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Maximal Sets of Points in Finite Projective Space, No *t*-Linearly Dependent

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Consider a finite $(t + r - 1)$-dimensional projective space $\mathrm{PG}(t + r - 1, s)$ based on the Galois field $\mathrm{GF}(s)$, where s is prime or power of a prime. A set of k distinct points in $PG(t + r - 1, s)$, no t-linearly dependent, is called a (k, t)-set and such a set is said to be maximal if it is not contained in any other (k^*, t)-set with $k^* > k$. The number of points in a maximal (k, t)-set with the largest k is denoted by $m_t(t + r, s)$. Our purpose in the paper is to investigate the conditions under which two or more points can be adjoined to the basic set of E_i, $i = 1, 2, \ldots, t + r$, where E_i is a point with one in i-th position and zeros elsewhere. The problem has several applications in the theory of fractionally replicated designs and information theory.

1. INTRODUCTION

Several interesting problems of a combinatorial nature have recently arisen in the study of finite projective spaces. One of these, due to Bose, concerns the maximum number of distinct points in finite projective geometry $\mathrm{PG}(t + r - 1, s)$ based on $\mathrm{GF}(s)$ so that no t among them lie on a $(t - 2)$-flat. Attempts made by several research workers including Bose, Segre, Tallini, and many others have been successful only in special cases. Very few general results or methods seem to be in the literature.

R. C. Bose [1] showed that the maximum number of factors in a symmetrical factorial design in which each factor operates at s levels, blocks are of size s^{t+r}, and no main effect or t-factor $(t > 1)$ or lower order interaction is confounded with block effects is given by the maximum number of distinct points in $PG(t + r - 1, s)$ no t-linearly dependent. These considerations have later led to an extensive use of fractional factorial designs and the study of their confounding properties has been approached from several closely related points of view. In a separate paper, Bose [2] proved that, for a fractionally replicated design $(1/s^d) \times s^k$, consisting of a single block with s^{t+r} plots or experimental units, $t + r = k - d$, the maximum possible value of k is $m_{2v}(2v + r, s)$ if no v-factor or lower order interaction is to be aliased with another v-factor or lower order interaction. In case it is required that no v-factor is to be aliased with a $(v + 1)$-factor or lower order interaction, then the maximum value of k is given by $m_{2v+1}(2v + r + 1, s)$. For given k and v, one needs to maximize d, that is, to take as high a fraction of the full factorial design as possible. The number $m_t(t + r, s)$ also plays a significant role in the information theory. If there is an s-ary channel capable of transmitting s distinct symbols, then, for a group code (k, d) with d information symbols and fixed redundancy k-d, the maximum value of k for which v-errors can be corrected with certainty is $m_{2v}(2v + r, s)$. Similarly, the maximum value of k for which v errors can be corrected and $v + 1$ errors detected is established by $m_{2v+1}(2v + r + 1, s)$. This interconnection between the theory of confounding and fractional replication developed by Fisher, Finney, Bose, and Kishan and theory of error correcting codes due to Hamming and Slepian has been elegantly brought out by Bose [2].

It was recently established [5] that

(1.1) $$m_t(t + r, s) = t + r + 1, \qquad \text{for} \quad t \geqslant s(r + 1),$$

a particular case for $r = 0$ having been established earlier by Bush [3], Maneri and Silverman [6], and Gulati [4]. It was also shown in [5] that

(1.2) $$m_t(t + r, s) \geqslant t + r + 2, \qquad \text{for} \quad t < s(r + 1),$$

(1.3) $$m_t(t + r, s) \leqslant t + r + s, \qquad \text{for} \quad sr \leqslant t < s(r + 1),\ r > 1.$$

We will establish that the upper bound (1.3) can be improved for $r \geqslant s$ and that only two points can be adjoined to the basic set of $(t + r)$-linearly independent points E_i, $i = 1, 2, \ldots, t + r$, where E_i is a point in $PG(t + r - 1, s)$ with a one in i-th position and zeros elsewhere.

