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Rewritable groups ☆

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ABSTRACT

A group G is said to have the n-rewritable property Q_n if for all elements $g_1, g_2, \ldots, g_n \in G$, there exist two distinct permutations $\sigma, \tau \in \mathrm{Sym}_n$ such that $g_{\sigma(1)}g_{\sigma(2)}\cdots g_{\sigma(n)} = g_{\tau(1)}g_{\tau(2)}\cdots g_{\tau(n)}$. We show here that if G satisfies Q_n, then G has a characteristic subgroup N such that $|G:N|$ and $|N'|$ are both finite and have sizes bounded by functions of n. This extends the result of Blyth (1988) in [3] which asserts that if G satisfies Q_n and if Δ is the finite conjugate center of the group, then $|G:\Delta|$ and $|\Delta'|$ are both finite with $|G:\Delta|$ bounded by a function of n. As a consequence, any group with Q_n satisfies the permutational property P_m with m bounded by a function of n.

1. Introduction

Let G be a multiplicative group and let $\Delta = \Delta(G)$ denote its finite conjugate center. Thus

$$\Delta(G) = \bigl\{ g \in G \bigm| \bigl|G : \mathbb{C}_G(g)\bigr| < \infty \bigr\}$$

is the set of all elements of G having just finitely many G-conjugates. It is clear that $\Delta(G)$ is a characteristic subgroup of G.

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Following [6], G is said to have the n-permutational property P_n if for all $g_1, g_2, \ldots, g_n \in G$ (in that order), there exists a nonidentity permutation $\sigma \in \mathrm{Sym}_n$ such that

$$g_1 g_2 \cdots g_n = g_{\sigma(1)} g_{\sigma(2)} \cdots g_{\sigma(n)}.$$

It was shown in [6] that if G has P_n, then $|G : \Delta|$ is finite and bounded by a function of n. Furthermore, the commutator subgroup Δ' of Δ is finite, but its order is not bounded by a function of n.

Somewhat later, the n-rewritable property Q_n was introduced in [3]. Here G is said to have Q_n if for all $g_1, g_2, \ldots, g_n \in G$ (in that order), there exist two distinct permutations $\sigma, \tau \in \mathrm{Sym}_n$ such that

$$g_{\sigma(1)} g_{\sigma(2)} \cdots g_{\sigma(n)} = g_{\tau(1)} g_{\tau(2)} \cdots g_{\tau(n)}.$$

Obviously, P_n implies Q_n, but examples in [3] show that the reverse implication is not true in general. Nevertheless, it was shown in [3] that if G has Q_n, then $|G : \Delta|$ is finite and bounded by a function of n. Furthermore, Δ' is finite and again its order is not bounded by a function of n. Groups with properties P_n and Q_n have been studied extensively, for example in [3–6,9,10].

There is a third property of groups that was considered somewhat earlier. Namely, G satisfies PI_n if the group algebra $K[G]$ satisfies a polynomial identity of degree n. This property actually depends on the characteristic of the field K, and we discuss the known results in more detail in the last section of this paper. For now, let us note that PI_n implies P_n (see Lemma 5.3), but that the converse is not true in general. Furthermore, the first step in [13], characterizing groups G with PI_n is to show that G has a characteristic subgroup N such that $|G : N|$ is finite and bounded by a function of n, and that $|N'|$ is finite and also bounded by a function of n. Obviously, such a subgroup N must be contained in Δ, but not necessarily equal to Δ.

The goal of this paper is to show that the techniques of [13] (see also [14, Section 5.2]) used to study PI_n can yield similar results for the properties P_n and Q_n. Indeed, the P_n theorem is almost a verbatim translation of the group ring argument. On the other hand, the Q_n result is appreciably more difficult and eventually depends upon the beautiful trick in [3]. With all of this, the main result here is

Theorem 1.1. *Let G be a group satisfying the rewritable property Q_n. Then G has a characteristic subgroup N, contained in $\Delta(G)$, such that $|G : N|$ and $|N'|$ are finite and have sizes bounded by functions of n.*

The bounds we obtain here are rather astronomical functions of n. On the other hand, the bounds for the permutation property P_n are significantly smaller. Thus, we treat the P_n case first, as a warm up for the more difficult Q_n argument.

Recall that a group G is said to be perfect if $G = G'$, and let us say that G is normally perfect if all normal subgroups of G are perfect. For example, any nonabelian simple group is perfect. Furthermore, if G is semisimple, that is a finite direct product of nonabelian simple groups, then all normal subgroups of G are semisimple and hence G is normally perfect. The following is an extension of the main result of [4]. Its proof is an immediate consequence of Theorem 1.1 and hence does not require aspects of the classification of finite simple groups or computer computations.

Corollary 1.2. *Let G be a normally perfect group satisfying the rewritable property Q_n. Then G has finite order bounded by a function of n.*

Proof. Since G satisfies Q_n, the main theorem implies that G has a normal subgroup N with both $|G : N|$ and $|N'|$ bounded by functions of n. But G is normally perfect and hence $N = N'$. Thus $|G| = |G : N| \cdot |N'|$ is bounded by a function of n. □

It is clear, in the above, that we only apply the normally perfect property of G to a specific characteristic subgroup of finite index. Finally, we have

Corollary 1.3. *If G satisfies the rewritable property Q_n, then G satisfies the permutational property P_m with m bounded by a function of n.*

Proof. Since G satisfies Q_n, the main theorem implies that G has a normal subgroup N with both $|G:N| = a$ and $|N'| = b$ bounded by functions of n. Thus, by [6, Lemma 3.4], G satisfies P_m with $m = a(b+1)$. $\square$

2. The permutational property

In this section, we use the methods of [13] to prove the P_n version of Theorem 1.1. For the convenience of the reader, we start by quoting some notation and several lemmas found in that paper.

Let G be a group and let T be a subset of G. Following [13], we say that T has finite index in G if there exist $x_1, x_2, \dots, x_n \in G$, for some finite n, with

$$G = Tx_1 \cup Tx_2 \cup \dots \cup Tx_n.$$

We then define the index $|G:T|$ to be the minimum possible such integer n. Observe that if T is a subgroup of G, then this agrees with the usual definition of index. We can construct subsets of finite index in G by means of

Lemma 2.1. *Let $S = \bigcup_1^k H_i g_i$ be a finite union of cosets of the subgroups H_i of G and assume that $S \neq G$. Then there exist $x_1, x_2, \dots, x_t \in G$, with $t = (k+1)!$, such that $\bigcap_1^t Sx_i = \emptyset$. In particular, if T is a subset of G with $G = S \cup T$, then $|G:T| \leqslant (k+1)!$.*

In order to apply the above, we need a result like

Lemma 2.2. *Let $S = \bigcup_1^k H_i g_i$ be a finite union of cosets of subgroups H_i of G. If $|G:H_i| > k$ for all i, then $S \neq G$.*

This is presumably due to [11]. Next, we relate subsets of finite index to the subgroups they generate. If T is a subset of G, let us write $T^* = T \cup \{1\} \cup T^{-1}$.

