



Integral points, divisibility between values of polynomials and entire curves on surfaces

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Abstract

We prove some new degeneracy results for integral points and entire curves on surfaces; in particular, we provide the first examples, to our knowledge, of a simply connected smooth variety whose sets of integral points are never Zariski-dense. Some of our results are connected with divisibility problems, i.e. the problem of describing the integral points in the plane where the values of some given polynomials in two variables divide the values of other given polynomials.

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1. Introduction

The purpose of this paper is to continue the investigation initiated in [6] on integral points on surfaces, concentrating now on rational quasi-projective surfaces. We shall in particular connect the distribution of integral points on such surfaces to divisibility problems: namely, given a set of pairs of polynomials $(f_i(X, Y), g_i(X, Y))$ in two variables, we shall study the set of integral points $(x, y) \in \mathbb{A}^2$ such that $f_i(x, y)$ divides the value $g_i(x, y)$ of g_i .

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In the one-variable situation, Siegel's finiteness theorem for integral points in the case of rational curves can be restated by saying that: *if $f(X), g(X) \neq 0$ are coprime polynomials with coefficients in a ring of S -integers, such that for infinitely many S -integers x , $f(x)$ divides $g(x)$ in the ring of S -integers, then the polynomial $f(X)$ has at most one (complex) root.*

One of the objects of the present work is to (partially) extend in dimension two this particular case of Siegel's result. In this context, we mention the S -unit theorem in three variables, which can be reformulated as follows: *the set of pairs of S -integers (x, y) such that x, y and $x + y - 1$ divide 1 in the ring of S -integers is not Zariski-dense on the plane.* Our Theorem 2 provides a generalisation of the S -unit theorem to the case of more general polynomials. A very special case appeared already in [7], where we considered values of linear forms in three variables dividing values of quadratic forms at S -unit points.

As mentioned, our results can also be stated in terms of distribution of points on algebraic surfaces. In some cases we explicitly state a degeneracy result for integral points on certain rational surfaces, including for instance cubic hypersurfaces in \mathbb{P}_3 (see Theorem 1). All our results constitute particular cases of the famous Vojta's conjecture (see [14] or [17]), asserting: *let \tilde{X} be a smooth projective variety, D a hypersurface with normal crossing singularities and K a canonical divisor for \tilde{X} . If $D + K$ is big, then no set of S -integral points on $\tilde{X} \setminus D$ is Zariski-dense.*

As usual in this theory, after the fundamental work of Vojta [14], analogous results are expected for entire curves on surfaces: namely, on such a class of quasi-projective varieties, no entire curve should be Zariski-dense. In some cases we shall also obtain such corresponding degeneration results for entire curves.

The principal tool in this work will be the Main Theorem of [6], which in turn is based on the Subspace Theorem of Schmidt and Schlickewei. As a consequence, our results will be ineffective; nevertheless in some cases, where the set of integral points (resp. an entire curve) is infinite (resp. non-constant) but not Zariski-dense, we shall find out the possible infinite families of integral points (resp. images of entire curves) lying on the given surface.

Let us now state formally our main results. We follow the standard notation concerning S -integral points, as in [6,7] or [3]: for a number field k and a finite set of places S , containing all the archimedean ones, let \mathcal{O}_S denote the corresponding ring of S -integers, \mathcal{O}_S^* its group of S -units. Given a projective variety \tilde{X} , embedded in a projective space \mathbb{P}_N , and a hypersurface $D \subset \tilde{X}$, both defined over k , let $X = \tilde{X} \setminus D$ be the complement of D in \tilde{X} . We say that a rational point $P \in X(k)$ is S -integral if for no valuation ν of k outside S , the reduction P_ν of P modulo ν lies in the reduction of D (we shall later define formally the precise notion of reduction of a sub-variety). Whenever D is the intersection of \tilde{X} with the hyperplane at infinity, so X can be identified with a closed sub-variety of \mathbb{A}^N , this definition coincides with the usual one.

Some of our results are stated as degeneracy for integral points on surfaces, others as finiteness or degeneracy of the set of points satisfying some divisibility conditions. While introducing them, we shall present their logical dependence; this will also be summarised at the end of this introduction.

Our first result concerns cubic hypersurfaces of \mathbb{P}_3 :

Theorem 1. *Let $\tilde{X} \subset \mathbb{P}_3$ be a smooth cubic surface defined over k ; let H_1, H_2 be hyperplane sections such that the curve $H_1 \cup H_2$ consists of six lines. Then the S -integral points on the affine surface $X := \tilde{X} \setminus (H_1 \cup H_2)$ are not Zariski-dense.*

We recall that, after enlarging if necessary the field k , one can always find two hyperplane sections on a smooth cubic surface \tilde{X} consisting of six lines. Let us note that Vojta's conjecture

for the specific case of cubic surfaces asserts the degeneracy of integral points after removing at least two hyperplane sections; hence our result proves Vojta's conjecture in this case, up to the condition that both hyperplane sections be completely reducible. Theorem 9 will show that the number of lines to remove cannot be lowered, and that the condition that the six lines lie on two planes cannot be omitted.

A finiteness statement, in place of the degeneracy result of Theorem 1, cannot hold: actually, the smooth projective cubic surface \tilde{X} contains twenty-seven lines (defined over a suitable extension of the number field k); after removing six of them lying on two planes, there still remain twenty-one lines on X , each having just two points at infinity. Hence, after enlarging if necessary the ring of S -integers, we can always obtain twenty-one infinite families of integral points. One could prove that for a "generic" cubic hypersurface, there are no other infinite families, i.e. the one-dimensional component of the Zariski-closure of $X(\mathcal{O}_S)$ is indeed the union of twenty-one lines.

A last consideration on the variety X is in order: let $L_i(x_0 : \dots : x_3) = 0$ (for $i = 1, 2$) be linear equations for the hyperplane sections H_i appearing in Theorem 1; then the rational function $f := L_1/L_2$ on \mathbb{P}_3 provides a surjective morphism $X \rightarrow \mathbb{G}_m$. It can be easily proved (see Theorem 11) that the torus \mathbb{G}_m is in fact the generalised Albanese variety of X , and $f : X \rightarrow \text{Alb}(X) = \mathbb{G}_m$ is the corresponding canonical map; in another language, the logarithmic irregularity of X is 1. While Faltings and Vojta proved the degeneracy of integral points on surfaces whose (logarithmic) irregularity is at least three, this is one of the rare examples, to our knowledge, where degeneracy can be proved although the irregularity is only one. We shall even present a more striking example in Corollary 2 to Theorem 2, of a surface where degeneracy of integral points still holds, although its Albanese variety is trivial.

Theorem 1 admits its natural counterpart in the complex analytic setting:

Theorem 1 bis. *Let $\tilde{X} \subset \mathbb{P}_3$ be a smooth complex cubic surface, and X , as before, the affine surface obtained from \tilde{X} by removing six lines lying on two planes. For every non-constant holomorphic map $f : \mathbb{C} \rightarrow X$, the image $f(\mathbb{C})$ is an algebraic curve.*

Again, this result is best-possible, as shown by Theorem 9 bis. As for Theorem 1, for a general surface X as above, one can prove that the image $f(\mathbb{C})$ is in fact a line on X .

Theorem 1 will be derived from a general statement concerning integral points on a suitable blow-up of the plane. The study of integral points on such rational surfaces, in turn, is motivated by questions of divisibility among values of polynomials at integral points. We shall explain in detail in Section 2 the precise connection between integrality on blow-ups and divisibility (see especially Lemma 1).

As mentioned, a typical example of a divisibility problem is the S -unit equation in three variables, where one is interested in pairs of S -integers $(x, y) \in \mathcal{O}_S^2$ such that the S -integers x , y and $x + y - 1$ divide 1, in the ring \mathcal{O}_S (i.e. they are S -units); it is well known that this problem can be reformulated in terms of integral points on the complement of four lines in general position in \mathbb{P}_2 . A natural generalisation consists of replacing the constant polynomial by arbitrary polynomials. In this direction we shall prove the following result, for which we need a definition:

Definition. Let $m \geq 1$ be an integer, $(f_1, g_1), \dots, (f_m, g_m)$ be pairs of polynomials in $k[X, Y]$. We say that the m pairs above are in *general position* if the following conditions are satisfied:

- for $1 \leq i \neq j \leq m$, the curves of equation $f_i = 0, f_j = 0$ have no point in common at infinity (after embedding $\mathbb{A}^2 \hookrightarrow \mathbb{P}_2$);
- for $1 \leq i < j < h \leq m$, the three affine curves $f_i = 0, f_j = 0, f_h = 0$ have no point in common;
- for each $i \in \{1, \dots, m\}$ such that g_i is non-constant, the affine curves $f_i = 0$ and $g_i = 0$ intersect transversely;
- for $1 \leq i < j \leq m$ and $h \in \{i, j\}$, the three curves $f_i = 0, f_j = 0, g_h = 0$ have no point in common.

With this notation we have:

Theorem 2. *Let $(f_1, g_1), (f_2, g_2), (f_3, g_3)$ be three pairs of nonzero polynomials in $\mathcal{O}_S[X, Y]$; suppose they are in general position in the above sense and that*

$$\deg f_i \geq \max\{1, \deg g_i\}$$

for $i = 1, 2, 3$. Then the set of points $(x, y) \in \mathcal{O}_S^2$ such that $f_i(x, y)|g_i(x, y)$ in \mathcal{O}_S is not Zariski-dense in \mathbb{A}^2 .

In the case $\deg(f_i) = 1, g_i \equiv 1$, the statement is equivalent to the S -unit theorem in three variables. Theorem 2 is best-possible, in the sense that its conclusion certainly does not hold if we just suppose $f_i(x, y)|g_i(x, y)$ for $i = 1, 2$: simply take $f_1 = x, f_2 = y, g_1 = g_2 = 1$. Also, it is easy to see that none of the four general position conditions in the definition can be omitted (although they might probably be replaced by weaker ones).

Of course, its analogue in complex analysis, which still holds, assures, under the above hypotheses for f_i, g_i , the algebraic dependence of any pair of entire functions φ, ψ such that the three functions $g_i(\varphi, \psi)/f_i(\varphi, \psi)$ are holomorphic. This is an extension of Borel’s theorem on pairs of entire functions φ, ψ such that $1/\varphi, 1/\psi, 1/(\varphi + \psi - 1)$ are holomorphic.