2. Maximal (k, t)-Sets

Definition 2.1. A set of k distinct points in $PG(t + r - 1, s)$ no t linearly dependent, is called a (k, t)-set.

DEFINITION 2.2. A (k, t)-set is said to be maximal if it is not contained in any (k^*, t)-set with $k^* > k$.

DEFINITION 2.3. The number of points in a maximal (k, t)-set with the largest k is denoted by $m_t(t + r, s)$.

THEOREM 2.1.

$$\text{For } r \geqslant s > 2,\ m_t(t + r, s) = t + r + 2, \text{ for } f(r) \leqslant t < s(r + 1),$$
$$m_t(t + r, s) \geqslant t + r + 3, \text{ for } t < f(r),$$

where

$$f(r) = \begin{cases} (s-1)(r+2) + \left[\dfrac{r+2}{s+1}\right], & \text{for } r = (s-2), (s-1) \bmod (s+1), \\ (s-1)(r+1) + (q+1) + \left[\dfrac{r+2}{s+1}\right], & \text{for } r = q \bmod (s+1) \text{ and} \\ & q = 0, 1, 2, \ldots, (s-3), \\ (s-1)(r+1) + \left[\dfrac{r+2}{s+1}\right], & \text{for } r = s \bmod (s+1), \end{cases} \tag{2.1}$$

where $[x]$ means the largest integer not exceeding x.

Proof. That the maximal set contains at least $t + r + 2$ points for $t < s(r + 1)$ is a consequence of (1.2). Consider an addition of three points Q_i, $i = 1,2,3$, to the basic set of $t + r$ linearly independent points, where

$$\begin{aligned} Q_1 &: (a_1, a_2, a_3, \ldots, a_{t+r}), \\ Q_2 &: (b_1, b_2, b_3, \ldots, b_{t+r}), \\ Q_3 &: (c_1, c_2, c_3, \ldots, c_{t+r}). \end{aligned} \tag{2.2}$$

The possibilities for the corresponding coordinates, up to multiples, are

x_1	x_2	x_3	x_4	x_5	x_6	$\cdots$	x_{s+2}	$\cdots\ \cdots$	x_{s^2}	x_{s^2+1}	x_{s^2+2}	x_{s^2+3}	$\cdots$	x_{s^2+s+1}
0	0	1	1	1	1	$\cdots$	1	$\cdots\ \cdots$	0	1	1	1	$\cdots$	1
0	1	0	1	w	w^2	$\cdots$	w^{s-2}	$\cdots\ \cdots$	1	0	1	w	$\cdots$	w^{s-2}
0	0	0	0	0	0	$\cdots$	0	$\cdots\ \cdots$	w^{s-2}	w^{s-2}	w^{s-2}	w^{s-2}	$\cdots$	w^{s-2}

where w is the primitive element of the field.

The above table, which exhausts all possibilities, may be represented in the following form:

$$\mathbf{V}_3 = \begin{bmatrix} \mathbf{0}^* & \mathbf{V}_2 & \mathbf{V}_2 & \mathbf{V}_2 & \cdots & \cdots & \mathbf{V}_2 \\ 1 & \mathbf{0} & \mathbf{1} & \mathbf{w} & \cdots & \cdots & \mathbf{w}^{s-2} \end{bmatrix}, \tag{2.3}$$

where $\mathbf{0}$, $\mathbf{1}$, $\mathbf{w}^i$ ($i = 1, 2, \ldots, s-2$) are each $1 \times (s+1)$ row vectors of the elements $0, 1, w, w^2, \ldots, w^{s-2}$, respectively, and $\mathbf{0}^*$ is a (2×1) column vector of 0's and

$$\mathbf{V}_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & \cdots & 1 \\ 1 & 0 & 1 & w & \cdots & \cdots & w^{s-2} \end{bmatrix}.$$