Lemma 2.3. *Let T be a subset of G with $|G:T| \leqslant k$. Then*

$$\left(T^*\right)^{4^k} = T^* \cdot T^* \cdots T^* \quad \left(4^k \text{ times}\right)$$

is the subgroup of G generated by T.

Finally, we introduce certain subsets of interest. Let G be a group. For each integer k, we define

$$\Delta_k = \Delta_k(G) = \left\{x \in G \;\middle|\; \left|G : \mathbb{C}_G(x)\right| \leqslant k\right\}.$$

Thus Δ_k is a normal subset of G and $\Delta_k = \Delta_k^*$ in the notation of the preceding lemma. Furthermore, $\Delta_a \cdot \Delta_b \subseteq \Delta_{ab}$ for all integers a and b, and of course, Δ_k need not be a subgroup of G. The following is a result of [16]. A somewhat shorter proof is contained in [13].

Lemma 2.4. *Let G be a group and let k be a positive integer.*

(i) *If $|G'| \leqslant k$, then $G = \Delta_k(G)$.*
(ii) *If $G = \Delta_k(G)$, then $|G'| \leqslant (k^4)^{k^4}$.*

As a consequence of the latter two results we have

Lemma 2.5. *Let k and l be positive integers and assume that $|G : \Delta_k| \leqslant l$. If N is the subgroup of G generated by Δ_k, then N is a characteristic subgroup of G with $|G : N| \leqslant l$, and with $|N'|$ finite and bounded by a function of k and l.*

Proof. Since $\Delta_k \subseteq N$, it follows that $|G : N| \leqslant |G : \Delta_k| \leqslant l$. Moreover since $\Delta_k = (\Delta_k)^*$, Lemma 2.3 implies that $N = (\Delta_k)^{4^l}$. But $\Delta_a \Delta_b \subseteq \Delta_{ab}$ for any integers a and b, so we have

$$N = (\Delta_k)^{4^l} = \Delta_k \cdot \Delta_k \cdots \Delta_k \quad \bigl(4^l \text{ times}\bigr) \subseteq \Delta_r(G)$$

where $r = k^{4^l}$, and this implies that $N = \Delta_r(N)$. Thus, by Lemma 2.4(ii), $|N'| \leqslant (r^4)^{r^4}$, and the latter is a finite function of k and l. $\square$

Following Ref. [13], we define a linear monomial in the noncommuting variables $\zeta_1, \zeta_2, \dots, \zeta_n$ to be a polynomial μ of the form $\zeta_{i_1}\zeta_{i_2}\cdots\zeta_{i_r}$ with all i_j distinct and with $r = \deg\mu$. Thus μ is linear in each variable that occurs in its expression, and $\mu = 1$ if and only if $\deg\mu = 0$. Furthermore, since there are $n!$ linear monomials of degree n, and since each such monomial has $n+1$ initial segments, it follows that there are at most $(n+1)!$ linear monomials in n variables. We now come to the main result of this section.

Theorem 2.6. *Let G be a group satisfying the permutational property P_n, and set $k = n!$. Then we have*

(i) $|G : \Delta_k(G)| \leqslant k \cdot (k+1)!$, *and*
(ii) *G has a characteristic subgroup N with $|G : N| \leqslant k \cdot (k+1)!$, and with $|N'|$ finite and bounded by a function of n.*

Proof. (i) We assume by way of contradiction that $|G : \Delta_k| > k \cdot (k+1)!$. Let $\mathcal{M}_1 = \emptyset$, and for $j \geqslant 2$, let $\mathcal{M}_j$ denote the set of all nonidentity (that is, not degree 0) linear monomials in the noncommuting variables $\zeta_j, \zeta_{j+1}, \dots, \zeta_n$. By the above, $|\mathcal{M}_j| \leqslant n! = k$. We show now by induction on $j = 1, 2, \dots, n$ that for any $x_j, x_{j+1}, \dots, x_n \in G$, we have either $x_j x_{j+1} \cdots x_n = x_{\sigma(j)} x_{\sigma(j+1)} \cdots x_{\sigma(n)}$ for some $1 \neq \sigma \in \operatorname{Sym}\{j, j+1, \dots, n\}$ or we have $\mu(x_j, x_{j+1}, \dots, x_n) \in \Delta_k$ for some $\mu \in \mathcal{M}_j$. Since G satisfies P_n, the result for $j = 1$ is clear.

Suppose the result holds for some $j < n$. Fix $x_{j+1}, x_{j+2}, \dots, x_n \in G$ and let $x \in G$ play the role of the jth variable. Let $\mu \in \mathcal{M}_{j+1}$. If $\mu(x_{j+1}, x_{j+2}, \dots, x_n) \in \Delta_k$ we are done. Thus we may assume that $\mu(x_{j+1}, x_{j+2}, \dots, x_n) \notin \Delta_k$ for all $\mu \in \mathcal{M}_{j+1}$.

Next, for each $\sigma \in \operatorname{Sym}\{j, j+1, \dots, n\}$ with $\sigma \neq 1$, let

$$S_\sigma = \{x = x_j \in G \mid x_j x_{j+1} \cdots x_n = x_{\sigma(j)} x_{\sigma(j+1)} \cdots x_{\sigma(n)}\}.$$

If $S_\sigma \neq \emptyset$ and σ fixes j, then we can cancel the initial x factor and conclude that

$$x_{j+1} x_{j+2} \cdots x_n = x_{\sigma(j+1)} x_{\sigma(j+2)} \cdots x_{\sigma(n)}$$

for some $1 \neq \sigma \in \operatorname{Sym}\{j+1, j+2, \dots, n\}$. Thus we may assume that if $S_\sigma \neq \emptyset$, then σ does not fix j.

