|\delta|=t}} c^{\lambda}_{2\cdot\alpha,(2\cdot\beta)',2*\gamma,(2*\delta)'},$$

where the sum runs over all partitions α and β with total size $n-t$, and partitions γ and δ with distinct parts and total size t. The operations $\cdot$ and $*$ are as defined in Section 2.1*, and the extended Littlewood–Richardson coefficients are as defined in Section* 2.2*.*

Proof. Similar arguments to those in the proof of Theorem 5.1 show that the multiplicity of V_k^λ in the hook component $M_k^{\otimes n}(t) \oplus M_k^{\otimes n}(t-1)$ is

$$\left\langle \left(\chi^{(n-t,1^t)} + \chi^{(n-t+1,1^{t-1})}\right)\uparrow_{S_n}^{S_{2n}}, \chi^\lambda \right\rangle.$$

By (3.1), this is equal to

$$\begin{aligned} &\left\langle \left(\chi^{(n-t)} \otimes \chi^{(1^t)}\right)\uparrow_{S_{n-t}\times S_t}^{S_n}\uparrow_{S_n}^{S_{2n}}, \chi^\lambda \right\rangle \\ &\quad= \left\langle \left(\chi^{(n-t)} \otimes \chi^{(1^t)}\right)\uparrow_{S_{n-t}\times S_t}^{B_{n-t}\times B_t}\uparrow_{B_{n-t}\times B_t}^{S_{2n}}, \chi^\lambda \right\rangle. \end{aligned}$$

By Lemmas 2.2 and 2.3, for every t,

$$\begin{aligned} &\left(\chi^{(n-t)} \otimes \chi^{(1^t)}\right)\uparrow_{S_{n-t}\times S_t}^{B_{n-t}\times B_t} \\ &\quad= \chi^{(n-t)}\uparrow_{S_{n-t}}^{B_{n-t}} \otimes \chi^{(1^t)}\uparrow_{S_t}^{B_t} = \sum_{i=0}^{n-t} \chi^{(i),(n-t-i)} \otimes \sum_{j=0}^{t} \chi^{(1^j),(1^{t-j})} \\ &\quad= \sum_{i=0}^{n-t} \left(\chi^{(i),\emptyset} \otimes \chi^{\emptyset,(n-t-i)}\right)\uparrow_{B_i\times B_{n-t-i}}^{B_{n-t}} \\ &\qquad \otimes \sum_{j=0}^{t} \left(\chi^{(1^j),\emptyset} \otimes \chi^{\emptyset,(1^{t-j})}\right)\uparrow_{B_j\times B_{t-j}}^{B_t}. \end{aligned}$$

Hence

$$\begin{aligned}
&\bigl(\chi^{(n-t)} \otimes \chi^{(1^t)}\bigr)\uparrow_{S_{n-t}\times S_t}^{S_{2n}} \\
&\quad = \sum_{i=0}^{n-t}\sum_{j=0}^{t}\bigl(\chi^{(i),\emptyset} \otimes \chi^{\emptyset,(n-t-i)} \otimes \chi^{(1^j),\emptyset} \\
&\qquad\qquad \otimes \chi^{\emptyset,(1^{t-j})}\bigr)\uparrow_{B_i\times B_{n-t-i}\times B_j\times B_{t-j}}^{S_{2i}\times S_{2(n-t-i)}\times S_{2j}\times S_{2(t-j)}} \uparrow_{S_{2i}\times S_{2(n-t-i)}\times S_{2j}\times S_{2(t-j)}}^{S_{2n}} \\
&\quad = \sum_{i=0}^{n-t}\sum_{j=0}^{t}\bigl(\chi^{(i),\emptyset}\uparrow_{B_i}^{S_{2i}} \otimes \chi^{\emptyset,(n-t-i)}\uparrow_{B_{n-t-i}}^{S_{2(n-t-i)}} \otimes \chi^{(1^j),\emptyset}\uparrow_{B_j}^{S_{2j}} \\
&\qquad\qquad \otimes \chi^{\emptyset,(1^{t-j})}\uparrow_{B_{t-j}}^{S_{2(t-j)}}\bigr)\uparrow_{S_{2i}\times S_{2(n-t-i)}\times S_{2j}\times S_{2(t-j)}}^{S_{2n}}.
\end{aligned}$$

Lemma 2.1 and the Littlewood–Richardson rule complete the proof. □

Let $M_k^{\otimes n}(i,j)$ be the component of $M_k^{\otimes n}(i)$ with j skew symmetric factors. The following result is a common refinement of Theorems 5.4 and 5.6.