Considering all possible linear combinations of three points, we observe that (i) there exists $s^2 + s + 1$ linear inequalities involving x_i's, (ii) each inequality contains $s + 1$ variables x_i's and each x_i appears in $s + 1$ linear inequalities, and (iii) $\binom{3}{u}(s-1)^{u-1}$, $u = 1, 2, 3$, possible linear combinations of Q_i, $i = 1, 2, 3$, exists each having at most $r + u - 1$ zero coordinates. This corresponds to the fact that three points determine a plane, that there are $s^2 + s + 1$ points in a plane, there exists $s + 1$ lines through a point, and a line contains exactly $s + 1$ points. Adding all the linear inequalities, one obtains

$$(s+1) \sum_{i=1}^{s^2+s+1} x_i \leqslant (s^2 + s + 1)r + (2s^2 - s - 1), \tag{2.4}$$

which reduces to

$$\sum_{i=1}^{s^2+s+1} x_i \leqslant s(r+2) - 3 + \left[\frac{r+2}{s+1}\right]. \tag{2.5}$$

Adding those inequalities in which x_j appears, we have

$$sx_j + \sum_{i=1}^{s^2+s+1} x_i \leqslant \begin{cases} (s+1)r + (s-1), & j = 1, 2, 3, \\ (s+1)r + (2s-1), & j > 3. \end{cases} \tag{2.6}$$

In general, the upper bound (2.5) is not sharp and one needs to consider several cases separately. However, if we consider the case for

$$r = (s+1)p + (s-1),$$

the bound (2.5) reduces to

$$\sum_{i=1}^{s^2+s+1} x_i \leqslant (s^2 + s + 1)p + (s^2 + s - 2), \tag{2.7}$$

an optimal solution for which is given by

$$x_j = \begin{cases} p, & j = 1, 2, 3, \\ p + 1, & j > 3. \end{cases}$$

If $r = (s + 1)p + s$, the upper bound (2.5) which reduces to

$$\sum_{i=1}^{s^2+s+1} x_i \leqslant (s^2 + s + 1)p + (s^2 + 2s - 2) \tag{2.9}$$

is not attained. However, if we consider improving (2.9) so as to read as

$$\sum_{i=1}^{s^2+s+1} x_i \leqslant (s^2 + s + 1)p + (s^2 + 2s - 2) - (s - 1),$$

that is,

$$\sum_{i=1}^{s^2+s+1} x_i \leqslant (s^2 + s + 1)p + (s^2 + s - 1), \tag{2.10}$$

then an optimal solution is given by

$$x_j = \begin{cases} p, & j = 1, 2, \\ p + 1, & j > 2. \end{cases} \tag{2.11}$$

Consider next the case for $r = (s + 1)p + q$, where $q = 0, 1, 2, \ldots, (s - 3)$. Then one obtains from (2.5) and (2.6)

$$\sum_{i=1}^{s^2+s+1} x_i \leqslant (s^2 + s + 1)p + s(q + 2) - 3 \tag{2.12}$$

and

$$sx_j + \sum_{i=1}^{x^2+s+1} x_i \leqslant \begin{cases} (s + 1)^2p + (s + 1)q + (s - 1), & j = 1, 2, 3, \\ (s + 1)^2p + (s + 1)q + (2s - 1), & j > 3. \end{cases} \tag{2.13}$$

If bound (2.12) is attained, then from (2.13) we have

$$sx_j \leqslant \begin{cases} s(p - 1) + (q + 2), & j = 1, 2, 3, \\ sp + (q + 2), & j > 3. \end{cases}$$