A concrete example is the following very special case of Theorem 2, which is however still strong enough to imply the S -unit theorem in three variables; it is obtained by taking $g_1 = g_2 = g_3$, which is not excluded by the general position assumption:

Corollary 1. *Let $g(X, Y) \in \mathcal{O}_S[X, Y]$ be a polynomial of degree ≤ 1 such that $g(0, 0) \neq 0, g(1, 0) \neq 0, g(0, 1) \neq 0$. The pairs $(x, y) \in \mathcal{O}_S^2$ such that $xy(1 - x - y)|g(x, y)$ are not Zariski-dense in \mathbb{A}^2 .*

More generally, the case of Theorem 2 in which $f_1, f_2, f_3, g_1, g_2, g_3$ have all degree one seems worth being mentioned. Under the already mentioned correspondence between integral points on suitable blow-ups of the projective plane and points satisfying divisibility conditions, such a statement is equivalent to the following:

Corollary 2. *Let L_1, \dots, L_4 be four lines in general position on the plane \mathbb{P}_2 and choose three points P_1, P_2, P_3 , with $P_i \in L_i$ for $i = 1, 2, 3, P_i \notin L_j$ if $j \neq i$. Let $\tilde{X} \rightarrow \mathbb{P}_2$ be the blow-up of the three points P_1, P_2, P_3 and let $D \subset \tilde{X}$ be the strict transform of $L_1 + \dots + L_4$. Let $X := \tilde{X} \setminus D$ be its complement. Then $X(\mathcal{O}_S)$ is not Zariski-dense.*

The corollary is obtained by applying Theorem 2 with the following data: we take the polynomials f_i, g_i (for $i = 1, 2, 3$) such that $f_i = 0$ is an equation for L_i and then we choose g_i so that $f_i = g_i = 0$ defines the point P_i . The line L_4 will be the line at infinity.

The interest of Corollary 2 lies in the geometrical fact:

Theorem 3. *The affine surface X appearing in Corollary 2 is simply connected. In particular, its generalised Albanese variety is trivial.*

So, we have an example of a *simply connected smooth algebraic variety whose integral points, over any ring of S -integers, are never Zariski-dense*. We think this is the first example of a simply connected variety for which such a statement has been proved. Since smooth simply connected curves are isomorphic to \mathbb{A}^1 or \mathbb{P}^1 , there cannot exist examples in dimension one. Moreover, no example can be found by using the powerful technique of Faltings–Vojta (see [15,16]) which, employing in an essential way the (generalised) Albanese variety, cannot give anything for simply connected varieties.

Of course, its analogue in Nevanlinna theory still holds: for every holomorphic map $f : \mathbb{C} \rightarrow X(\mathbb{C})$, the image $f(\mathbb{C})$ is contained in an algebraic curve. In the compact case, some examples of smooth simply connected algebraic varieties admitting no entire Zariski-dense curves were already known; for instance Fermat hypersurfaces with that property were constructed by Toda and Green (see [8,13]). Ours might be the first example of an affine surface.

A variation of Theorem 2 concerns homogeneous forms in three variables instead of polynomials. For simplicity we state only a particular case of a much more general result which could be proved by our methods:

Theorem 4. *Let F_1, \dots, F_r be absolutely irreducible homogeneous forms in three variables, of the same degree, with coefficients in a ring of integers \mathcal{O}_S . Let G be another absolutely irreducible homogeneous form in three variables, still defined over \mathcal{O}_S . Suppose that:*

- (1) *for every point p where $F_i = G = 0$ for some i , p is a smooth point of both curves $F_i = 0$ and $G = 0$ and the corresponding tangents are distinct;*
- (2) *for distinct $1 \leq i < j < h \leq r$, there is no non-trivial solution to the equation $F_i = F_j = F_h = 0$;*
- (3) *for distinct $1 \leq i < j \leq r$ there is no non-trivial solution to $F_i = F_j = G$.*

Suppose moreover that

$$\deg(F_i) \geq \deg(G) \quad \text{for } i = 1, \dots, r \text{ and } r \geq 5.$$

Then the integral points $(x, y, z) \in \mathcal{O}_S^3$ such that

$$F_i(x, y, z) | G(x, y, z) \quad \text{for } i = 1, \dots, r, \tag{1}$$

in the ring \mathcal{O}_S , are not Zariski-dense in \mathbb{P}^2 .

(By abuse of notation, we have identified the vector $(x, y, z) \in \mathcal{O}_S^3$ with its corresponding point $(x : y : z)$ in \mathbb{P}^2 ; this will be possible since both sides on (1) are homogeneous and $\deg F_i \geq \deg G$, so if (x, y, z) is a solution of (1) and $\lambda \in \mathcal{O}_S$ is a common divisor of x, y, z , then a fortiori

$(x/\lambda, y/\lambda, z/\lambda)$ will be another solution.) Again, the conditions on the degree and on the number of forms are sharp, as shown by Corollary 3, Section 4.

The particular case when $\deg(F_i) = \deg(G) = 2$ for all i can be interpreted in terms of integral points on Del Pezzo surfaces of degree 4. We present below a couple of theorems concerning such surfaces and will explain in the sequel their relation with Theorem 4. Let us first recall some classical facts about Del Pezzo surfaces of degree four (see, e.g., [1]). A Del Pezzo surface of degree four can be obtained from the projective plane by blowing up five points in general position: it can be embedded in a four-dimensional projective space, as a smooth complete intersection of two quadrics; let us denote it by $\tilde{X} \subset \mathbb{P}_4$. It contains exactly sixteen lines, and infinitely many conics. A hyperplane section of \tilde{X} is a quartic curve in a three-dimensional space, which will not be in general a plane curve; some special hyperplane sections, however, will be reducible, consisting of the union of two (plane) conics; it is easy to see that there are infinitely many such reducible hyperplane sections. Now, if five generic hyperplane sections are removed from \tilde{X} , one obtains a closed (affine) variety $X \subset \mathbb{G}_m^4$ (since the complement in \mathbb{P}_4 of five hyperplanes in general position is isomorphic to \mathbb{G}_m^4), and an application of the S -unit theorem in several variables proves the degeneracy of its integral points. In the particular case where the hyperplane sections are reducible, we improve on such a consequence of the S -unit theorem; the proof will make use of a theorem of Levin [12]:

Theorem 5. *Let $\tilde{X} \subset \mathbb{P}_4$ be a Del Pezzo surface of degree four. Let H_1, \dots, H_5 be five hyperplane sections, each of the form $H_i = C_i + C'_i$, for smooth conics C_i, C'_i . Suppose they are in general position, in the sense that no three of them intersect. Set $X := \tilde{X} \setminus (C_1 \cup \dots \cup C_5)$. Then the integral points for the set $X(\mathcal{O}_S)$ are not Zariski-dense.*

So, while the standard application of the S -unit theorem requires the removal of all the ten conics from \tilde{X} to ensure degeneracy of integral points, we provide the same conclusion after removing just five of them.

Actually we could prove that if the hyperplane sections are sufficiently generic (but still reducible), all but finitely many integral points lie in the union of up to eleven lines. Also, such infinite families cannot be avoided, since any such surface contains eleven lines with at most two points at infinity.

Finally, let us note that the quasi-projective variety X in Theorem 5 can be non-affine (and of course is never projective); as previously remarked, some questions about divisibility of values of polynomials in two variables amount to problems on distribution of integral points on surfaces which are neither projective nor affine.

Let us present a further result on Del Pezzo surfaces of degree four: recall that by the S -unit equation theorem, the integral points on the complement of four pairwise linearly equivalent ample divisors on a surface are degenerate. Again, when one of them is reducible, we can improve on this classical result, obtaining:

Theorem 6. *Let $\tilde{X} \subset \mathbb{P}_4$ be a Del Pezzo surface of degree four. Let H_1, \dots, H_4 be hyperplane sections, such that no three of them intersect. Suppose one of them, say H_4 , is reducible, so $H_4 = C + C'$, for two curves C, C' (either two conics, or a cubic and a line). Then the integral points on $X := \tilde{X} \setminus (H_1 \cup H_2 \cup H_3 \cup C)$ are degenerate.*

Again, both results above admit the obvious analogue in complex function theory, i.e. the algebraic degeneracy of entire curves on X .

As announced, Theorem 1 is a corollary of a general result concerning divisibility of values of homogeneous forms at integral points. To state it, we shall adhere to the following standard notation concerning ideals in a ring: for S -integers $\alpha_1, \dots, \alpha_h \in \mathcal{O}_S$, we shall denote by $(\alpha_1, \dots, \alpha_h)$ the ideal they generate in the ring \mathcal{O}_S . (We shall give soon-after another equivalent formulation, without mentioning ideals.)

Theorem 7. *Let $n \geq 6$ be an integer, and for $i \in \mathbb{Z}/n\mathbb{Z}$, let F_i be a linear form in three variables, such that any three of them are linearly independent. The set of points $(x : y : z) \in \mathbb{P}_2(k)$ satisfying, for each $i \in \mathbb{Z}/n\mathbb{Z}$, the equality of ideals*

$$F_i(x, y, z) \cdot (x, y, z) = (F_{i-1}(x, y, z), F_i(x, y, z)) \cdot (F_i(x, y, z), F_{i+1}(x, y, z)) \tag{2}$$

is not Zariski-dense in \mathbb{P}_2 .

In the above formula, by homogeneity one can suppose that x, y, z are S -integers; otherwise, one can interpret the ideals in the above equation in the sense of fractional ideals of k .

Let us now see another way of stating Theorem 7, without mentioning ideals. Enlarging S by adding a finite number of places, we can ensure that the ring \mathcal{O}_S is a unique factorisation domain, so it makes sense to define the greatest common divisor of two elements $\alpha, \beta \in \mathcal{O}_S$ (as a generator of the ideal (α, β)). Also, we can suppose that any three of the linear forms are independent modulo any place outside S (since by hypothesis they are independent over k). Then, for coprime S -integers x, y, z , any three of the values $F_i(x, y, z)$ will be coprime. Let us put for every $i \in \mathbb{Z}/n\mathbb{Z}$,

$$\beta_i = \gcd(F_i(x, y, z), F_{i-1}(x, y, z));$$

then for $i \neq j$, the S -integers β_i, β_j are coprime, in particular $\gcd(\beta_i, \beta_{i+1}) = 1$ so that we can write

$$F_i(x, y, z) = \beta_i \beta_{i+1} \alpha_i,$$

for an S -integer α_i . Then Theorem 7 asserts the degeneracy of the points $(x : y : z) \in \mathbb{P}_2(k)$ such that α_i is a unit for all $i \in \mathbb{Z}/n\mathbb{Z}$.

We shall see that Theorem 1 follows formally from the case $n = 6$ of Theorem 7, and that this and all other results presented so far can be interpreted as the degeneracy of integral points on certain blow-ups of the projective plane; these facts, in turn, consist of particular cases of Vojta’s conjecture for surfaces. On the other hand, the analogous result for $n \leq 5$ does not hold (see Theorem 10), in accordance to the fact that the hypotheses of Vojta’s conjecture also fail in that case.