Theorem 5.7. *For every $0 \leqslant i \leqslant n$, $0 \leqslant j \leqslant n$ and $\lambda \in \mathrm{Par}_k(2n)$, the multiplicity of V_k^λ in $M_k^{\otimes n}(i,j) \oplus M_k^{\otimes n}(i-1,j)$ is*

$$\sum_{\substack{|\alpha|+|\beta|+|\gamma|+|\delta|=n \\ |\beta|+|\delta|=j \\ |\gamma|+|\delta|=i}} c^{\lambda}_{2\cdot\alpha,(2\cdot\beta)',2*\gamma,(2*\delta)'},$$

where the sum is over all partitions $\alpha, \beta, \gamma, \delta$ with total size n such that β and δ have distinct parts and total size j, and γ and δ have total size i.

Proof. Lemma 5.3, used as in the proof of Theorem 5.4, shows that the factors of $M_k^{\otimes n}(i,j) \oplus M_k^{\otimes n}(i-1,j)$ in the decomposition given in Theorem 5.6 are those for which $|\beta| + |\delta| = j$. □

Corollary 5.8. *Let $\lambda \subseteq (k^k)$ be a partition of $2n$. For every $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant n$, the multiplicity of V_k^λ in $M_k^{\otimes n}(i,j)$ is equal to the multiplicity of $V_k^{\lambda'}$ in $M_k^{\otimes n}(i, n-j)$.*

Proof. It suffices to show that for every $0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant n$, the multiplicity of V_k^λ in $M_k^{\otimes n}(i,j) \oplus M_k^{\otimes n}(i-1,j)$ is equal to the multiplicity of $V_k^{\lambda'}$ in

$M_k^{\otimes n}(i, n-j) \oplus M_k^{\otimes n}(i-1, n-j)$. By Theorem 5.7, the multiplicity of V_k^λ in $M_k^{\otimes n}(i, j) \oplus M_k^{\otimes n}(i-1, j)$ is

$$\sum_{\substack{|\alpha|+|\beta|+|\gamma|+|\delta|=n \\ |\beta|+|\delta|=j \\ |\gamma|+|\delta|=i}} c^{\lambda}_{2\cdot\alpha,(2\cdot\beta)',2*\gamma,(2*\delta)'}.$$

By (2.1) and the definition of $c^{\lambda}_{\alpha,\beta,\gamma,\delta}$, this is equal to

$$\sum_{\substack{|\alpha|+|\beta|+|\gamma|+|\delta|=n \\ |\beta|+|\delta|=j \\ |\gamma|+|\delta|=i}} c^{\lambda'}_{(2\cdot\alpha)',2\cdot\beta,(2*\gamma)',2*\delta},$$

which is the multiplicity of $V_k^{\lambda'}$ in $M_k^{\otimes n}(i, n-j) \oplus M_k^{\otimes n}(i-1, n-j)$. $\square$

Acknowledgments

The authors thank R. Howe, N. Wallach and an anonymous referee for their useful comments.

Appendix A

A.1. Proof of Lemma 2.2

Lemma 2.2 follows from a more general result.

For partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_m)$, let $\lambda \oplus \mu$ be the skew shape defined by

$$\lambda \oplus \mu := (\lambda_1 + \mu_1, \lambda_2 + \mu_1, \ldots, \lambda_k + \mu_1, \mu_1, \mu_2, \ldots, \mu_m) \big/ \big(\mu_1^k\big).$$

Theorem A.1. *If (λ, μ) is a bipartition of n then the restriction*

$$\chi^{\lambda,\mu}\downarrow^{B_n}_{S_n} = \chi^{\lambda\oplus\mu}.$$

Proof. The characters $\chi^{\lambda\oplus\mu}$ and $\chi^{\lambda,\mu}$, evaluated at elements of S_n, have the same recursive formula (Murnaghan–Nakayama rule). For $\chi^{\lambda,\mu}$ see [16, Theorem 4.3]. For $\chi^{\lambda\oplus\mu}$ see [9, Theorem 5.6.16]. $\square$