Since x_j are non-negative integers, it follows that

$$x_j \leqslant \begin{cases} p - 1, & j = 1, 2, 3, \\ p, & j > 3. \end{cases}$$

This shows that

$$\sum_{i=1}^{s^2+s+1} x_i = (s^2 + s + 1)p - 3,$$

a contradiction to (2.12). Thus the bound is not attained, but it considers refining (2.12) so as to read as

$$\sum_{i=1}^{s^2+s+1} x_i \leqslant (s^2+s+1)p + s(q+2) - 3 - (s-q-2), \quad (2.14)$$

it follows that

$$x_j \leqslant \begin{cases} p, & j = 1, 2, 3, \\ p+1, & j > 3. \end{cases} \quad (2.15)$$

The problem now reduces to solving the linear inequalities separately for each value of s in question subject to (2.15) above. We will demonstrate here the solutions for $s = 3$ and 5. The solutions for $s = 2$ and $s = 4$ are given in [5]. One can find solutions for $s > 5$ by adopting a similar procedure. In this case, (2.5) is improved to

$$\sum_{i=1}^{s^2+s+1} x_i \leqslant (s-1)(r+1) + (r+q) + \left[\frac{r+2}{s+1}\right]. \quad (2.16)$$

Since $\sum_i x_i = t + r$, it follows that, if

$$t \leqslant (s-1)(r+1) + q + \left[\frac{r+2}{s+1}\right], \quad (2.17)$$

three points can be added but, if

$$t \geqslant (s-1)(r+1) + (q+1) + \left[\frac{r+2}{s+1}\right], \quad (2.18)$$

only two points can be included in the maximal set.

(i) $s = 3$. The possibilities for the corresponding coordinates, up to multiples are

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}
0	0	1	1	1	0	1	1	1	0	1	1	1
0	1	0	1	2	1	0	1	2	1	0	1	2
1	0	0	0	0	1	1	1	1	2	2	2	2

Since any strict linear combination of Q_i, $i = 1, 2, 3$, yield at most $r + u - 1$ zero coordinates, we have

$$\mathbf{A}_2\mathbf{X}_2 \leqslant r\mathbf{E}_2 + \mathbf{D}_2,$$

where

$$\mathbf{A}_2 = \left[\begin{array}{c|cccc|cccc|cccc} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \qquad \mathbf{D} = \left[\begin{array}{c} 0 \\ 0 \\ 1 \\ 1 \\ \hline 1 \\ 1 \\ 2 \\ 2 \\ \hline 1 \\ 1 \\ 2 \\ 2 \\ \hline 0 \end{array}\right].$$

These may recursively be represented as follows:

$$\mathbf{A}_2 = \left[\begin{array}{c|ccc} \mathbf{E}_1 & \mathbf{A}_1 & \mathbf{A}_1 & \mathbf{A}_1 \\ \mathbf{0} & \mathbf{A}_1 & \mathbf{B}_1 & \mathbf{C}_1 \\ \mathbf{0} & \mathbf{A}_1 & \mathbf{C}_1 & \mathbf{B}_1 \\ \hline 0 & \mathbf{E}_1' & \mathbf{0}' & \mathbf{0}' \end{array}\right], \qquad \mathbf{D}_2 = \left[\begin{array}{c} \mathbf{D}_1 \\ \mathbf{D}_1 + \mathbf{E}_1 \\ \mathbf{D}_1 + \mathbf{E}_1 \\ \hline 0 \end{array}\right],$$

where $\mathbf{D}_1' = (0,0,1,1)$, $\mathbf{E}_1' = (1, 1, 1, 1)$, $= \mathbf{0}' = (0, 0, 0, 0)$,

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

and $\mathbf{X}_2' = (x_1, x_2, \ldots, x_{13})$, $\mathbf{E}_2' = (\mathbf{E}_1', \mathbf{E}_1', \mathbf{E}_1', 1)$.