We state below the complex-analytic analogue of Theorem 7:

Theorem 7 bis. *Let $n \geq 6$ be an integer and for each $i \in \mathbb{Z}/n\mathbb{Z}$ let F_i be a linear form in three variables, with complex coefficients, such that no three of them are linearly dependent. Let $f, g, h : \mathbb{C} \rightarrow \mathbb{C}$ be three entire functions without common zeros. Suppose the following holds: for every index $i \in \mathbb{Z}/n\mathbb{Z}$ and every zero $p \in \mathbb{C}$ of the holomorphic function $F_i(f, g, h)$, p is also a zero of either $F_{i-1}(f, g, h)$ or $F_{i+1}(f, g, h)$ and*

$$\text{ord}_p F_i(f, g, h) = \max(\text{ord}_p F_{i-1}(f, g, h), \text{ord}_p F_{i+1}(f, g, h)).$$

Then the meromorphic functions $f/h, g/h$ are algebraically dependent.

Theorem 7 extends to forms F_1, \dots, F_n of arbitrary degree, provided some general position conditions are imposed.

A last application of our methods concerns algebraic families of S -unit equations, of the kind treated by the authors in [7] and by Levin in [11]; we fix three polynomials $f(T), g(T), h(T) \in \mathcal{O}_S[T]$ and consider the S -unit equation

$$f(t)u + g(t)v = h(t) \tag{3}$$

to be solved in S -units u, v and S -integers t . In [7] the authors considered the linear case, where $\text{deg}(f) = \text{deg}(g) = \text{deg}(h) = 1$, while Levin treated in [11] the case when $\text{deg}(f) + \text{deg}(g) = \text{deg}(h)$. In both cases, the statement was reduced to a question of integral points on the complement of a suitable divisor in the product $\mathbb{P}_1 \times \mathbb{P}_1$. In the present work, we shall consider a different compactification, given by a Hirzebruch¹ surface (as it was done for a different Diophantine problem in [5]), and prove

Theorem 8. *Let $f(T), g(T), h(T)$ be three polynomials of the same degree with S -integral coefficients, without common zeros. There exists a finite set $\Phi \subset \mathcal{O}_S^*$ such that for all solutions $(t, u, v) \in \mathcal{O}_S \times \mathcal{O}_S^* \times \mathcal{O}_S^*$ to Eq. (3), one at least among $u, v, u/v$ belongs to Φ . In particular, the solutions are not Zariski-dense in the surface defined by (3).*

It is easily seen that for every zero t_0 of $f(t)g(t)h(t)$, there exists an infinite family of solutions, up to enlarging S ; for instance, if $f(t_0) = 0$, just put $v = -h(t_0)/g(t_0)$ and take for u any S -unit. For such families, either u or v or u/v is fixed. In some cases, there might also exist infinite families with variable t .

The next statements will show that the integral points on certain open surfaces are indeed Zariski-dense: they will prove that our Theorems 1, 4, 7 are in a sense best-possible. Several other statements of the same flavour will be given in Section 4.

We begin by showing that in Theorems 1, 1 bis one cannot omit neither the condition on the number of lines to remove, nor the condition that they lie on two planes:

Theorem 9 (Counter-example). *Let \tilde{X} be a smooth cubic surface, defined over a number field k ; let L_1, \dots, L_5 be five lines lying on two hyperplane sections of \tilde{X} ; put $X = \tilde{X} \setminus (L_1 \cup \dots \cup L_5)$. There exists a finite extension k' of k and a finite set S of places of k' such that $X(\mathcal{O}_S)$ is Zariski-dense. Also, there exists a configuration of nine lines L'_1, \dots, L'_9 and a Zariski-dense set of S -integral points on $\tilde{X} \setminus (L'_1 \cup \dots \cup L'_9)$.*

Its complex-analytic analogue reads:

Theorem 9 bis. *Let \tilde{X} be a smooth cubic surface, L_1, \dots, L_5 be five lines on \tilde{X} , lying on two hyperplanes; there exists a holomorphic map $f : \mathbb{C} \rightarrow \tilde{X}$ omitting all the lines L_1, \dots, L_5 and*

¹ The so-called Hirzebruch surfaces were introduced by Friederich Hirzebruch in [Math. Annalen, 1951]; later Aldo Andreotti realised that all minimal rational surfaces are of this kind.

having Zariski-dense image. Also, there exists a configuration of nine lines L'_1, \dots, L'_9 and a holomorphic map $g : \mathbb{C} \rightarrow \tilde{X}$ omitting L'_1, \dots, L'_9 and having Zariski-dense image.

This counter-example can be strengthened, using the work of Buzzard and Lu [4]; we could even prove that the complement of five lines in a smooth cubic surface is holomorphically dominated by \mathbb{C}^2 .

We now show that Theorem 7 is also best-possible, in the sense that it fails in the case $n \leq 5$:

Theorem 10. *Let n be an integer with $1 \leq n \leq 5$; let F_1, \dots, F_n be linear forms in three variables in general position, defined over a number field k . There exists a ring of S -integers \mathcal{O}_S , for a finite set of places S , such that the set of integers $(x, y, z) \in \mathcal{O}_S^3$ satisfying the relation (2) is Zariski-dense (in \mathbb{P}_2 or, equivalently, in \mathbb{A}^3).*

To summarise: we dispose of three main “non-divisibility theorems”, namely Theorems 2, 4, and 7, stating that certain divisibility conditions are satisfied only in proper Zariski-closed sets of the plane. These results formally imply some theorems on integral points on surfaces: Corollary 2 (on the simply connected surface) follows from Theorem 2; Theorem 1 (on cubic surfaces) follows from Theorem 7; Theorem 5 (on Del Pezzo surfaces of degree four) can be derived, at least in a particular case, from Theorem 4. One extra result (Theorem 8) is stated in terms of S -unit equations with parametric coefficients.

The rest of the paper is organised as follows: In the next section we first show how divisibility and integrality on quasi-projective surfaces are related, and then prove our main Diophantine theorems, the degeneracy of integral points on certain surfaces.

In Section 3 we prove our claims of geometric type, for instance the simply-connectedness of the surface appearing in Corollary 2, and the determination of the Albanese variety of the surface appearing in Theorem 1. We also classify the infinite families of solutions to the S -unit equation (3).

Finally, the last section contains results in the opposite direction: we shall prove that integral points on certain surfaces form a Zariski-dense set, thus proving that some of our degeneracy results are in a sense best-possible: for instance, one cannot omit the hypothesis appearing in Theorem 1 that the six lines at infinity lie on two hyperplanes, and one cannot improve on the hypothesis $n \geq 6$ in Theorem 7.

2. Proofs of main results

In this section, we shall prove the degeneracy of integral points on the surfaces considered in the introduction, Theorems 2 and 4 on divisibility and Theorem 8 on parametric S -unit equations. We shall also show the relations between them. The corresponding results on entire curves can be proved in the same way, by replacing Schmidt’s Subspace Theorem by Cartan’s Second Main Theorem, as explained by Levin [12]. Hence, we shall omit the proofs in the analytic setting, and work only over number fields.

In the sequel, k will be a fixed number field, S a finite set of places of k containing the archimedean ones, \mathcal{O}_S (resp. \mathcal{O}_S^*) will denote the ring of S -integers (resp. group of S -units) of k . We recall that for every non-archimedean place v of k , with residue field k_v , there is a well-defined reduction map $\mathbb{P}_n(k) \rightarrow \mathbb{P}_n(k_v)$. In particular, for a projective variety $X \subset \mathbb{P}_n$, defined over k , and a rational point $P \in X(k)$, we can speak of the reduction of P modulo v . If $D \subset X$ is a closed sub-variety, we can always define in two different ways its reduction modulo v (which

will be a variety defined over the residue field k_ν of ν as follows: (1) Set-theoretically: extend in some way the valuation ν to the algebraic closure \bar{k} of k and consider the set of the reduction of the points $P \in D(\bar{k})$; it is the set of points \bar{k}_ν -rational of an algebraic variety D_ν defined over k_ν . (2) Alternatively, note that for all but finitely many places ν , one can simply reduce modulo ν the coefficients of a given system of equations for D , obtaining a system of equations for D_ν . Since we shall consider only finitely many sub-varieties $D \subset \tilde{Y}$, and in each of our statements we are allowed to disregard a fixed but arbitrary finite set of places of k , one can always refer to the reduction modulo ν in the second sense.

Given a hypersurface $D \subset \tilde{X}$ defined over k , we say that a rational point $P \in \tilde{X} \setminus D(k)$ is ν -integral if its reduction modulo ν does not lie in the reduction of D modulo ν ; we say it is S -integral if it is ν -integral for every place ν outside S .

The main tool for reducing questions of divisibility between values of polynomials to integrality for points on rational varieties is represented by the following lemma:

Lemma 1. *Let $\tilde{Y} \subset \mathbb{P}_N$ be a smooth projective surface defined over k , C_1, C_2 be curves on \tilde{Y} intersecting transversely at points P_1, \dots, P_n , also defined over k . Let φ, ψ be rational functions on \tilde{Y} , defined over k , such that local equations (on an affine open set $U \subset \tilde{Y}$) for C_1 (resp. C_2) are given by $\varphi = 0$ (resp. $\psi = 0$). Let $\pi : \tilde{X} \rightarrow \tilde{Y}$ be the blow-up of \tilde{Y} over the points P_1, \dots, P_n ; for $i = 1, 2$ denote by \hat{C}_i , the strict transform of C_i under π . Let ν be a non-archimedean place of k such that the following conditions are satisfied:*

- *the reductions modulo ν of C_1, C_2 intersect transversely;*
- *the reductions modulo ν of the functions φ, ψ induce local equations for the reductions of C_1, C_2 ;*
- *the blow-up map π induces an isomorphism modulo ν of the complement of the exceptional divisors in \tilde{X} with the complement of $\{P_1, \dots, P_n\}$ in \tilde{Y} .*

Then for every point $P \in \tilde{Y}(k)$ lying in the domain U of φ and ψ , not in $C_1 \cup C_2$, the following are equivalent:

1. $\nu(\varphi(P)) \leq \nu(\psi(P))$.
2. $\pi^{-1}(P)$ is ν -integral with respect to \hat{C}_1 .