Now suppose $S_\sigma \neq \emptyset$ and let $x \in S_\sigma$ so that

$$x x_{j+1} x_{j+2} \cdots x_n = x_{\sigma(j)} x_{\sigma(j+1)} \cdots x \cdots x_{\sigma(n)}.$$

In particular, if we set $\rho = x_{j+1}x_{j+2}\cdots x_n$, then the above is equivalent to

$$\rho = x_{j+1}x_{j+2}\cdots x_n = x^{-1}(x_{\sigma(j)}x_{\sigma(j+1)}\cdots)x\cdots x_{\sigma(n)} = x^{-1}\lambda_\sigma x\bar{\lambda}_\sigma,$$

where λ_σ and $\bar{\lambda}_\sigma$ depend only upon σ. Indeed, since $\sigma(j) \neq j$, λ_σ is a linear monomial in $\mathcal{M}_{j+1}$ evaluated at $x_{j+1}, x_{j+2}, \ldots, x_n$, and therefore, by assumption, $\lambda_\sigma \notin \Delta_k$. We conclude from the above displayed equation that $x^{-1}\lambda_\sigma x = \rho(\bar{\lambda}_\sigma)^{-1}$ and it follows that if S_σ is nonempty, then it consists of precisely one right coset of $\mathbb{C}_G(\lambda_\sigma)$, say $S_\sigma = \mathbb{C}_G(\lambda_\sigma)h_\sigma$.

Write $S = \bigcup_\sigma S_\sigma = \bigcup_\sigma \mathbb{C}_G(\lambda_\sigma)h_\sigma$. Since $\lambda_\sigma \notin \Delta_k$, it follows that $|G : \mathbb{C}_G(\lambda_\sigma)| > k$, and since there are at most k cosets in the above union for S, we conclude from Lemma 2.2 that $S \neq G$. Thus by Lemma 2.1, $G\backslash S$ has index $\leqslant (k+1)!$ in G.

Finally, set $\mathcal{M}_j \setminus \mathcal{M}_{j+1} = \mathcal{F}_j$ and let $\mu \in \mathcal{F}_j$ so that μ involves the variable ζ_j. Write $\mu = \mu'\zeta_j\mu''$, where μ' and μ'' are linear monomials in the variables $\zeta_{j+1}, \zeta_{j+2}, \ldots, \zeta_n$. Then $\mu(x, x_{j+1}, \ldots, x_n) \in \Delta_k$ if and only if

$$x = x_j \in \mu'(x_{j+1}, \ldots, x_n)^{-1}\Delta_k\mu''(x_{j+1}, \ldots, x_n)^{-1} = \Delta_k g_\mu,$$

a fixed right translate of Δ_k, since Δ_k is a normal subset of G. Thus, if $T = \bigcup_{\mu\in\mathcal{F}_j} \Delta_k g_\mu$, then the inductive assumption implies that $G = S \cup T$. Indeed, for any $g \in G$, either $g, x_{j+1}, \ldots, x_n$ satisfies the permutational property and $g \in S_\sigma$ for some $\sigma \neq 1$, or $\mu(g, x_{j+1}, \ldots, x_n) \in \Delta_k$ for some $\mu \in \mathcal{F}_j$ and $g \in T$.

It follows that $T \supseteq G \setminus S$, so

$$|G : T| \leqslant |G : G \setminus S| \leqslant (k+1)!.$$

But $T = \bigcup_{\mu\in\mathcal{F}_j} \Delta_k g_\mu$ and $|\mathcal{F}_j| \leqslant |\mathcal{M}_j| \leqslant k$, so we see that

$$|G : \Delta_k| \leqslant |\mathcal{F}_j| \cdot |G : T| \leqslant k \cdot (k+1)!,$$

a contradiction by assumption. Therefore the inductive statement is proved.

In particular, the inductive result holds for $j = n$ and this is a contradiction. Indeed, when $j = n$ there are no nonidentity permutations in $\mathrm{Sym}\{n\}$ and $\mathcal{M}_n = \{\zeta_n\}$. Thus we conclude that $x \in \Delta_k$ for all $x \in G$, contradicting $G \neq \Delta_k$. In other words, the assumption that $|G : \Delta_k| > k \cdot (k+1)!$ is false and (i) is proved.

(ii) Here we set $l = k \cdot (k+1)!$, and we let N be the characteristic subgroup of G generated by Δ_k. Since $|G : \Delta_k| \leqslant l$, Lemma 2.5 yields the result. □

As a consequence, we obtain the main result of [6].

Corollary 2.7. *Let the group G satisfy the permutational property P_n, and set $k = n!$. Then $|G : \Delta(G)| \leqslant k \cdot (k+1)!$ and $|\Delta(G)'|$ is finite.*

Proof. Since $\Delta_k(G) \subseteq \Delta(G)$, Theorem 2.6 implies that $|G : \Delta(G)| \leqslant |G : \Delta_k(G)| \leqslant k \cdot (k+1)!$. Furthermore, since $\Delta(G)$ is a subgroup of G, it follows easily that $|\Delta(G) : \Delta_k(G)| \leqslant |G : \Delta_k(G)| \leqslant k \cdot (k+1)!$ and thus $\Delta(G) = \bigcup_i \Delta_k(G)x_i$ is a finite union of translates of Δ_k. Now each $x_i \in \Delta$ has only finitely many conjugates in G and hence there exists an integer a with $x_i \in \Delta_a(G)$ for all i. In other words, $\Delta(G) = \Delta_k(G)\Delta_a(G) \subseteq \Delta_{ka}(G)$ and we conclude from Lemma 2.4 that $\Delta(G)'$, the commutator subgroup of $\Delta(G)$, is finite. □

As was observed in [6], the order of $\Delta(G)'$ cannot be bounded by a function of n. For example, let $G = A \rtimes B$ be a finite Frobenius group, where A is the abelian kernel and B is a cyclic complement of order b. Then G satisfies P_{2b}, $G = \Delta(G)$, and $G' = A$ can be arbitrarily large independent of b. Finally, we have

Corollary 2.8. *Let G be a normally perfect group satisfying the permutational property P_n. Then G has finite order bounded by a function of n.*

This is, of course, an extension of one of the main results of [2], and its proof follows from Theorem 2.6 and uses the argument of Corollary 1.2. The specific bound can be obtained via a close reading of the proof of Lemma 2.5.

3. The rewritable property

Now we move on to consider the rewritable property. Our goal in this section is to prove the main theorem of this paper, namely Theorem 1.1, and some appropriate corollaries. As will be apparent, the proof here is appreciably harder than that of Theorem 2.6. It uses a mixture of the subsets of finite index techniques from [13], along with a key trick from Ref. [3]. We simplify matters a bit by replacing the sequences found in [3] by certain sets, but the idea is basically the same. As will also be apparent, the bounds here are rather astronomical functions of n, so we will not be concerned with small improvements in the bounds. We begin with

Theorem 3.1. *Let G be a group satisfying the rewritable property Q_n. Then there exist functions k and l of n, such that $|G : \Delta_k(G)| \leqslant l$.*

Proof. We first define a sequence of constants that depend upon n, namely

$$m = n!, \qquad p = m^2, \qquad q = p \cdot 2^p, \qquad k = m \cdot q^p \quad \text{and} \quad l = k \cdot (k+1)!.$$

Our goal here is to show that $|G : \Delta_k| \leqslant l$. As in the proof of Theorem 2.6, we assume that $|G : \Delta_k| > l$ and derive a contradiction. Indeed, we proceed in essentially the same manner, but using somewhat different notation.