Proof of Lemma 2.2. (a) Let (λ, μ) be a bipartition of n. By Frobenius reciprocity and Theorem A.1,

$$\begin{aligned}\left\langle\chi^{(n)}\uparrow_{S_n}^{B_n}, \chi^{\lambda,\mu}\right\rangle &= \left\langle\chi^{(n)}, \chi^{\lambda,\mu}\downarrow_{S_n}^{B_n}\right\rangle = \left\langle\chi^{(n)}, \chi^{\lambda\oplus\mu}\right\rangle \\ &= \begin{cases}1, & \max\{\ell(\lambda),\ell(\mu)\}\leqslant 1,\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

The last equality follows from the Littlewood–Richardson rule, reformulated for skew shapes [15, (7.64)]. By this rule, $\langle\chi^{(n)}, \chi^{\lambda\oplus\mu}\rangle$ is nonzero (and equal to 1) if and only if $\lambda\oplus\mu$ is a horizontal strip (i.e., each column contains at most one box).

(b) The proof for $\chi^{(1^n)}$ is similar. □

A.2. Proof of Lemma 5.3

Proof. Let $\sigma_i := (i, -i) \in B_n$ $(1 \leqslant i \leqslant n)$, and let η be the sum $\sum_{i=1}^n \sigma_i \in \boldsymbol{C}[B_n]$. Consider the tensor product $w = w_1 \otimes w_2 \otimes \cdots \otimes w_n \in M_k^{\otimes n}$, where each w_i is either a symmetric or a skew symmetric matrix. Then according to the B_n-action, defined in Section 5.2,

$$\sigma_i(w) = \begin{cases} w, & \text{if } w_i \text{ is symmetric},\\ -w, & \text{if } w_i \text{ is skew symmetric.}\end{cases}$$

Hence, for every vector $v \in M_k^{\otimes n}(i)$,

$$\eta(v) = (n-2i)v. \tag{A.1}$$

On the other hand, the set $\{\sigma_i \mid 1 \leqslant i \leqslant n\}$ is a conjugacy class in B_n. Thus the element $\eta = \sum_{i=1}^n \sigma_i$ is in the center of $\boldsymbol{C}[B_n]$. By Schur's lemma, for every vector v in the irreducible B_n-module $S^{\mu,\nu}$,

$$\eta(v) = c^{\mu,\nu}\cdot v, \quad \text{where } c^{\mu,\nu} = \frac{\chi^{\mu,\nu}(\eta)}{\chi^{\mu,\nu}(\mathrm{id})} = \frac{n\chi^{\mu,\nu}(\sigma_1)}{\chi^{\mu,\nu}(\mathrm{id})}.$$

Let f^λ, $f^{\mu,\nu}$ be the number of standard Young tableaux (bitableaux) of shapes λ, (μ,ν), respectively. Recall that

$$\chi^{\mu,\nu}(\mathrm{id}) = f^{\mu,\nu} = \binom{n}{|\nu|} f^\mu f^\nu,$$

and that $\chi^{\mu,\nu}(\sigma_1)$ is equal to the number of standard Young bitableaux of shape (μ,ν), in which the digit 1 is in the first diagram μ, minus the number of those in which 1 is in the second diagram ν. Thus

$$\chi^{\mu,\nu}(\sigma_1) = \binom{n-1}{|\nu|} f^\mu f^\nu - \binom{n-1}{|\nu|-1} f^\mu f^\nu = \frac{n-2|\nu|}{n}\binom{n}{|\nu|} f^\mu f^\nu.$$

It follows that

$$c^{\mu,\nu} = \frac{n\chi^{\mu,\nu}(\sigma_1)}{\chi^{\mu,\nu}(\mathrm{id})} = n - 2|\nu|;$$

and therefore

$$\eta(v) = (n - 2|\nu|)v \quad (\forall v \in S^{\mu,\nu}). \tag{A.2}$$

Combining (A.1) with (A.2) completes the proof. □

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