Proof. Let us suppose that 1. holds; we want to prove that $\pi^{-1}(P)$ is ν -integral with respect to \hat{C}_1 . If P does not reduce to C_1 modulo ν , we are done (in fact P is ν -integral with respect to the divisor $\pi^*(C_1)$, which contains the strict transform \hat{C}_1). Then we can suppose that $\nu(\varphi(P)) > 0$; by hypothesis 1. we also have $\nu(\psi(P)) > 0$, so P reduces modulo ν to an intersection point of $C_1 \cap C_2$, say P_i . By the hypotheses of the lemma, the functions φ, ψ are local parameters on the surface \tilde{Y} at the point P_i . The blown up surface can be defined locally at P as the closed subset of $\tilde{Y} \times \mathbb{P}_1$ defined by the equation $\varphi(y)\xi_i = \psi(y)\eta_i$; the exceptional divisor E_i is defined by $\varphi = \psi = 0$ and \hat{C}_1 by $\eta_i = 0$. We have to prove that $\pi^{-1}(P)$ does not reduce to $\eta_i = 0$ modulo ν . From the equality $\nu(\varphi(P)) + \nu(\xi_i) = \nu(\psi(P)) + \nu(\eta_i)$ and the inequality 1. it follows that $\nu(\xi_i) \geq \nu(\eta_i)$, i.e. $(\xi_i : \eta_i)$ is not congruent to $(1 : 0)$ modulo ν , which, as remarked, is what we want. The other implication, which will not be used in the proof, can be proved by the same reasoning. \square

We now prove Theorem 7, from which we shall deduce Theorem 1. Theorem 7 will be shown to be equivalent, via the above lemma, to the following:

Proposition 1. *Let $n \geq 6$ be an integer; for every index $i \in \mathbb{Z}/n\mathbb{Z}$, let $H_i \subset \mathbb{P}_2$ be a line defined over k . Suppose that no three of them intersect. Let, for each index $i \in \mathbb{Z}/n\mathbb{Z}$, P_i be the intersection point $H_i \cap H_{i+1}$. Let $\tilde{X} \rightarrow \mathbb{P}_2$ be the blow-up of the plane over the points P_1, \dots, P_n and let $D_i \subset \tilde{X}$ be the corresponding strict transform of H_i . Finally put $D = D_1 + \dots + D_n$. Then no set of S -integral points of $\tilde{X} \setminus D$ is Zariski-dense.*

Proof. We apply the Main Theorem of [6], in its generalised form given in [7], Theorem 2.1, hence we follow the notation of [6,7]. We take $r = n$ and $p_i = 1$ for all $i = 1, \dots, n$. To check the validity of the hypothesis of Theorem 2.1 in [7], we have to compute the intersection matrix of the divisor $D = D_1 + \dots + D_n$. For this purpose, note that since each divisor D_i is the strict transform of the line H_i in \mathbb{P}_2 , which passes through two blown up points, its self-intersection equals $1 - 2 = -1$. Also, for $i \neq j$, D_i intersects D_j in one point unless i, j are consecutive. Finally we have, for all $i \in \mathbb{Z}/n\mathbb{Z}$, $D_i^2 = -1$, $D_i \cdot D_{i+1} = 0$; $D_i \cdot D_j = 1$ for $j \neq i, i + 1, i - 1$. Then $D \cdot D_i = n - 4$ and $D^2 = n(n - 4)$, so $D^2 > 0$, and $D \cdot C \geq 0$ for all irreducible curves C in \tilde{X} , as required by the hypotheses of Theorem 2.1 in [7]. We note that for a generic choice of the lines H_i in the plane, D will be ample, but in some degenerate cases it will not be so (for instance, if all the n intersection points $H_i \cap H_{i+1}$ lie on a conic C , then $D \cdot C = 0$; of course this does not happen generically since $n \geq 6$). The equation for $\xi_i =: \xi$ becomes

$$\xi^2 + 2(n - 4)\xi - n(n - 4) = 0. \tag{4}$$

The inequality $2D^2\xi > D \cdot D_i \xi^2 + 3D^2$, appearing in the hypothesis of the Main Theorem of [6] (or Theorem 2.1 of [7]), now reads

$$2n(n - 4)\xi > (n - 4)\xi^2 + 3n(n - 4).$$

After simplifying the common factor $(n - 4)$ and using Eq. (4) for expressing ξ^2 as $\xi^2 = n(n - 4) - 2(n - 4)\xi$, the above inequality becomes $2(2n - 4)\xi > n^2 - n$, i.e.

$$\xi > \frac{n^2 - n}{2(2n - 4)}.$$

We have to prove that this inequality is satisfied by the minimal positive solution (actually the only positive solution) of (4). Now, this amounts to proving that the value of the polynomial on the left side in (4), calculated at the point $(n^2 - n)/2(2n - 4)$, is negative, i.e.

$$\left(\frac{n^2 - n}{2(2n - 4)} \right)^2 + 2(n - 4) \frac{n^2 - n}{2(2n - 4)} < n(n - 4).$$

The above inequality simplifies to $7n^3 - 70n^2 + 207n - 192 > 0$, which is easily seen to be equivalent (for positive integers n) to $n \geq 6$, concluding the proof. \square

Deduction of Theorem 7 from Proposition 1. Let F_1, \dots, F_n be the linear forms appearing in Theorem 7 and let, for $i \in \mathbb{Z}/n\mathbb{Z}$, $H_i \subset \mathbb{P}_2$ be the line of equation $F_i(x, y, z) = 0$. By the

hypotheses of Theorem 7 such lines are in general position. Let $\pi : \tilde{X} \rightarrow \mathbb{P}_2$ be the blow-up defined in the above proposition, E_1, \dots, E_n the corresponding exceptional curves, D the hypersurface defined in the statement of Proposition 1. Note that π is an isomorphism between $\tilde{X} \setminus (D \cup E_1 \cup \dots \cup E_n)$ and $\mathbb{P}_2 \setminus (H_1 \cup \dots \cup H_n)$. As we mentioned in the introduction, we can enlarge the set S of places in such a way that the reductions of H_1, \dots, H_n modulo of every place outside S are still in general position. Also, we can suppose that the isomorphism between the affine surfaces $\tilde{X} \setminus (D \cup E_1 \cup \dots \cup E_n)$ and $\mathbb{P}_2 \setminus (H_1 \cup \dots \cup H_n)$, given by the restriction of π , is defined over \mathcal{O}_S , together with its inverse, so it induces an isomorphism modulo every place outside S .

Let now $(x, y, z) \in \mathcal{O}_S^3$ be a point satisfying the relation (2) and such that $F_i(x, y, z) \neq 0$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. If we prove that such a point defines an S -integral point $\pi^{-1}(x : y : z)$ on \tilde{X} with respect to D , we have finished the proof that Proposition 1 implies Theorem 7.

To this end, we use Lemma 1. Let v be a place of k outside S , fix an index $i \in \mathbb{Z}/n\mathbb{Z}$ and a point $P = (x : y : z) \in \mathbb{P}_2(k)$ satisfying (2); we have to prove that the reduction of $\pi^{-1}(P)$ modulo v does not lie on D_i . If $F_i(P) \not\equiv 0 \pmod{v}$, then we have finished, since $\pi^{-1}(P)$ is integral even with respect to the whole pull-back $\pi^*(H_i)$. Then suppose $F_i(P) \equiv 0 \pmod{v}$. By the relation (2), there exists an index $j \in \{i + 1, i - 1\}$ such that $v(F_i(P)) = v(F_j(P))$. Suppose for instance $j = i + 1$; this means that P reduces to $H_i \cap H_{i+1}$ modulo v . By the hypothesis that the forms F_1, \dots, F_n are in general position modulo v , for all other indices $h \in \{1, \dots, n\} \setminus \{i, i + 1\}$, we have $v(F_h(P)) = 0$. Let us choose one such index, say $i - 1$, and consider the rational functions $\varphi := F_i/F_{i-1}$, $\psi := F_j/F_{i-1} = F_{i+1}/F_{i-1}$. Then local equations for H_i (resp. H_{i+1}) in a neighbourhood of $H_i \cap H_{i+1}$ are given by $\varphi = 0$ (resp. $\psi = 0$). Lemma 1 applies, proving that $\pi^{-1}(P)$ is integral with respect to D_i , finishing the proof. \square

Proof of Theorem 1. We suppose the hypotheses of Theorem 1 hold; in particular, $\tilde{X} \subset \mathbb{P}_3$ is a cubic surface defined over a number field, H_1, H_2 are two completely reducible hyperplane sections. It is well known (see [1]) that \tilde{X} can be obtained by blowing up the plane at six points P_1, \dots, P_6 in general position (no three on a line, no six on a conic); after enlarging the field of definition, we can suppose that such points are defined over the given number field k . The configuration of six lines lying on two planes can be obtained as the strict transform of the six lines on the plane connecting P_i, P_{i+1} for $i \in \mathbb{Z}/6\mathbb{Z}$. Then apply Theorem 7 to obtain the degeneracy of the set of integral points considered in Theorem 1. \square

Proof of Theorem 8. Let us consider the affine surface defined by Eq. (3); it admits a smooth compactification in $\mathbb{P}_2 \times \mathbb{P}_1$ obtained in the following way: write $\tilde{f}, \tilde{g}, \tilde{h} \in k[T_0, T_1]$ for the forms of degree $d = \deg(f) = \deg(g) = \deg(h)$ satisfying $\tilde{f}(1, T) = f(T)$, $\tilde{g}(1, T) = g(T)$, $\tilde{h}(1, T) = h(T)$. Then our surface is an open affine set of the surface defined in $\mathbb{P}_2 \times \mathbb{P}_1$, with coordinates $((U : V : W), (T_0 : T_1))$, by the bi-homogeneous equation

$$\tilde{X}: \quad U \cdot \tilde{f}(T_0, T_1) + V \cdot \tilde{g}(T_0, T_1) = W \cdot \tilde{h}(T_0, T_1).$$

It is endowed with a canonical projection $\tilde{X} \rightarrow \mathbb{P}_1$ whose fibers are projective lines. It turns out to be isomorphic to the d -th Hirzebruch surface (see [1]), independently of the degree d polynomials f, g, h (provided they have no common zero, which is assumed here). The solutions $(u, v, t) \in \mathcal{O}_S^* \times \mathcal{O}_S^* \times \mathcal{O}_S$ to Eq. (3) give rise to rational points $((u : v : 1), (1 : t))$ which are integral with respect to the divisor $D: T_0 \cdot UVW = 0$. It consists of four components D_1, D_2, D_3 and D_4 , where D_4 is given by $T_0 = 0$, so it is a fiber for the projection $\tilde{X} \rightarrow \mathbb{P}_1$, while $D_1,$

D_2, D_3 are pull-backs of lines in \mathbb{P}_2 (via the natural projection $\tilde{X} \rightarrow \mathbb{P}_2$). Then $D_4^2 = 0$, D_1, D_2, D_3 are linearly equivalent and satisfy $D_i \cdot D_j = d \geq 1$ for $1 \leq i, j \leq 3$. We are in the situation of applying Theorem 1.1 of [7] (or the Main Theorem of [6]) to obtain the degeneracy of the set of solutions. Hence all but finitely many solutions lie on a finite set of curves, which, by Siegel’s theorem on integral points on curves, are parametrised either by \mathbb{A}^1 or by \mathbb{G}_m . It is easy to see that there are no curves parametrised by \mathbb{A}^1 on the affine surface $\tilde{X} \setminus D$. Those parametrised by \mathbb{G}_m , i.e. of vanishing Euler characteristic, are classified by Theorem 12; they give rise to the infinite families with fixed u, v or u/v , and this completes the proof. \square