Let $\mathcal{M}_n = \emptyset$ and for each $j = 1, 2, \ldots, n-1$, let $\mathcal{M}_j$ denote the set of all nonidentity linear monomials in the noncommuting variables $\zeta_1, \zeta_2, \ldots, \zeta_j$. Thus $|\mathcal{M}_j| \leqslant |\mathcal{M}_{n-1}| \leqslant n! = m$. We show by inverse induction on $1 \leqslant j \leqslant n$ that if $x_1, x_2, \ldots, x_j \in G$, then either there exist distinct permutations $\sigma, \tau \in \mathrm{Sym}_j$ with

$$x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(j)} = x_{\tau(1)} x_{\tau(2)} \cdots x_{\tau(j)},$$

or there exists a linear monomial $\mu \in \mathcal{M}_j$ with $\mu(x_1, x_2, \ldots, x_j) \in \Delta_k$. Since G satisfies Q_n, the statement is trivially true for $j = n$.

Suppose now that the inductive result holds for some $j+1$ with $2 \leqslant j+1 \leqslant n$. We show that it holds for j. To this end, let $x_1, x_2, \ldots, x_j$ be fixed elements of G. If there exists a linear monomial $\mu \in \mathcal{M}_j$ with $\mu(x_1, x_2, \ldots, x_j) \in \Delta_k$, then we are done. Thus, we can assume that this is not the case. In particular, if $M = \{\mu(x_1, x_2, \ldots, x_j) \mid \mu \in \mathcal{M}_j\} \subseteq G$, then $M \cap \Delta_k = \emptyset$ and $|M| \leqslant |\mathcal{M}_j| \leqslant m$.

For each $\sigma \in \mathrm{Sym}_{j+1}$, let us write

$$\begin{aligned}\phi_\sigma(\zeta_1, \ldots, \zeta_j, \zeta_{j+1}) &= \zeta_{\sigma(1)} \cdots \zeta_{\sigma(j)} \zeta_{\sigma(j+1)} \\ &= \mu_\sigma(\zeta_1, \ldots, \zeta_j) \cdot \zeta_{j+1} \cdot \overline{\mu}_\sigma(\zeta_1, \ldots, \zeta_j),\end{aligned}$$

so that μ_σ and $\overline{\mu}_\sigma$ are linear monomials in $\zeta_1, \zeta_2, \ldots, \zeta_j$, possibly of degree 0. Furthermore, for any $x \in G$, set

$$f_\sigma(x) = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(j+1)} = \phi_\sigma(x_1, \ldots, x_j, x),$$

where x plays the role of x_{j+1}. Thus $f_\sigma(x) = \lambda_\sigma x \bar{\lambda}_\sigma$, where $\lambda_\sigma = \mu_\sigma(x_1, \ldots, x_j)$ and $\bar{\lambda}_\sigma = \bar{\mu}_\sigma(x_1, \ldots, x_j)$. For convenience, we will use $\deg \lambda_\sigma$ to denote the degree of the monomial μ_σ. Note that λ_σ and $\bar{\lambda}_\sigma$ depend only upon σ and not upon x.

Now, let $\mathcal{P}$ denote the set of all pairs $\pi = (\sigma, \tau)$ of distinct permutations $\sigma, \tau \in \mathrm{Sym}_{j+1}$ and with $\deg \lambda_\sigma \geqslant \deg \lambda_\tau$. Thus $|\mathcal{P}| < [(j+1)!]^2 \leqslant [n!]^2 = m^2 = p$. Now, for each $\pi = (\sigma, \tau) \in \mathcal{P}$, we define

$$d_\pi = \lambda_\tau^{-1} \lambda_\sigma, \quad \text{and} \quad e_\pi = \bar{\lambda}_\tau (\bar{\lambda}_\sigma)^{-1}.$$

Claim 1. *If $x \in G \setminus \Delta_k M^{-1}$, then there exists $\pi = (\sigma, \tau) \in \mathcal{P}$ with $f_\sigma(x) = f_\tau(x)$, and hence with $x^{-1} d_\pi x = e_\pi$.*

Proof. Let $x \in G$ and observe that the inductive result holds for the $(j+1)$-tuple $x_1, x_2, \ldots, x_j, x_{j+1}$ with $x_{j+1} = x$. First, suppose that there exists $\mu \in \mathcal{M}_{j+1}$ with $\mu(x_1, x_2, \ldots, x_{j+1}) \in \Delta_k$. Then, by assumption, $\mu \notin \mathcal{M}_j$, and hence μ involves the variable ζ_{j+1}, say

$$\mu(\zeta_1, \ldots, \zeta_j, \zeta_{j+1}) = \mu'(\zeta_1, \ldots, \zeta_j) \cdot \zeta_{j+1} \cdot \mu''(\zeta_1, \ldots, \zeta_j).$$

Then $\mu(x_1, \ldots, x_j, x) \in \Delta_k$ if and only if

$$\begin{aligned} x = x_{j+1} &\in \mu'(x_1, \ldots, x_j)^{-1} \cdot \Delta_k \cdot \mu''(x_1, \ldots, x_j)^{-1} \\ &= \Delta_k \cdot \mu'(x_1, \ldots, x_j)^{-1} \cdot \mu''(x_1, \ldots, x_j)^{-1} \end{aligned}$$

since Δ_k is a normal subset of G. But

$$\mu'(x_1, \ldots, x_j)^{-1} \cdot \mu''(x_1, \ldots, x_j)^{-1} = \left[\mu''(x_1, \ldots, x_j) \cdot \mu'(x_1, \ldots, x_j)\right]^{-1}$$

and $\mu''\mu'$ is clearly a member of $\mathcal{M}_j$. Thus, in this case, we see that $x \in \Delta_k M^{-1}$.