The solutions corresponding to the maximal t are given below:

$r = 4p$

$$x_i = \begin{cases} p - 1, & i = 4, \\ p + 1, & i = 5, 8, 12, \\ p, & \text{otherwise.} \end{cases}$$

$r = 4p + 1$

$$x_i = \begin{cases} p + 1, & i = 4, 6, 7, 9, 12, 13, \\ p, & \text{otherwise.} \end{cases}$$

$r = 4p + 2$

$$x_i = \begin{cases} p, & i = 1, 2, 3, \\ p + 1, & i > 3. \end{cases}$$

$r = 4p + 3$

$$x_i = \begin{cases} p, & i = 1, 2, \\ p + 1, & i > 2. \end{cases}$$

(ii) $s = 5$. The possible coordinates of Q's up to multiples are

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	x_{15}	x_{16}
0	0	1	1	1	1	1	0	1	1	1	1	1	0	1	1
0	1	0	1	2	3	4	1	0	1	2	3	4	1	0	1
1	0	0	0	0	0	0	1	1	1	1	1	1	2	2	2

x_{17}	x_{18}	x_{19}	x_{20}	x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}	x_{27}	x_{28}	x_{29}	x_{30}	x_{31}
1	1	1	0	1	1	1	1	1	0	1	1	1	1	1
2	3	4	1	0	1	2	3	4	1	0	1	2	3	4
2	2	2	3	3	3	3	3	3	4	4	4	4	4	4

Consideration of all possible linear combinations of three points, we have

$$\mathbf{A}_2\mathbf{X}_2 \leqslant r\mathbf{E}_2 + \mathbf{D}_2$$

where

$$\mathbf{A}_2 = \left[\begin{array}{c|ccccc} \mathbf{E}_1 & \mathbf{B}_1 & \mathbf{B}_1 & \mathbf{B}_1 & \mathbf{B}_1 & \mathbf{B}_1 \\ \mathbf{0} & \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \mathbf{B}_4 & \mathbf{B}_5 \\ \mathbf{0} & \mathbf{B}_1 & \mathbf{B}_3 & \mathbf{B}_5 & \mathbf{B}_2 & \mathbf{B}_4 \\ \mathbf{0} & \mathbf{B}_1 & \mathbf{B}_4 & \mathbf{B}_2 & \mathbf{B}_5 & \mathbf{B}_3 \\ \mathbf{0} & \mathbf{B}_1 & \mathbf{B}_5 & \mathbf{B}_4 & \mathbf{B}_3 & \mathbf{B}_2 \\ \hline 0 & \mathbf{E}_1' & \mathbf{0}' & \mathbf{0}' & \mathbf{0}' & \mathbf{0}' \end{array}\right], \qquad \mathbf{D}_2 = \left[\begin{array}{c} \mathbf{D}_1 \\ \mathbf{D}_1 + \mathbf{E}_1 \\ \mathbf{D}_1 + \mathbf{E}_1 \\ \mathbf{D}_1 + \mathbf{E}_1 \\ \mathbf{D}_1 + \mathbf{E}_1 \\ \hline 0 \end{array}\right],$$

where $\mathbf{E}_1'$ and $\mathbf{0}'$ are each 1×6 row vectors of 1 and 0, respectively, and $\mathbf{D}_1' = (0, 0, 1, 1, 1, 1)$ where $\mathbf{B}_i$, $i = 1, 2, 3, 4, 5$, are described below:

$$\mathbf{X}_2' = (x_1, x_2, \ldots, x_{31}), \qquad \mathbf{E}_2' = (\mathbf{E}_1', \mathbf{E}_1', \mathbf{E}_1', \mathbf{E}_1', \mathbf{E}_1', 1),$$

$$\mathbf{B}_1 = \begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 1&0&0&1&0&0 \\ 0&0&0&0&0&1 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \end{bmatrix}, \quad \mathbf{B}_3 = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&1&0 \\ 0&0&1&0&0&0 \\ 1&0&0&0&0&1 \\ 0&0&0&1&0&0 \end{bmatrix},$$

$$\mathbf{B}_4 = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&1 \\ 1&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \end{bmatrix}, \quad \text{and} \quad \mathbf{B}_5 = \begin{bmatrix} 0&1&1&1&1&1 \\ 1&0&1&0&0&0 \\ 0&1&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&1&0&0&0&0 \\ 1&1&0&0&0&0 \end{bmatrix}.$$