To prove Theorem 2, we need yet another consequence of our Main Theorem from [6], which we immediately state and prove:

Proposition 2. *Let \tilde{X} be a smooth projective surface, D_1, \dots, D_r, H be reduced and irreducible divisors on \tilde{X} , no three of them intersecting. Suppose there exist positive integers p_1, \dots, p_r, c, h such that for $1 \leq i < j \leq r$, $p_i D_i \cdot H = ch$, $p_i p_j D_i \cdot D_j = c^2$ and $H^2 = h^2$. Suppose moreover that $D_i^2 = 0$. If $r \geq 3$, then the integral points on $\tilde{X} \setminus (H \cup D_1 \cup \dots \cup D_r)$ are not Zariski-dense.*

Proof. It is clear that the result follows from the particular case when $r = 3$, so we shall suppose $r = 3$. Again, we apply the Main Theorem of [6]. Let us denote by D_4 the divisor H and let $p_4 = p$ be a positive integer to be chosen later. Put

$$D = p_1 D_1 + \dots + p_4 D_4 = p_1 D_1 + p_2 D_2 + p_3 D_3 + p H.$$

We shall now verify that the hypotheses of the Main Theorem in [6] are satisfied. First of all, let us remark again that the condition that D is ample, appearing in the statement of the Main Theorem in [6] (and which, in any case, will be satisfied in the applications to Theorems 2 and 4) is not really needed, and can be replaced by the condition that $D^2 > 0$ and D is nef; this fact appears explicitly in [7] and [12]. Let us compute the intersection products. For $i \leq 3$ we have

$$D \cdot D_i = \frac{2c^2}{p_i} + \frac{pch}{p_i}, \tag{5}$$

while for the intersection number $D \cdot D_4 = D \cdot H$ we have

$$D \cdot H = 3ch + ph^2. \tag{6}$$

Then

$$D^2 = 6c^2 + 6pch + p^2h^2. \tag{7}$$

For $i \leq 3$, the equation for ξ_i , i.e. $(D - \xi_i D_i)^2 = 0$, gives

$$\xi_i = \frac{D^2}{2(D \cdot D_i)}.$$

The inequality $2\xi_i D^2 > D \cdot D_i \xi_i^2 + 3p_i D^2$, appearing in the hypotheses of the Main Theorem of [6] (or Theorem 2.1 of [7]), becomes

$$2D^2 \frac{D^2}{2(D.D_i)} > (D.D_i) \left(\frac{D^2}{2(D.D_i)} \right)^2 + 3p_i D^2,$$

which simplifies to $D^2 > 4p_i(D.D_i)$, i.e. $6c^2 + 6pch + p^2h^2 > 4(2c^2 + pch)$, or

$$p^2h^2 + 2pch > 2c^2. \tag{8}$$

For $i = 4$, the equation for $\xi = \xi_4$ is

$$h^2\xi^2 - 2(D.H)\xi + D^2 = 0, \tag{9}$$

which provides the relation

$$\xi^2 = (2(D.H)\xi - D^2)/h^2. \tag{10}$$

Solving Eq. (9) for ξ , taking into account that by (6) and (7) one has $(D.H)^2 - D^2H^2 = 3c^2h^2$, we obtain for the minimal solution ξ the expression

$$\xi = \frac{3ch + ph^2 - \sqrt{3}ch}{h^2} = (3 - \sqrt{3})\frac{c}{h} + p. \tag{11}$$

Now, the condition $2\xi D^2 > (D.H)\xi^2 + 3pD^2$, appearing in the hypotheses of the Main Theorem of [6], becomes, after substituting ξ^2 by its expression in (10),

$$2\xi D^2 > (D.H) \left(\frac{2(D.H)\xi - D^2}{h^2} \right) + 3pD^2.$$

The above inequality becomes $2\xi((D.H)^2 - D^2h^2) < (D.H)D^2 - 3ph^2D^2$, which, by (11), (6), (7) and the already used fact that the discriminant $(D.H)^2 - D^2H^2$ equals $3c^2h^2$, amounts to

$$6c^2h[(3 - \sqrt{3})c + ph] < (3ch - 2ph^2)(6c^2 + 6pch + p^2h^2). \tag{12}$$

Remember that we were searching for a positive integer p such that both inequalities (8) and (12) are satisfied. Now, it is easy to see that such inequalities are indeed satisfied for $p = \frac{3c}{4h}$. This number might be a rational non-integral one; but, from the relation $p_i D_i H = ch$, appearing in the present assumptions, and the fact that $h = (H^2)^{1/2}$ is fixed (i.e. independent of p_1, p_2, p_3), one easily sees that if p_1, p_2, p_3 are replaced by suitable positive multiples, the assumptions of the theorem will still be satisfied and one can secure that the number $c/(4h)$ is indeed an integer. This remark concludes the proof. \square

Proof of Theorem 2. Let $(f_1, g_1), (f_2, g_2), (f_3, g_3)$ be the pairs of polynomials appearing in Theorem 2, and let D'_i (for $i = 1, 2, 3$) be the curves in \mathbb{P}_2 whose local equations in \mathbb{A}^2 are $f_i = 0$. Let H' be the line at infinity of \mathbb{P}_2 . For each $i = 1, 2, 3$, let $\{P_{i,1}, \dots, P_{i,l_i}\}$ be the intersection of the affine curves $f_i = 0$ and $g_i = 0$; note that $l_i \leq \deg(f_i)^2$, by the assumption $\deg(f_i) \geq \deg g_i$. If $\deg g_i < \deg f_i$ complete the set $\{P_{i,1}, \dots, P_{i,l_i}\}$ by adding smooth points $P_{i,l_i+1}, \dots, P_{i,\deg(f_i)^2}$ of the curve $f_i = 0$, but outside the two other curves $f_j = 0$ (for $1 \leq j \leq 3$,

$j \neq i$). We thus obtain $\sum_{i=1}^3 \deg(f_i)^2$ points on the affine plane \mathbb{A}^2 . The general position assumptions guarantee they are indeed distinct; also, they guarantee that no three among the four projective curves D'_1, D'_2, D'_3, H' intersect and no two intersect on any of the points $P_{i,j}$. Let \tilde{X} be the blow-up of the projective plane over the points $P_{i,j}$; let D_i be the strict transform of D'_i , and let $H \subset \tilde{X}$ be the curve corresponding to H' . After enlarging if necessary the finite set of places S , we can ensure that at each point $P_{i,j}$ the assumptions of Lemma 1 are satisfied (with $\varphi = f_i, \psi = g_i$). Suppose now $(x, y) \in \mathcal{O}_S^2$ is a point satisfying $f_i(x, y)|g_i(x, y)$ for $i = 1, 2, 3$ (so in particular $f_i(x, y) \neq 0$, which excludes that $(x : y : 1)$ is one of the blown up points). Let $p \in \tilde{X}$ be its corresponding point on the blown up surface. Since x and y are S -integers, the point $(x : y : 1)$ is integer with respect to H' , so the corresponding point p is integer with respect to H . By Lemma 1, the point p is integer also with respect to D_i for $i = 1, 2, 3$. We now show that we can apply Proposition 2 to conclude that such points are not Zariski-dense. Letting $d_i > 0$ be the degree of f_i , we observe that D_i is the strict transform of a curve of degree d_i in \mathbb{P}_2 blown up at d_i^2 points, so its self-intersection vanishes; the intersection product $D_i D_j$ (for $1 \leq i < j \leq 3$) equals $d_i d_j$, since no common point of D'_i, D'_j is blown up. Also, since no point in $D'_i \cap H'$ is blown up, we have $D_i \cdot H = d_i$. Now put $p_i = d_1 d_2 d_3 / d_i$. It is immediate to check that the assumptions of Proposition 2 are satisfied with $c = d_1 d_2 d_3$ and $h = 1$. \square

The above proof also shows the correspondence between the divisibility problems considered in Theorem 2 and integrality on certain surfaces; in such a correspondence, the case $\deg f_i = \deg g_i = 1$ for $i = 1, 2, 3$ leads to the surface considered in the corollary to Theorem 2.

Proof of Theorem 4. The proof is very similar to the above one and to the proof of Theorem 7. Let, for each $i = 1, \dots, r, D'_i \subset \mathbb{P}_2$ be the projective curve of equation $F_i = 0$, and let $d = \deg F_i = \deg D'_i$. Let $P_{i,1}, \dots, P_{i,d \cdot \deg G}$ be the intersection points of D'_i with the curve $G = 0$ (by assumption there are exactly $d \cdot \deg G$ of them). Let $\tilde{X} \rightarrow \mathbb{P}_2$ be the blow-up of the plane over all these points. Again by Lemma 1, we are reduced to proving the degeneracy of integral points on $\tilde{X} \setminus (D_1 \cup \dots \cup D_r)$, where D_i is the strict transform of D'_i for $i = 1, \dots, r$. In the case $\deg G = \deg F_i$, all the self-products D_i^2 vanish, and the products $D_i D_j$ for $i \neq j$ are all equal to $d^2 > 0$. Then the degeneracy of integral points follows directly from Theorem 1, part (b), of [6]. Hence the theorem is proved in this case. If $\deg F > \deg G$, just remark that the theorem becomes weaker if we blow up more points on the curves D'_i ; hence, let us choose, for each i , any set of $\deg F_i (\deg F_i - \deg G)$ points on the curve D'_i , outside each curve D'_j (for $j \neq i$) and the curve $G = 0$. Considering the corresponding surface \tilde{X} , and letting again D_i be the strict transforms of D'_i , we reduce to the previous case. \square

Proof of Theorem 5. We note that the five conics C_1, \dots, C_5 appearing in the statement have self-intersection zero. It easily follows from Riemann–Roch that they have Kodaira dimension one, i.e. $h^0(\tilde{X}, nC_i) \gg n$ for $n \rightarrow \infty$ (actually this fact can be proved directly even without Riemann–Roch). Then Theorem 11.9A of [12] applies and gives the desired conclusion. \square

We now show the relation between the just proved Theorem 5 and Theorem 4. We can obtain a Del Pezzo surface $\tilde{X} \subset \mathbb{P}_4$ by blowing up five points P_1, \dots, P_5 in general position in \mathbb{P}_2 , and embedding it in \mathbb{P}_4 via the linear system of cubics passing through P_1, \dots, P_5 . The lines on \tilde{X} correspond to the five blown up points, to the lines on \mathbb{P}_2 connecting two such points, and to the unique conic containing P_1, \dots, P_5 ; they all have self-intersection -1 . Every hyperplane

section of \tilde{X} corresponds to a cubic curve in \mathbb{P}_2 , passing through P_1, \dots, P_5 . If a hyperplane section is formed by two conics, the corresponding cubic is formed by a conic passing through four of the five points P_1, \dots, P_5 and a line passing through the remaining one. So, a particular case of the configuration considered in Theorem 5 arises when C_1, \dots, C_5 are obtained from conics in \mathbb{P}_2 , with C_i passing through all the P_j but P_i . Then let, for $i = 1, \dots, 5$, $F_i = 0$ be an equation for the conics corresponding to C_i on the plane \mathbb{P}_2 , so F_i is a quadratic form in three variables. Let $G = 0$ be an equation for the unique conic passing through P_1, \dots, P_5 , so G too is a quadratic form. Again by Lemma 1, the rational points on \tilde{X} , integral with respect to the strict transforms of C_1, \dots, C_5 correspond to solutions $(x, y, z) \in \mathcal{O}_S^3$ of the divisibility problem $F_i(x, y, z) \mid G(x, y, z)$. Then Theorem 4 applies and reproves Theorem 5 in this case.