In particular, if $x \in G \setminus \Delta_k M^{-1}$, then there exist distinct $\sigma, \tau \in \mathrm{Sym}_{j+1}$ with $f_\sigma(x) = f_\tau(x)$. By symmetry, we can assume that $\deg \lambda_\sigma \geqslant \deg \lambda_\tau$, so that $\pi = (\sigma, \tau) \in \mathcal{P}$. Finally,

$$\lambda_\sigma x \bar{\lambda}_\sigma = f_\sigma(x) = f_\tau(x) = \lambda_\tau x \bar{\lambda}_\tau$$

implies that $x^{-1}(\lambda_\tau^{-1}\lambda_\sigma)x = \bar{\lambda}_\tau(\bar{\lambda}_\sigma)^{-1}$, or equivalently $x^{-1} d_\pi x = e_\pi$. $\square$

Next, let $\mathcal{Q}$ denote the set of all pairs (π, P) where $\pi \in \mathcal{P}$, P is a subset of $\mathcal{P}$ (possibly empty), and $\pi \notin P$. We say that the length of the pair (π, P) is the size of P, and hence all these lengths are nonnegative and bounded by $|\mathcal{P}| - 1 < p$. Furthermore, $|\mathcal{Q}| < |\mathcal{P}| \cdot 2^{|\mathcal{P}|} \leqslant p \cdot 2^p = q$. For convenience, we say that (π, P) is a Blyth pair if there exists an element $g \in G$ with $g \in \mathbb{C}_G(d_\kappa)$ for all $\kappa \in P$ and with $g^{-1} d_\pi g = e_\pi$. For each Blyth pair, we choose one such group element g having the listed properties, and we let B_0 denote the set of all such choices. Thus B_0 is a subset of G with $|B_0| \leqslant |\mathcal{Q}| < q$. In particular, if $B = B_0 \cup \{1\}$, then $|B| \leqslant q$. Furthermore, since $1 \in B$, we have $B^{i+1} = B^i \cdot B \supseteq B^i$ for all integers i.

Claim 2. *Let $S = \bigcup_{u \in M} \mathbb{C}_G(u) B^p$ and let $T = \Delta_k M^{-1} B^p$. Then $G \neq S \cup T$.*

Proof. Notice that S is a union of $|M| \cdot |B^p| \leqslant |M| \cdot |B|^p \leqslant m \cdot q^p = k$ cosets of the various $\mathbb{C}_G(u)$ with $u \in M$. Thus, since each such group element u is not contained in Δ_k, we have $|G : \mathbb{C}_G(u)| > k$, and $S \neq G$ by Lemma 2.2. We conclude therefore from Lemma 2.1 that $G \setminus S$ has index $\leqslant (k+1)!$ in G.

In particular, if $G = S \cup T$, then $|G : T| \leqslant (k+1)!$. But T is a union of at most $|M| \cdot |B|^p \leqslant k$ right translates of Δ_k, so this implies that $|G : \Delta_k| \leqslant k \cdot (k+1)! = l$, contrary to our original assumption. □

Since $G \neq S \cup T$, we can choose a group element $g \in G \setminus (S \cup T)$ which we fix for the remainder of this argument. Now we say that the pair $(\pi, P) \in \mathcal{Q}$ is a g-Blyth pair of length ℓ if $|P| = \ell$ and if there exists $h \in gB^{-\ell}$ with $h^{-1}(d_\pi)h = e_\pi$ and with $h \in \mathbb{C}_G(d_\kappa)$ for all $\kappa \in P$. Since $g \notin T$, Claim 1 implies that there exists a g-Blyth pair of length 0.

Finally, let us choose a g-Blyth pair (π', P') of largest possible length ℓ' so that $\ell' \leqslant |\mathcal{P}| - 1 < p - 1$, and let $h' \in gB^{-\ell'}$ be the corresponding group element. Then $(h')^{-1} d_{\pi'} h' = e_{\pi'}$ and $h' \in \mathbb{C}_G(d_{\kappa'})$ for all $\kappa' \in P'$. We attempt to construct a g-Blyth pair (π, P) of larger length $\ell = \ell' + 1$. To start with, (π', P') is a Blyth pair, so there exists $b \in B$ with $b^{-1} d_{\pi'} b = e_{\pi'}$ and with $b \in \mathbb{C}_G(d_{\kappa'})$ for all $\kappa' \in P'$. In particular, if we set $h = h'b^{-1}$, then $h \in \mathbb{C}_G(d_\kappa)$ for all $\kappa \in P = P' \cup \{\pi'\}$. Furthermore, note that $|P| = \ell' + 1 = \ell$ since $\pi' \notin P'$. Also $h = h'b^{-1} \in gB^{-\ell'}B^{-1} = gB^{-\ell}$. Finally, since $g \notin T = \Delta_k M^{-1} B^p$ and $\ell \leqslant p$, we see that $h \notin \Delta_k M^{-1}$. Claim 1 therefore implies that there exists a pair $\pi = (\sigma, \tau) \in \mathcal{P}$ with $h^{-1} d_\pi h = e_\pi$.

The pair (π, P) now satisfies almost all of the conditions for being a g-Blyth pair of length $\ell > \ell'$. But ℓ' was chosen maximal, so (π, P) cannot satisfy the remaining condition, namely $\pi \notin P$. In other words, we must have $\pi \in P$ and therefore $h \in \mathbb{C}_G(d_\pi)$. But $h^{-1} d_\pi h = e_\pi$, so $d_\pi = e_\pi$. Thus $\lambda_\tau^{-1}\lambda_\sigma = d_\pi = e_\pi = \bar{\lambda}_\tau(\bar{\lambda}_\sigma)^{-1}$, and hence $\lambda_\sigma \bar{\lambda}_\sigma = \lambda_\tau \bar{\lambda}_\tau$.

Recall that

$$\phi_\sigma = \zeta_{\sigma(1)} \cdots \zeta_{\sigma(j)} \zeta_{\sigma(j+1)} = \mu_\sigma(\zeta_1, \ldots, \zeta_j) \cdot \zeta_{j+1} \cdot \overline{\mu}_\sigma(\zeta_1, \ldots, \zeta_j),$$

with $\lambda_\sigma = \mu_\sigma(x_1, \ldots, x_j)$ and $\bar{\lambda}_\sigma = \overline{\mu}_\sigma(x_1, \ldots, x_j)$. In particular, $\mu_\sigma \overline{\mu}_\sigma$ is a product of all the variables $\zeta_1, \zeta_2, \ldots, \zeta_j$ in some order. Similarly, $\lambda_\tau = \mu_\tau(x_1, \ldots, x_j)$ and $\bar{\lambda}_\tau = \overline{\mu}_\tau(x_1, \ldots, x_j)$.

Suppose first that $\mu_\sigma \overline{\mu}_\sigma = \mu_\tau \overline{\mu}_\tau$. Since $\deg \lambda_\sigma \geqslant \deg \lambda_\tau$, it follows that μ_τ must be an initial segment of μ_σ, say $\mu_\sigma = \mu_\tau \rho$ for some linear monomial ρ. Then $\mu_\tau \rho \overline{\mu}_\sigma = \mu_\sigma \overline{\mu}_\sigma = \mu_\tau \overline{\mu}_\tau$, so $\overline{\mu}_\tau = \rho \overline{\mu}_\sigma$. If $\rho = 1$, then $\mu_\sigma = \mu_\tau$ and $\overline{\mu}_\sigma = \overline{\mu}_\tau$, so $\phi_\sigma = \phi_\tau$ and therefore $\sigma = \tau$, certainly a contradiction. Thus $\rho \neq 1$ and hence $\rho \in \mathcal{M}_j$. Note that $d_\pi = \lambda_\tau^{-1}\lambda_\sigma = \rho(x_1, \ldots, x_j) \in M$, and h centralizes d_π, so $h \in \mathbb{C}_G(d_\pi)$ and $g \in h \cdot B^p \in \mathbb{C}_G(d_\pi)B^p \subseteq S$, again a contradiction.