Solutions which correspond to the maximal t with Q's in the set are:

$r = 6p$

$$x_i = \begin{cases} p-1, & i = 5, 6, 16, 20, 25, 26, \\ p+1, & i = 4, 7, 8, 12, 14, 18, 19, 22, 23, 29, \\ p, & \text{otherwise.} \end{cases}$$

$r = 6p + 1$

$$x_i = \begin{cases} p-1, & i = 6, 25, 26, \\ p+1, & i = 4, 7, 8, 12, 13, 14, 18, 19, 22, 23, 24, 27, 29, \\ p, & \text{otherwise.} \end{cases}$$

$r = 6p + 2$

$$x_i = \begin{cases} p-1, & i = 26, \\ p, & i = 1, 2, 3, 5, 6, 10, 11, 15, 16, 21, 25, 30, 31, \\ p+1, & \text{otherwise.} \end{cases}$$

$r = 6p + 3$

$$x_i = \begin{cases} p, & i = 1, 2, 3, 6, 11, 16, 21, 26, 31, \\ p+1, & \text{otherwise.} \end{cases}$$

$r = 6p + 4$

$$x_i = \begin{cases} p, & i = 1, 2, 3, \\ p+1, & i > 3. \end{cases}$$

$r = 6p + 5$

$$x_i = \begin{cases} p, & i = 1, 2, \\ p+1, & i > 2. \end{cases}$$

3. Maximal Sets with $n \geqslant 3$ Additional Points

In this section, we sketch briefly some conditions on t which must be satisfied for the existence of $n(\leqslant t)$ points $Q_1, Q_2, \ldots, Q_n$ such that any t points in the set

(3.1) $$\{E_i, Q_1, Q_2, \ldots, Q_n\}, \qquad i = 1, 2, \ldots, t + r,$$

are linearly independent. Clearly, no t points in this set are linearly dependent if a strict linear combination of u, $1 \leqslant u \leqslant t$, of Q_j, $j = 1, 2, \ldots, n$, has at most $r + u - 1$ zero coordinates. If Q_j's are written in the form of $(t + r, n)$ matrix, then column vectors denoted by x_i, can be interpreted as points in $PG(n - 1, s)$. Considering all possible linear combinations of n points, we observe that

(i) there are $(s^n - 1)/(s - 1)$ linear inequalities involving x_i's,

(ii) each inequality contains $(s^{n-1} - 1)/(s - 1)$ variables, and each x_i appears in$(s^{n-1} - 1)/(s - 1)$ inequalities, and

(iii) $\binom{n}{u}(s - 1)^{u-1}$, $u = 1, 2, \ldots, n$, linear combinations of Q_j's, $j = 1, 2, \ldots, n$, exist, each having at most $r + u - 1$ zero coordinates. This corresponds to the fact that $(s^n - 1)/(s - 1)(n - 1)$-flats incident with each point and $(s^{n-1} - 1)/(s - 1)(n - 1)$-flats incident with each pair of points.

Adding all the linear inequalities, we have

$$\left(\frac{s^{n-1} - 1}{s - 1}\right) \sum_i x_i \leqslant \sum_{u=1}^{u=n} \binom{n}{u} (s - 1)^{u-1}(r + u - 1), \tag{3.2}$$

which yields

$$\left(\frac{s^{n-1} - 1}{s - 1}\right) \sum_i x_i \leqslant \left(\frac{s^n - 1}{s - 1}\right) (r - 1) + n \cdot s^{n-1}. \tag{3.3}$$

This further reduces to

$$\sum_i x_i \leqslant s(r + n - 1) - n + \left[\frac{(s - 1)(r + n - 1)}{s^{n-1} - 1}\right]. \tag{3.4}$$