Proof of Theorem 6. Let H_1, \dots, H_4 be the hyperplane sections appearing in the statement, and suppose $H_4 = C + C'$ is reducible. Note that $H_i^2 = H_i \cdot H_j$ ($= 4$) for $1 \leq i, j \leq 3$ and $H_i \cdot C \geq 1$ for all $i = 1, 2, 3$. Then we can apply Theorem 1 of [7], concluding the proof. \square

3. Geometry on certain open surfaces

The aim of this section is two-fold:

(1) We first want to calculate the generalised Albanese variety of the affine surfaces appearing in Theorem 1 and in the corollary to Theorem 2 and prove the claim we made in the introduction: namely that such Albanese varieties are one-dimensional for the surface appearing in Theorem 1 and trivial for the one appearing in the corollary; the latter is even simply connected (Theorem 3 below).

(2) Second, we want to classify the possible infinite families of integral points on some of the open surfaces considered so far. In this respect, note that our results from Section 2 provide the degeneracy of the set of integral points on such surfaces X ; this means that all but finitely many integral points lie on the union of finitely many curves on X . Now, Siegel’s theorem on integral points on curves states that any such curve is rational, and moreover can be parametrised either by \mathbb{A}^1 or by \mathbb{G}_m .

We intend in this section to classify completely the non-constant morphisms $\mathbb{A}^1 \rightarrow X$ and $\mathbb{G}_m \rightarrow X$, for some of the surfaces X considered in the preceding section. Clearly, the most difficult case will concern the multiplicative group \mathbb{G}_m , i.e. the curves of vanishing Euler characteristic on X .

All the statements in this section are of geometric nature; in particular they hold over any field k of characteristic zero, which one can suppose algebraically closed.

We begin by studying generalised Albanese varieties. Recall that given a smooth complete variety \tilde{X} and a hypersurface $D \subset \tilde{X}$ with normal-crossing singularities (if any), putting $X = \tilde{X} \setminus D$, one can define the generalised Albanese variety of X (or quasi-Albanese, for some authors) in the following way: it is a semi-abelian variety G endowed with a morphism $\pi : X \rightarrow G$ with the following universal property: for every semi-abelian variety G' and every morphism $f : X \rightarrow G'$, there exists a morphism $g : G \rightarrow G'$ with $f = g \circ \pi$. The pair (G, π) can also be defined analytically, by integration of the 1-forms on \tilde{X} with at most logarithmic poles along D .

In this paper, all the varieties we consider are rational, so the Albanese varieties we are interested in will be linear tori.

We start with the analysis of cubic surfaces; our reference is Chapter IV from [1]. A smooth cubic surface \tilde{X} contains twenty-seven lines, each having self-intersection -1 ; the maximal number of pairwise disjoint lines in \tilde{X} is six. By blowing them down, one obtains the projective

plane. Vice versa, \tilde{X} can be defined also as the blow-up of the projective plane over six points $P_1, \dots, P_6 \in \mathbb{P}_2$ in general position (i.e. no three of them on a line and not all of them on a conic). Let us denote by $\pi : \tilde{X} \rightarrow \mathbb{P}_2$ the corresponding blowing up map. Then \tilde{X} is embedded in \mathbb{P}_3 via the linear system of cubics passing through P_1, \dots, P_6 ; the twenty-seven lines on \tilde{X} correspond to the six exceptional divisors, the six conics containing five points among P_1, \dots, P_6 and the fifteen lines joining two points P_i, P_j , for $1 \leq i < j \leq 6$. The six lines on the two hyperplane sections appearing in the statement can be obtained as follows: for each $i \in \mathbb{Z}/6\mathbb{Z}$, consider the line L_i joining P_i and P_{i+1} ; denoting by \hat{L}_i its strict transform, the sum $\hat{L}_1 + \hat{L}_3 + \hat{L}_5$ will form a hyperplane section H_1 on \tilde{X} , while the other will be $H_2 = \hat{L}_2 + \hat{L}_4 + \hat{L}_6$. Let us denote by E_i the exceptional divisor above P_i ; it is a line on \tilde{X} intersecting both L_i and L_{i+1} , but no other lines of the forms L_j . From this picture it immediately follows that $L_i^2 = -1$ for each $i \in \mathbb{Z}/6\mathbb{Z}$. Actually, each line on the cubic \tilde{X} has self-intersection -1 .

The above considerations will be crucial in the sequel; in particular in the proof of the following:

Theorem 11. *Let $\tilde{X} \subset \mathbb{P}_3$ be a smooth cubic hypersurface, $X \subset \tilde{X}$ be the open subset obtained by removing six lines on \tilde{X} lying on two planes. The generalised Albanese variety of X is a one-dimensional torus.*

Proof. Let H_1, H_2 be the hyperplane sections mentioned in the statement; each is defined by a linear equation $F_i(x_0, \dots, x_3) = 0$ in \mathbb{P}_3 ; the rational function $\pi = F_1/F_2$ induces a never vanishing regular function on X , still denoted by π , so a morphism $\pi : X \rightarrow \mathbb{G}_m$. We should prove that every other morphism $X \rightarrow \mathbb{G}_m$ factors through π . Letting L_1, L_3, L_5 be the three components of H_1 and L_2, L_4, L_6 the components of H_2 , the divisor of π (viewed as a rational function on \tilde{X}) is

$$(\pi) = L_1 + L_3 + L_5 - L_2 - L_4 - L_6.$$

In particular, on the Picard group $\text{Pic}(\tilde{X})$ we have the relation $L_1 + L_3 + L_5 = L_2 + L_4 + L_6$. We shall prove that all the linear relations in the group generated by L_1, \dots, L_6 derive from the above one, so such group is free-abelian of rank 5. For this task, consider any non-trivial linear combination of L_1, \dots, L_6 omitting one term, for instance L_6 . So take an index $1 \leq i \leq 5$, integers a_1, \dots, a_i , with $a_i \neq 0$, such that $a_1 L_1 + \dots + a_i L_i = 0$ in $\text{Pic}(\tilde{X})$. Taking the intersection product with E_i , we have $(a_1 L_1 + \dots + a_i L_i) \cdot E_i = a_i \neq 0$, which shows that the linear combination under consideration does not vanish in the Picard group.

Let now $f : X \rightarrow \mathbb{G}_m$ be another morphism. The divisor of f must also have its support in $L_1 \cup \dots \cup L_6$. Let us write

$$(f) = \sum_{i \in \mathbb{Z}/6\mathbb{Z}} m_i L_i - \sum_{j \in \mathbb{Z}/6\mathbb{Z}} n_j L_j$$

where the two functions $\mathbb{Z}/6\mathbb{Z} \ni i \mapsto m_i \in \{0, 1, \dots\}$ and $\mathbb{Z}/6\mathbb{Z} \ni i \mapsto n_i \in \{0, 1, \dots\}$ have disjoint supports. Since the divisor class of $\sum_{i \in \mathbb{Z}/6\mathbb{Z}} m_i L_i - \sum_{j \in \mathbb{Z}/6\mathbb{Z}} n_j L_j$ in the Picard group vanishes, by the above consideration we must have $\sum_{i \in \mathbb{Z}/6\mathbb{Z}} m_i L_i - \sum_{j \in \mathbb{Z}/6\mathbb{Z}} n_j L_j = N(L_1 + L_3 + L_5 - L_2 - L_4 - L_6)$ for some integer $N \in \mathbb{Z}$. Then the function f/π^N has a trivial divisor, so is a constant $\lambda \neq 0$. In other words, $f = \lambda \cdot \pi^N$ is obtained from π by composition with an endomorphism of \mathbb{G}_m , which proves that the Albanese variety of X is isomorphic to \mathbb{G}_m . \square

Let us now consider the surface X appearing in Corollary 2; our goal is to prove Theorem 3, stating that X is simply connected. Recall that $X = \tilde{X} \setminus D$, where \tilde{X} is the plane blown up on three points P_1, P_2, P_3 , and D is the strict transform of four lines L_1, \dots, L_4 , with $P_i \in L_i$ for $i = 1, 2, 3$. Let D_i be the strict transform of L_i (for $i = 1, 2, 3$), and let E_i (for $i = 1, 2, 3$) be the exceptional divisor crossing D_i . Finally, let $Y \subset X$ be the surface

$$Y := \tilde{X} \setminus (D_1 \cup \dots \cup D_4 \cup E_1 \cup \dots \cup E_3) = X \setminus (E_1 \cup E_2 \cup E_3).$$

It is clear that $Y \simeq \mathbb{P}_2 \setminus (L_1 \cup \dots \cup L_4)$. Now, to prove Theorem 3 we shall use the following facts, stated as Lemmas 2, 3, 4:

Lemma 2. *Let X be a (connected) complex manifold, $Z \subset X$ a proper closed complex submanifold, $Y = X \setminus Z$ its complement, $p \in Y$ a point. Then the inclusion $i : Y \hookrightarrow X$ induces a surjective homomorphism $i_* : \pi_1(Y, p) \rightarrow \pi_1(X, p)$ between the corresponding fundamental groups.*

Proof. We shall use the functorial property of the fundamental group. Let $\pi : X' \rightarrow X$ be the universal cover of X , $\Gamma \simeq \pi_1(X, p)$ be the deck-transformation group, and let $Y' := \pi^{-1}(Y)$ be the pre-image of Y in X' . Then $Y' = X' \setminus Z'$, where $Z' := \pi^{-1}(Z)$ is a complex analytic sub-variety. In particular Z' has real codimension at least two, so Y' is connected and Γ is the automorphism group of the cover $Y' \rightarrow Y$. Letting $\pi' : Y'' \rightarrow Y'$ be the universal cover of Y' , we obtain by composition the universal cover $\pi \circ \pi' : Y'' \rightarrow Y$ of Y . Then the group Γ is realised as a quotient of the deck-transformations group of $Y'' \rightarrow Y$, i.e. of $\pi_1(Y, p)$, as wanted. It is easy to see that the surjective homomorphism $\pi_1(Y, p) \rightarrow \Gamma = \pi_1(X, p)$ so constructed coincides with the homomorphism i_* (but actually we do not even need this fact in the sequel). \square

An alternative, more concrete proof, is based on the following fact: *Let X be a real manifold of dimension ≥ 2 , $Z \subset X$ a codimension two sub-variety. Let $p \in X \setminus Z$ be a point, $\gamma : [0, 1] \rightarrow X$ a loop with base point p . Then γ is homotopic to a loop in $X \setminus Z$ (with the same base point).* This fact is well known, and appears for instance as Lemma 3.12 in [4].