We conclude that $\mu_\sigma \overline{\mu}_\sigma \neq \mu_\tau \overline{\mu}_\tau$ and therefore $\lambda_\sigma \bar{\lambda}_\sigma = \lambda_\tau \bar{\lambda}_\tau$ yields a nontrivial rewriting of $x_1, x_2, \ldots, x_j$. Thus the induction step is proved.

In particular, the inductive result holds when $j = 1$. But here there is only one element in Sym_1 and $\mathcal{M}_1$ consists of the unique monomial $\mu(\zeta_1) = \zeta_1$. We conclude therefore that for all $x \in G$, $x = \mu(x) \in \Delta_k$, so $G = \Delta_k$, and this contradicts our assumption that $|G : \Delta_k| > l$. Thus $|G : \Delta_k| \leqslant l$ and the theorem is proved. □

Theorem 1.1 and hence Corollary 1.2 now follow immediately from the above and Lemma 2.5. As a consequence, using the argument of Corollary 2.7, we also obtain the main result of [3], namely

Corollary 3.2. *Let the group G satisfy the rewritable property Q_n. Then $|\Delta(G)'|$ and $|G : \Delta(G)|$ are finite, with the latter bounded by a function of n.*

4. A generalization

There are numerous generalizations of the P_n and Q_n properties in the literature. Here, we offer an interesting generalization of n-permutational groups. Let m and n be positive integers and suppose that $\mathcal{A}$ is a set of n-tuples $(a_1, a_2, \ldots, a_n)$ of elements of G with $|\mathcal{A}| = m$. A group G is said to be (m, n)-permutational with respect to $\mathcal{A}$ if for every n-tuple $(x_1, x_2, \ldots, x_n)$ of elements of G there exist an n-tuple $(a_1, a_2, \dots, a_n) \in \mathcal{A}$ and a nonidentity permutation $\sigma \in \mathrm{Sym}_n$ such that

$$x_1 a_1 x_2 a_2 \dots x_n a_n = x_{\sigma(1)} a_{\sigma(1)} x_{\sigma(2)} a_{\sigma(2)} \dots x_{\sigma(n)} a_{\sigma(n)}.$$

Using the same proof as that of Theorem 2.6, we obtain

Proposition 4.1. *If G is an (m, n)-permutational group with respect to $\mathcal{A}$, then setting $k = m \cdot n!$, we have*

(i) $[G : \Delta_k] \leqslant k \cdot (k+1)!$, *and*
(ii) *G has a characteristic subgroup N with $|G : N| \leqslant k \cdot (k+1)!$, and with $|N'|$ finite and bounded by a function of n.*

Proof. We assume by way of contradiction that $[G : \Delta_k] > k \cdot (k+1)!$. Let $\mathcal{M}_1 = \emptyset$ and, for $j \geqslant 2$, let $\mathcal{M}_j$ denote the set of all linear monomials in the noncommuting variables $\zeta_j, \zeta_{j+1}, \dots, \zeta_n$. As we have observed, this implies that $|\mathcal{M}_j| \leqslant n!$. We show now by induction on $j = 1, 2, \dots, n$ that, for any $x_j, x_{j+1}, \dots, x_n \in G$, there exists $(a_1, a_2, \dots, a_n) \in \mathcal{A}$ such that either

$$x_j a_j x_{j+1} a_{j+1} \cdots x_n a_n = x_{\sigma(j)} a_{\sigma(j)} x_{\sigma(j+1)} a_{\sigma(j+1)} \cdots x_{\sigma(n)} a_{\sigma(n)}$$

for some $1 \neq \sigma \in \mathrm{Sym}\{j, j+1, \dots, n\}$ or $\mu(x_j a_j, x_{j+1} a_{j+1}, \dots, x_n a_n) \in \Delta_k$ for some monomial $\mu \in \mathcal{M}_j$. Since G is an (m, n)-permutational group, the result for $j = 1$ is given.

Suppose the result holds for some $j < n$. Fix $x_{j+1}, x_{j+2} \dots, x_n \in G$ and let x play the role of the jth variable. Let $\mu \in \mathcal{M}_{j+1}$. If $\mu(x_{j+1} a_{j+1}, \dots, x_n a_n) \in \Delta_k$ for some $(a_1, a_2, \dots, a_n) \in \mathcal{A}$, then we are done. Thus we may assume that $\mu(x_{j+1} a_{j+1}, \dots, x_n a_n) \notin \Delta_k$ for all $\mu \in \mathcal{M}_{j+1}$ and for all $(a_1, a_2, \dots, a_n) \in \mathcal{A}$.

Next, for each $1 \neq \sigma \in \mathrm{Sym}\{j, j+1, \dots, n\}$ and $\alpha = (a_1, a_2, \dots, a_n) \in \mathcal{A}$, let

$$S_{\sigma,\alpha} = \{x = x_j \in G \mid x_j a_j x_{j+1} a_{j+1} \cdots x_n a_n = x_{\sigma(j)} a_{\sigma(j)} x_{\sigma(j+1)} a_{\sigma(j+1)} \cdots x_{\sigma(n)} a_{\sigma(n)}\}.$$

If $S_{\sigma,\alpha} \neq \emptyset$ and σ fixes j, then we can cancel the beginning $x_j a_j$ factors and conclude that

$$x_{j+1} a_{j+1} \cdots x_n a_n = x_{\sigma(j+1)} a_{\sigma(j+1)} \cdots x_{\sigma(n)} a_{\sigma(n)}$$

for some $1 \neq \sigma \in \mathrm{Sym}\{j+1, \dots, n\}$. Thus we can assume that if $S_{\sigma,\alpha} \neq \emptyset$, then σ does not fix j.