To derive conditions analogous to (2.6), we recall that each x_i appears in $(s^{n-1} - 1)/(x - 1)$ inequalities. Adding inequalities containing x_i, we obtain

$$\left(\frac{s^{n-1} - 1}{s - 1}\right) x_i + \frac{s^{n-2} - 1}{s - 1} \sum_{j \neq i} x_j$$
$$\leqslant \begin{cases} \left(\dfrac{s^{n-1} - 1}{s - 1}\right) (r - 1) + (n - 1)\, s^{n-2}, & i = 1, 2, \ldots, n, \\ \left(\dfrac{s^{n-1} - 1}{x - 1}\right) (r - 1) + ns^{n-2}, & i > n, \end{cases} \tag{3.5}$$

which reduces to

$$s^{n-2}x_i + \left(\frac{s^{n-2}-1}{s-1}\right)\sum_j x_j$$

$$\leqslant \begin{cases} \left(\dfrac{s^{n-1}-1}{s-1}\right)(r-1) + (n-1)\, s^{n-2}, & i = 1, 2,\ldots, n, \\ \left(\dfrac{s^{n-1}-1}{s-1}\right)(r-1) + ns^{n-2}, & i > n. \end{cases} \tag{3.6}$$

The inequalities derived in (3.6) are useful in investigating upper bound on x_j for each j, $j = 1, 2,\ldots, (s^{n-1}-1)/(s-1)$.

The upper bound (3.4) is attained in few special cases. Consider, for example,

$$r = \left(\frac{s(s^{n-2}-1)}{s-1} - n + m\right) \bmod \left(\frac{s^{n-1}-1}{s-1}\right), \qquad 2 \leqslant m \leqslant n,$$

that is (3.7)

$$r = \left(\frac{s^{n-1}-1}{s-1}\right)p + \frac{s(s^{n-2}-1)}{s-1} - n + m$$
$$= \left(\frac{s^{n-1}-1}{s-1}\right)(p+1) - n + m - 1.$$

Thus,

$$r + n - 1 = \left(\frac{s^{n-1}-1}{s-1}\right)(p+1) + (m-2). \tag{3.8}$$

Clearly,

$$\left[\frac{(r+n-1)(s-1)}{s^{n-1}-1}\right] = p + 1. \tag{3.9}$$

The upper bound (3.4) reduces to

$$\sum_i x_i \leqslant \left(\frac{s^n-1}{s-1}\right)(p+1) + s(m-2) - n. \tag{3.10}$$

An optimal solution for $m = 2$ is evidently given by

$$x_i = \begin{cases} p, & i = 1, 2,\ldots, n, \\ p+1, & i > n. \end{cases} \tag{3.11}$$

Since $\sum_i x_i = t + r$, it follows that if

$$t \geqslant (s-1)(r+n-1) + \left[\frac{(r+n-1)(s-1)}{s^{n-1}-1}\right], \tag{3.12}$$

the set of $t + r$ E_i's, $i = 1, 2,\ldots, t + r$, will include at most $(n-1)$ points.

For $3 \leqslant m \leqslant n$, the bound (3.10) can be improved. Consider

$$\sum_i x_i \leqslant \left(\frac{s^n - 1}{s - 1}\right)(p + 1) + s(m - 2) - n - (m - 2)(s - 1),$$

which reduces to

$$\sum_i x_i \leqslant \left(\frac{s^n - 1}{s - 1}\right)(p + 1) - (n - m + 2), \tag{3.13}$$

an optimal solution for which is given by

$$x_i = \begin{cases} p, & i = 1, 2,..., (n - m + 2);\ m = 3, 4,..., n, \\ p + 1, & i > (n - m + 2). \end{cases} \tag{3.14}$$

In conclusion, we wish to remark that the values of r considered here are by no means exhaustive and one needs to consider several cases separately. The problem lies in the refinement of the upper bound (3.4) and exhibiting a solution to the basic inequalities.

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