Lemma 3. *The fundamental group of the complement of four lines in general position in \mathbb{P}_2 is isomorphic to \mathbb{Z}^3 .*

Proof. This is a well-known theorem of Zariski, proved² in [18, Chap. VIII]. \square

Lemma 4. *The surface X admits no finite cyclic unramified connected cover of degree > 1 .*

Proof. Let $X' \rightarrow X$ be a cyclic unramified cover of degree $d > 1$. By Riemann’s Existence Theorem,³ see Théorème 5.1 in [9], X' has a structure of algebraic variety such that the covering map is algebraic. Then the function field of X' is obtained by taking the d -th root of a rational function $f \in \mathbb{C}(\tilde{X})^*$ on \tilde{X} . The property that $X' \rightarrow X$ is unramified means that all zeros and

² Zariski stated a much more general result in [18], on the fundamental group of the complement of any (reducible) curve with normal crossing singularities; as remarked for instance by Mumford in the notes to the latest edition of Zariski’s book [18], the proof of this general statement is incorrect, and was settled only later by Abhyankar. Nevertheless, in the case of an arrangement of lines, considered in Corollary 3, Zariski’s proof is sound.

³ Proved in its general form by Grauert and Remmert.

poles of f in X have multiplicity divisible by d . This means precisely that: for each irreducible curve $C \subset \tilde{X}$ outside $D_1 \cup \dots \cup D_4$, the multiplicity of C in the divisor (f) is divisible by d . Considering the blow-up map $\pi : \tilde{X} \rightarrow \mathbb{P}_2$, which is birational, the function f defines a rational function $(\pi^{-1})^*(f) =: g$ on \mathbb{P}_2 . Hence the function field of X' is obtained from the function field $\mathbb{C}(\mathbb{P}_2) = \mathbb{C}(x, y)$ by adding the d -th root of g . Using the fact that the Picard group of \mathbb{P}_2 is cyclic we can always multiply g by a suitable d -th power having all its poles in L_4 obtaining a new function, still denoted by g , which is regular outside L_4 . Hence g can be viewed as a polynomial in x, y , after identifying $\mathbb{P}_2 \setminus L_4 \simeq \mathbb{A}^2$. Let $g_i = 0$ be a (linear) equation for $L_i \cap \mathbb{A}^2$ ($i = 1, 2, 3$). The condition on the divisor of f (in \tilde{X}) can be stated in terms of g by saying that each irreducible factor of g distinct from g_i occurs with multiplicity divisible by d . This will not be enough to conclude, but now observe that if some g_i occurred in the factorisation of g with multiplicity not divisible by d , then the entire pull-back of L_i would lie in the ramification locus of the cover $\tilde{X}' \rightarrow \tilde{X}$, contrary to our hypothesis (recall that we are assuming that E_i does not lie on the ramification locus). Then we have that each irreducible factor of g occurs with multiplicity divisible by d , so g is a perfect d -th power, and the same is true of f . But this means that the cover $X' \rightarrow X$ is disconnected, concluding the proof. \square

Proof of Theorem 3. We shall use the notation of Corollary 2, already used in the proofs above. Let $Y \subset X$ be the open set $X \setminus (E_1 \cup E_2 \cup E_3) \simeq \mathbb{P}_2 \setminus (L_1 \cup \dots \cup L_4)$. Then by Lemma 2, the fundamental group of X is a quotient of the fundamental group of Y . Now, Lemma 3 assures it is a quotient of the group \mathbb{Z}^3 ; to prove that such a quotient is trivial, it is sufficient to prove that it admits no non-trivial finite cyclic quotients. Geometrically, this means that X admits no non-trivial (finite) cyclic cover, which is the content of Lemma 4, finishing the proof. \square

An alternative proof of Theorem 3, avoiding the use of Lemmas 3 and 4, consists in noticing that the topology of such surfaces is independent of the particular choice of the lines L_1, \dots, L_4 and the points P_1, \dots, P_3 . Then, it suffices to prove the theorem for one single such surface; this can be done for instance via Lemma 2, by proving that each generator of the fundamental group of $Y = X \setminus (E_1 \cup E_2 \cup E_3)$ becomes homotopically trivial in X .

Our last goal in this section is to conclude the proof of Theorem 8, classifying the possible infinite families of solutions to Eq. (3). This amounts to finding the non-constant morphisms from \mathbb{G}_m to open sets in the Hirzebruch surface considered in the context of parametric S -unit equations:

Theorem 12. *Let $f(T), g(T), h(T) \in k[T]$ be three polynomials of the same degree, pairwise coprime. Let $X \subset \mathbb{G}_m^2 \times \mathbb{A}^1$ be the surface defined by Eq. (3), where (u, v) are coordinates in \mathbb{G}_m^2 , and t is the coordinate in \mathbb{A}^1 . Then every morphism $\mathbb{A}^1 \rightarrow X$ is constant. Let $\mathbb{G}_m \ni x \mapsto (u(x), v(x), t(x)) \in X$ be a morphism; then at least one among the three functions $u, v, u/v$ is constant. For every zero t_0 of $f(t)g(t)h(t)$, the fiber of t_0 in X is a curve isomorphic to \mathbb{G}_m , so in particular there exists a non-constant morphism $(u, v, t) : \mathbb{G}_m \rightarrow X$ with $t(x) \equiv t_0$.*

Proof. Let $\mathbb{G}_m \ni x \mapsto (u(x), v(x), t(x)) \in X$ be a non-constant morphism. Suppose first that the regular function $t : \mathbb{G}_m \rightarrow \mathbb{A}^1$ is constant, equal to t_0 . Then the curve in \mathbb{G}_m^2 defined by the equation $f(t_0)u + g(t_0)v = h(t_0)$ has Euler characteristic 1 (it is isomorphic to $\mathbb{P}_1 \setminus \{0, 1, \infty\}$) unless $f(t_0)g(t_0)h(t_0) = 0$, in which case it is isomorphic to \mathbb{G}_m (so its Euler characteristic vanishes). In the first case, the morphism would be constant; in the second case, if $f(t_0) = 0$, then by hypothesis $g(t_0) \neq 0 \neq h(t_0)$, so necessarily $v(x) \equiv -h(t_0)/g(t_0)$ would be constant; if

$g(t_0) = 0$, then u would be constant, for the same reason; finally, if $h(t_0) = 0$, then $u(x)/v(x)$ would be constant, equal to $-g(t_0)/f(t_0)$. Consider now the case that the function t is non-constant. It might have one or two poles. Let $p \in \{0, \infty\}$ be a pole of t ; if p is a pole of u or v , after dividing both sides in (3) by u (resp. v), we obtain another equation of the same type, namely $f(t) + g(t)(v/u) = h(t)(1/u)$ (or $f(t)(u/v) + g(t) = h(t)(1/v)$), where the new functions (v/u) , $(1/u)$ (or (u/v) , $(1/v)$) would have no pole in p . Hence we are reduced to the case where one pole of t , say ∞ , is not a pole of u, v ; if we prove that in this particular case u or v is constant, we would have proved in general that one among $u, v, u/v$ is so. Then, suppose that $u(x) = \lambda x^{-a}, v(x) = \mu x^{-b}$, for $\lambda, \mu \in k^*, a \geq 0, b \geq 0$. Then the equation becomes

$$\frac{\lambda f(t(x))}{x^a} + \frac{\mu g(t(x))}{x^b} = h(t(x)).$$

Comparing the order of pole at infinity, we deduce that $\min\{a, b\} = 0$, as wanted. \square

We note that, for $d = \deg(f) = \deg(g) = \deg(h) = 1$, there always exists a non-constant morphism $\mathbb{G}_m \rightarrow X$, with a non-constant t , i.e. which remains non-constant under composition with the canonical projection $X \rightarrow \mathbb{A}^1$. In fact, after change of variable in t (i.e. composition with an automorphism of \mathbb{A}^1) Eq. (3) with $d = 1$ takes the form

$$tu + (1 - t)v = t + a$$

for a suitable $a \in k^*$. Then, the map $\mathbb{G}_m \ni x \mapsto (u, v, t) = (x, x, x - a)$ is a non-constant morphism. For $d \geq 2$ and generic choice of the polynomials $f(t), g(t), h(t)$, the image of every morphism $\mathbb{G}_m \rightarrow X$ will be contained in a fibre $t = t_0$ for the projection $X \rightarrow \mathbb{A}^1$; for special choices of $f(t), g(t), h(t)$ there will be exactly one exception.

4. Density of integral points on certain surfaces

In this paragraph, k will again denote a number field. Our aim is to prove that some of the main results are in a sense best-possible, by proving that certain surfaces possess Zariski-dense sets of integral points.

The main ideas and tools in this section originated in the paper [2], and were vastly generalised by Hassett and Tschinkel in [10]. In particular, we shall repeatedly use the following lemma, which is a particular case of Theorem 2.3 of [2]:

Lemma 5. *Let k be a number field, $\mathcal{O}_S \subset k$ be a ring of S -integers such that the group \mathcal{O}_S^* is infinite. Let $C \subset \mathbb{P}_2$ be a smooth conic defined over k , $A, B \in C(\bar{k})$ be rational points on C (possibly $A = B$), such that the divisor $A + B$ on C is defined over k . Then the set of S -integral points on the open curve $C \setminus \{A, B\}$ is either empty or infinite.*

The idea of the proofs of all results in this section is to produce a conic fibration on the relevant surfaces, and apply Lemma 5 to prove that infinitely many conics in such fibrations possess infinitely many integral points. This is the same idea of Hassett and Tschinkel.

We begin by proving that the condition $d \geq 6$ in Theorem 7 is really necessary. In fact, the next result proves that the conclusion of Theorem 7 fails if $d = 5$; the case $d \leq 4$ is easier and its proof will be left to the reader.