Now suppose $S_{\sigma,\alpha} \neq \emptyset$ and let $x \in S_{\sigma,\alpha}$ so that

$$x a_j x_{j+1} a_{j+1} x_{j+2} a_{j+2} \cdots x_n a_n = x_{\sigma(j)} a_{\sigma(j)} x_{\sigma(j+1)} a_{\sigma(j+1)} \cdots x a_j \cdots x_{\sigma(n)} x_{\sigma(n)}.$$

In particular, if we set $\rho = x_{j+1} a_{j+1} x_{j+2} a_{j+2} \cdots x_n a_n$, then we have

$$\begin{aligned}
\rho &= x_{j+1} a_{j+1} x_{j+2} a_{j+2} \cdots x_n a_n \\
&= (x a_j)^{-1} (x_{\sigma(j)} a_{\sigma(j)} x_{\sigma(j+1)} a_{\sigma(j+1)} \cdots)(x a_j) \cdots x_{\sigma(n)} a_{\sigma(n)} \\
&= (x a_j)^{-1} \lambda_{\sigma,\alpha} (x a_j) \bar{\lambda}_{\sigma,\alpha},
\end{aligned}$$

where $\lambda_{\sigma,\alpha}$ and $\bar{\lambda}_{\sigma,\alpha}$ depend only upon σ and α. Indeed, since $\sigma(j) \neq j$, $\lambda_{\sigma,\alpha}$ is a linear monomial in $\mathcal{M}_{j+1}$ evaluated at $x_{j+1} a_{j+1}, x_{j+2} a_{j+2}, \dots, x_n a_n$, and therefore, by assumption, $\lambda_{\sigma,\alpha} \notin \Delta_k$.

Since the above displayed equation is equivalent to

$$x^{-1} \lambda_{\sigma,\alpha} x = a_j \rho (\bar{\lambda}_{\sigma,\alpha})^{-1} a_j^{-1},$$

it is clear that $S_{\sigma,\alpha}$ consists of precisely one right coset of $\mathbb{C}_G(\lambda_{\sigma,\alpha})$, say $S_{\sigma,\alpha} = \mathbb{C}_G(\lambda_{\sigma,\alpha})h_{\sigma,\alpha}$. Write $S = \bigcup_{\sigma,\alpha} S_{\sigma,\alpha} = \bigcup_{\sigma,\alpha} \mathbb{C}_G(\lambda_{\sigma,\alpha})h_{\sigma,\alpha}$. Since $\lambda_{\sigma,\alpha} \notin \Delta_k$, it follows that $|G : \mathbb{C}_G(\lambda_{\sigma,\alpha})| > k$, and since there are at most $m \cdot n! = k$ cosets in the above union for S, we conclude from Lemma 2.2 that $S \neq G$. Thus by Lemma 2.1, $G \backslash S$ has index $\leqslant (k+1)!$ in G.

Finally, set $\mathcal{M}_j \setminus \mathcal{M}_{j+1} = \mathcal{F}_j$ and let $\mu \in \mathcal{F}_j$ so that μ involves the variable ζ_j. Thus we can write $\mu = \mu' \zeta_j \mu''$, where μ' and μ'' are linear monomials in the variables $\zeta_{j+1}, \zeta_{j+2}, \dots, \zeta_n$. If $\alpha = (a_1, a_2, \dots, a_n) \in \mathcal{A}$, then $\mu(x_j a_j, x_{j+1} a_{j+1}, \dots, x_n a_n) \in \Delta_k$ if and only if

$$x a_j = x_j a_j \in \mu'(x_{j+1}a_{j+1}, \dots, x_n a_n)^{-1} \Delta_k \mu''(x_{j+1}a_{j+1}, \dots, x_n a_n)^{-1} = \Delta_k g_{\mu,\alpha},$$

since Δ_k is a normal subset of G. In particular, this occurs if and only if $x \in \Delta_k g_{\mu,\alpha} {a_j}^{-1}$, a fixed right translate of Δ_k.

Thus if $T = \bigcup_{\mu,\alpha} \Delta_k g_{\mu,\alpha} {a_j}^{-1}$, where the union is over all $\mu \in \mathcal{F}_j$ and $\alpha \in \mathcal{A}$, then the inductive assumption implies that $G = S \cup T$. Indeed, suppose $g \in G$. If there exist σ and α with

$$x_j a_j x_{j+1} a_{j+1} \cdots x_n a_n = x_{\sigma(j)} a_{\sigma(j)} x_{\sigma(j+1)} a_{\sigma(j+1)} \cdots x_{\sigma(n)} a_{\sigma(n)}$$

and $x_j = g$, then $g \in S_{\sigma,\alpha} \subseteq S$. On the other hand, if there exist μ and α with

$$\mu(x_j a_j, x_{j+1} a_{j+1}, \dots, x_n a_n) \in \Delta_k$$

and $x_j = g$, then $g \in \Delta_k g_{\mu,\alpha} {a_j}^{-1} \subseteq T$. It follows that $T \supseteq G \setminus S$, and hence $[G : T] \leqslant [G : G \setminus S] \leqslant (k+1)!$. But T is a union of at most $|\mathcal{A}| \cdot |\mathcal{F}_j| \leqslant m \cdot n! = k$ right translates of Δ_k, so this yields $|G : \Delta_k| \leqslant k \cdot |G : T| \leqslant k \cdot (k+1)!$, contrary to our assumption. Consequently, the induction step is proved.

In particular, the inductive result holds when $j = n$. Here, there are no nonidentity permutations in $\mathrm{Sym}\{n\}$, and $\mathcal{M}_n = \{\zeta_n\}$. We conclude that, for each $x \in G$, there exists $\alpha \in \mathcal{A}$ with $x a_n \in \Delta_k$ and hence with $x \in \Delta_k a_n^{-1}$. In other words, we have $G = \bigcup_\alpha \Delta_k a_n^{-1}$, where a_n is the nth entry of α, and thus $|G : \Delta_k| \leqslant m \leqslant k$. Therefore the assumption $[G : \Delta_k] > k(k+1)!$ is false, and part (i) of the proposition is proved. Of course, part (ii) is an immediate consequence of (i) and Lemma 2.5. $\square$

Obviously, there are corollaries of the above concerning the structure of $\Delta(G)$ for (m,n)-permutational groups, and the nature of G under the additional assumption that G is normally perfect. However, we will not bother to list these here.

5. Polynomial identities

In this final section, we briefly discuss the known results on group algebras satisfying a polynomial identity. To start with, let K be a field and let $\mathcal{F} = K\langle \zeta_1, \zeta_2, \zeta_3, \dots \rangle$ be the free K-algebra in the noncommuting variables $\zeta_1, \zeta_2, \zeta_3, \dots$. A K-algebra R is said to satisfy the polynomial identity $f(\zeta_1, \zeta_2, \dots, \zeta_k) \in \mathcal{F}$ if $f(r_1, r_2, \dots, r_k) = 0$ for all $r_1, r_2, \dots, r_k \in R$. For example, any commutative algebra satisfies $\zeta_1\zeta_2 - \zeta_2\zeta_1$. In general, we think of polynomial identities as weakened versions of commutativity. A simple linearization argument shows that

Lemma 5.1. *If R satisfies a polynomial identity of degree n, then R satisfies a multilinear identity of the form*

$$f(\zeta_1, \zeta_2, \dots, \zeta_n) = \sum_{\sigma \in \mathrm{Sym}_n} k_\sigma \zeta_{\sigma(1)} \zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)},$$

with $k_\sigma \in K$ and $k_1 = 1$.