Theorem 13. *Let L_1, \dots, L_5 be lines on \mathbb{P}_2 , defined over k , in general position. Let, for $i \in \mathbb{Z}/5\mathbb{Z}$, P_i be the point of intersection $L_i \cap L_{i+1}$. Let $\tilde{X} \rightarrow \mathbb{P}_2$ be the blow-up of the plane over P_1, \dots, P_5 and denote by \hat{L}_i the strict transforms of the lines L_i . Then for a suitable finite set S of places of k , the integral points on the surface $\tilde{X} \setminus (\hat{L}_1 \cup \dots \cup \hat{L}_5)$ are Zariski-dense.*

Proof. We shall actually prove a little more, namely the density of integral points on \tilde{X} with respect to the divisor $\hat{L}_1 + \dots + \hat{L}_5 + E_5$, where E_5 is the exceptional divisor over P_5 (in other words: we do not need to blow up the points P_5). First of all, let us enlarge the set S (if necessary) so that the group of units \mathcal{O}_S^* is infinite and the configuration of lines $L_1 + \dots + L_5$ has good reduction outside S . By this, we mean that their reductions modulo every place outside S are still in general position. Denote by M the line joining P_1 to P_4 .

Consider the pencil \mathbb{P} of conics passing through P_1, \dots, P_4 . It is parametrised by the projective line \mathbb{P}_1 ; those conics which do not reduce (modulo any place outside S) to the reducible conic $\mathcal{C}_1 := L_2 + L_4$ nor to the conic $\mathcal{C}_2 := L_3 + M$ are parametrised by $\mathbb{P} \setminus \{\mathcal{C}_1, \mathcal{C}_2\} \simeq \mathbb{G}_m$; in particular, there exist infinitely many such conics. Denote by $\mathbb{P}' \subset \mathbb{P}(k)$ this set of conics. For every such conic $\mathcal{C} \in \mathbb{P}'$, let $A_{\mathcal{C}}, B_{\mathcal{C}}$ be the second point of intersection with L_5 and L_1 respectively (they might coincide, in which case $A_{\mathcal{C}} = B_{\mathcal{C}} = P_5$; also, we might have $A_{\mathcal{C}} = P_4$ or $B_{\mathcal{C}} = P_1$, in case \mathcal{C} is tangent to L_5 at P_4 or to L_1 at P_1). We pose to prove that all the integral points on \mathcal{C} with respect to $A_{\mathcal{C}}, B_{\mathcal{C}}$ give rise to integral points on \tilde{X} with respect to $\hat{L}_1 + \dots + \hat{L}_5 + E_5$. In fact, let Q be a point on \mathcal{C} which is integral with respect to $A_{\mathcal{C}}, B_{\mathcal{C}}$, and let \hat{Q} be its corresponding point on \tilde{X} . We have to prove that \hat{Q} does not reduce to $\hat{L}_1 + \dots + \hat{L}_5 + E_5$. For this purpose, we note that no element $\mathcal{C} \in \mathbb{P}'$ is tangent to L_2, L_3 or L_4 , nor becomes tangent after reducing modulo any place outside S . Then \hat{Q} cannot reduce to $\hat{L}_2 + \dots + \hat{L}_4$; the integrality of the point Q with respect to $A_{\mathcal{C}}, B_{\mathcal{C}}$ just means that its \hat{Q} does not reduce to $\hat{L}_1 + \hat{L}_5 + E_5$. Hence, to prove Theorem 13 it is sufficient to prove that such conics have infinitely many integral points. By Lemma 5, it is sufficient to find one integral point. Now, by our assumptions on S , the point P_2 gives rise in the strict transform $\hat{\mathcal{C}}$ of \mathcal{C} in \tilde{X} to a point $\hat{P}_2 := E_2 \cap \hat{\mathcal{C}}$ which is integral with respect to $\hat{L}_1 + \dots + \hat{L}_5 + E_5$, so in particular with respect to $A_{\mathcal{C}} + B_{\mathcal{C}}$, concluding the proof. \square

Corollary 3. *Let F_1, \dots, F_5 be linear forms in three variables, in general position, defined over k . There exists a non-degenerate quadratic form G , defined over k , and a finite set S of places of k such that the points $(x, y, z) \in \mathcal{O}_S^3$ with $F_i(x, y, z) | G(x, y, z)$ are Zariski-dense in \mathbb{P}_2 .*

Proof. Putting, as above $P_i = F_i^{-1}(0) \cap F_{i+1}^{-1}(0)$, for $i \in \mathbb{Z}/5\mathbb{Z}$, let \mathcal{C} be the (unique smooth) conic passing through P_1, \dots, P_5 . Let $G = 0$ be a homogeneous equation for \mathcal{C} . Define again X to be the complement of the strict transform of the lines $F_i = 0$ in the blow-up of the plane over P_1, \dots, P_5 . The integral points on X correspond to the solution of the above divisibility problem. Now, by Theorem 13, such integral points are Zariski-dense, under a suitable ring extension of \mathcal{O}_S . \square

From the general position assumption, the points $(x, y, z) \in \mathcal{O}_S^3$ with coprime coordinates and satisfying $F_i(x, y, z) | G(x, y, z)$, will also satisfy

$$F_1(x, y, z) \cdots F_5(x, y, z) | G^2(x, y, z).$$

So, after the above corollary, we can obtain a dense set of integral points where a degree-five homogeneous form divides a form of degree four.

We now turn our attention to cubic surfaces again; we want to prove that Theorem 1 is best-possible; more precisely, we prove Theorem 9, asserting that: (1) if we remove just five lines on two hyperplane sections from a cubic surface, the integral points are Zariski-dense (in a suitable ring of S -integers); (2) one can always remove nine lines in such a way that the integral points on the complement are still dense.

Proof of Theorem 9. Let us begin by the first assertion. We recall from the preceding section that a cubic surface can always be realised as the blow-up of six points P_1, \dots, P_6 in general position on the plane. Up to enlarging k and S we can suppose they are all defined over k and they remain in general position when reduced modulo every place outside S . The twenty-seven lines correspond to the six blown up points, to the fifteen lines joining two of them, to the six conics passing through five of them. Suppose we are given five lines on two hyperplane sections: then either the two hyperplane sections share a common line, or the union of the two hyperplane sections consists of six lines (and we are considering just five of them).

Let us consider the first case, where a line is common to the two hyperplane sections. We can realise the first hyperplane section as the image of the lines $L(P_1, P_2)$, $L(P_3, P_4)$, $L(P_5, P_6)$; the second one as the image of the lines $L(P_3, P_4)$ (in common with the preceding hyperplane section), $L(P_2, P_6)$, $L(P_1, P_5)$. Then consider the pencil Λ of conics through P_1, P_2, P_5, P_6 ; those which do not reduce, modulo any place outside S , to the singular conic $C_1 := L(P_1, P_2) + L(P_5, P_6)$ nor to $C_2 := L(P_2, P_6) + L(P_1, P_5)$ correspond to the integral points on $\Lambda \setminus \{C_1, C_2\} \simeq \mathbb{G}_m$, so there are infinitely many of them. Fix one such conic C ; its image on the cubic surface intersects the union of the two hyperplane sections on the (image of the) line $L(P_3, P_4)$, so on a divisor of the form $A + B$, consisting on one or two points (A might coincide with B). Each such conic possesses at least one integral point with respect to this divisor, namely P_1 .⁴ Then, by Lemma 5, C contains infinitely many points which are integral with respect to $A + B$. We then obtain infinitely many curves on the surface, each admitting infinitely many integral points, thus providing a Zariski-dense infinite set as wanted.

In the second case, we argue in a very similar way. We can suppose, as before, that the first hyperplane section is given by the lines $L(P_1, P_2)$, $L(P_3, P_4)$, $L(P_5, P_6)$; let the second one be defined by $L(P_2, P_3)$, $L(P_4, P_5)$, $L(P_6, P_7)$. Consider the complement of the first five lines, i.e. all the above but $L(P_1, P_6)$; we claim that, even after removing the two lines corresponding to the blown up points P_1, P_6 , the integral points are Zariski-dense. To prove this, consider the pencil Λ of conics passing through P_2, \dots, P_5 ; for every such conic $C \in \Lambda$, let $A + P_2$ be its intersection with $L(P_1, P_2)$, $B + P_5$ its intersection with $L(P_5, P_6)$; we do not exclude that $A = B$, nor $A = P_2$, nor $B = P_5$. Integral points on $C \setminus \{A, B\}$ give rise to integral points on the surface we are interested in. Note that $\Lambda \simeq \mathbb{P}_1$, and the conics in $\Lambda(k)$ which do not reduce to $L(P_2, P_3) + L(P_4, P_5)$ nor to $L(P_3, P_4) + L(P_2, P_5)$ modulo any place outside S correspond to rational points on the line Λ which are integral with respect to two points, i.e. to integral points on \mathbb{G}_m , hence they are Zariski-dense, over a suitable ring of S -integers. By Lemma 5, each such conic contains infinitely many integral points, with respect to the divisor $A + B$, since it contains

⁴ Note, however, that the point P_1 on \mathbb{P}_2 corresponds to a line on the cubic surface; this line intersects the image of C on one point, corresponding on the tangent direction of C at P_1 ; this direction cannot be the direction of the line $L(P_1, P_2)$, nor can reduce to this modulo any place outside S .

the integral point P_3 (or P_4). Then the integral points on the surface are dense, so this case is settled too.

Let us now prove the second sentence, namely that one can sometimes remove up to nine lines, and still obtain a Zariski-dense set of integral points. Again, consider the plane blown up at six points P_1, \dots, P_6 in general position. The three lines $L(P_1, P_2)$, $L(P_3, P_4)$, $L(P_5, P_6)$, as mentioned, give rise to corresponding lines on the cubic surface; adding the six lines which correspond to the blown-up points, we obtain a configuration of nine lines whose complement is isomorphic to the complement of the mentioned three lines on the plane. This complement is then isomorphic to \mathbb{G}_m^2 , whose integral points are indeed Zariski-dense, over a suitable ring of S -integers, concluding the proof. \square

Concluding remarks. As mentioned, in Section 2 we omitted the proofs in the Nevanlinna theoretic setting, since they are essentially the same as in the arithmetic case, up to the replacement of Schmidt’s Subspace Theorem by Cartan’s Second Main Theorem. In the present section, however, as far as density is concerned, we can go further than the exact analogue of the arithmetic case: one can prove that the open surfaces considered in Theorems 9, 13 and Corollary 3, not only contain Zariski-dense entire curves, but are even holomorphically dominated by \mathbb{C}^2 . This last fact could be proved by combining the above technique of producing conic fibrations with recent work of Buzzard and Lu [4].

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