A multilinear polynomial of particular importance is the standard polynomial given by

$$s_n(\zeta_1, \zeta_2, \ldots, \zeta_n) = \sum_{\sigma \in \mathrm{Sym}_n} (-1)^\sigma \zeta_{\sigma(1)}\zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)},$$

where $(-1)^\sigma$ is the sign of the permutation. Indeed, the main result of [1] asserts

Theorem 5.2. *Let S be a commutative K-algebra and let $R = \mathbf{M}_n(S)$ be the ring of $n \times n$ matrices over S. Then R satisfies s_{2n}, the standard identity of degree $2n$, but no polynomial identity of smaller degree.*

As a consequence of the above, we have

Lemma 5.3. *Let $K[G]$ be the group algebra of the group G over the field K.*

(i) *If $K[G]$ satisfies a polynomial identity of degree n, then G satisfies the permutational property P_n.*
(ii) *If G has an abelian subgroup A of finite index n, then $K[G]$ satisfies s_{2n}, the standard identity of degree $2n$.*

Proof. (i) We can assume that $K[G]$ satisfies the multilinear polynomial identity $f(\zeta_1, \zeta_2, \ldots, \zeta_n)$ given by Lemma 5.1. In particular, if $g_1, g_2, \ldots, g_n \in G \subseteq K[G]$, then $f(g_1, g_2, \ldots, g_n) = 0$. But notice that the coefficient $k_1 = 1$, so $g = g_1 g_2 \cdots g_n$ is a group element summand of $f(g_1, g_2, \ldots, g_n)$. Thus, since the elements of G are K-linearly independent in $K[G]$, there must exist at least one other term in $f(g_1, g_2, \ldots, g_n)$ that gives rise to g. In other words, there exists $1 \neq \sigma \in \mathrm{Sym}_n$ with $g_1 g_2 \cdots g_n = g = g_{\sigma(1)} g_{\sigma(2)} \cdots g_{\sigma(n)}$, and hence G satisfies P_n.

(ii) Write $V = K[G]$. Then V is a faithful right $K[G]$-module, via right multiplication. Furthermore, via left multiplication, V is a free left $K[A]$-module with right coset representatives of A in G as a free basis. Since left and right multiplications commute as operators on V, it follows that $K[G]$ is contained isomorphically in $\mathrm{End}_{K[A]}(V) \cong \mathbf{M}_n(K[A])$. But $K[A]$ is commutative, so $\mathbf{M}_n(K[A])$ satisfies s_{2n}, by Theorem 5.2, and therefore so does $K[G]$. □

Part (i) above is, of course, our comment in the Introduction that PI_n implies P_n. On the other hand, part (ii) leads us to ask whether $K[G]$ satisfying a polynomial identity is equivalent to G having an abelian subgroup of finite index. This turns out to be the case in characteristic 0.

If G is a finite group and K is an algebraically closed field of characteristic 0, then the group algebra $K[G]$ is semisimple and hence is a direct sum of full matrix rings $\mathbf{M}_d(K)$, where d is the degree of the corresponding irreducible representation. In particular, if $K[G]$ satisfies a polynomial identity of degree n, then Theorem 5.2 implies that $d \leqslant n/2$. In other words, all irreducible representations of G have degree bounded by $n/2$. Using the character theory of finite groups, and then lifting the result to arbitrary groups, [8] proved

Theorem 5.4. *Let K be a field of characteristic 0. If $K[G]$ satisfies a polynomial identity of degree n, then G has an abelian subgroup A of finite index, with $|G : A|$ bounded by a function of n.*

Since group algebras in characteristic $p > 0$ are not necessarily semiprimitive, they cannot be studied entirely via their representation theory. Thus, more combinatorial arguments were needed, starting with papers [12] and [15]. For convenience, we say that a group A is p-abelian if A' is a finite p-group. Then the main result of [13] asserts

Theorem 5.5. *Let K be a field of characteristic $p > 0$ and let G be a group.*

(i) *If G has a p-abelian subgroup A of finite index, then $K[G]$ satisfies a polynomial identity of degree $2|G : A| \cdot |A'|$.*

(ii) *If $K[G]$ satisfies a polynomial identity of degree n, then G has a p-abelian subgroup A of finite index with $|G:A|\cdot|A'|$ bounded by a function of n.*

The proof of part (ii) above did not use the P_n property, but it is certainly familiar. It starts by showing that there exist integers k and l, depending on n, with $|G:\Delta_k|\leqslant l$. As usual, we suppose, by way of contradiction, that $|G:\Delta_k|>l$. Now let $K[G]$ satisfy the multilinear polynomial identity $f(\zeta_1,\zeta_2,\ldots,\zeta_n)$ given by Lemma 5.1 and, for each $j=1,2,\ldots,n$, define $f_j(\zeta_j,\zeta_{j+1},\ldots,\zeta_n)$ to be the polynomial determined by

$$f(\zeta_1,\zeta_2,\ldots,\zeta_n)=\zeta_1\zeta_2\cdots\zeta_{j-1}f_j \\ +\text{terms not starting with } \zeta_1\zeta_2\cdots\zeta_{j-1},$$

so that $f_1=f$, $f_n=\zeta_n$, and f_j is multilinear of degree $n-j+1$. It is then shown, by induction on $j=1,2,\ldots,n$, that for all $g_j,g_{j+1},\ldots,g_n\in G$, we have either $f_j(g_j,g_{j+1},\ldots,g_n)=0$, or there exists a nonidentity linear monomial μ with $\mu(g_j,g_{j+1},\ldots,g_n)\in\Delta_k$.

Finally, in view of the preceding two theorems, it is easy to construct examples to show that P_n does not imply PI_n (see also [7]). Indeed, let q be any prime and let G_q be an infinite central product of nonabelian groups of order q^3. Then $|G'_q|=q$, G_q is a q-group, and G_q has no abelian subgroups of finite index. Thus G_q has no p-abelian subgroups of finite index, for any $p\neq q$, and hence $K[G_q]$ does not satisfy a polynomial identity unless K has characteristic q. To remedy the latter, choose a second prime r, construct the analogous group G_r, and let $G=G_q\times G_r$. Then $K[G]$ cannot satisfy a polynomial identity of any degree over any field K. Furthermore, since $|G'|=qr$, we know from [6] that G satisfies P_{qr}.

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