Artinian, Non-Noetherian Rings

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The structure of the rings of the title is studied with an eye to giving a representative set of examples. A uniqueness theorem is proved for the components appearing in Peirce decompositions with respect to a principal idempotent, and a Krull-Schmidt-type theorem is proved for the direct sum decompositions of these rings.

1. INTRODUCTION

Do the left Artinian, left non-Noetherian rings (notation: left $A$ not $N$) form an interesting class, or just a few isolated examples? This question was motivated by Hopkins' famous theorem that a left $A$ not $N$ ring cannot have a one or two-sided identity element. [H, (6.7)] (A more customary statement of Hopkins' theorem is that, for rings with identity, left $A$ $\Rightarrow$ left $N$.)

Let $D(R)$ be the maximal divisible, torsion subgroup of a left $A$ ring $R$. Then $D(R) = R = R \cdot D(R)$ (1.1)

(see [F, (72.3)]. The significance of $D(R)$ is given by the following result of Fuchs and Szele [F, (73.3)]; or [K, (10.10)]. Let $R$ be a left $A$ ring. Then

$$R \text{ is left } N \iff D(R) = 0.$$ (1.2)

Let $S(R) = R/D(R)$. It follows from (1.2) that $S(R)$ is a left $A$ and $N$ ring, and we will call $S(R)$ the support ring of $R$. The importance of $S(R)$ in the structure of $R$ is shown by the theorem: Only finitely many indecomposable, nonisomorphic left $A$ not $N$ rings $R$ can be supported by any given left $A$ and $N$ ring $S$ (proved in Sect. 6). Here indecomposable ring means a ring $\neq 0$ which is not the ring-direct sum of two nonzero rings.

We often focus on indecomposable $R$ to rule out uninteresting trivialities such as rings which are $Z(p^\infty)$ groups with zero multiplication, and direct sums of these rings with arbitrary left $A$ and $N$ rings.
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Much of the effort in this investigation is directed to the question, "Which left $A$ and $N$ rings $S$ can support some indecomposable left $A$ not $N$ ring $R$?" We prove, in Section 5, that such an $S$ cannot have a left or right identity element. This is stronger than Hopkin's theorem that $R$ cannot have a left or right identity element. Also, the additive group of $S$ (and $R$, too) must be primary. (This is a result of Szasz; see Sect. 2.) However there is a surprising abundance of rings $S$ which will support indecomposable left $A$ not $N$ rings $R$: (1) any direct sum of 2 or more indecomposable, nonidempotent (e.g., nilpotent) rings, (2) any indecomposable left $A$ and $N$ ring $S$ which has nonzero multiplication kernel, by which we mean the kernel of $\mu_S : S \times S \to S$, and perhaps most surprising, (3) any direct sum of rings $S$ which will individually support an indecomposable $R$. (Of course, in (1) and (3), $(R, +)$ must be kept primary.) This is proved in Section 7, and specific examples are given in Section 8 by making use of properties of flat modules (over rings with identity).

This paper is organized into three parts. Part I, "PPD and Krull–Schmidt," begins with an examination of the Peirce decomposition

$$(R, +) = (1 - e) Re \oplus eRe \oplus eR(1 - e) \oplus (1 - e)(1 - e)$$

of a left $A$ ring with respect to a principal idempotent $e$, that is, an idempotent of $R$ which maps to the identity element of the semisimple ring $R/\text{rad } R$. In Section 2 we complete Hopkins' description of the relation between chain conditions in $R$ and those in the terms on the right of (1.3).

Left $A$ rings without identity usually have more than one principal idempotent. Thus in Section 3, we ask about uniqueness of the terms on the right side of (1.3) up to appropriate isomorphism. The result is: Let $e'$ be another principal idempotent of the left $A$ ring $R$. Then:

$$eRe \cong e'R'e' \text{ and } (1 - e)R(1 - e) \cong (1 - e')R(1 - e') \text{ as rings,}$$

and when these isomorphic rings are identified, $eR(1 - e) \cong e'R(1 - e')$ as bimodules over $eRe$ and $(1 - e)R(1 - e)$. Similarly $$(1 - e)eR \cong (1 - e')e'R.$$ It is easy to verify that (1.4) is equivalent to the simpler assertion: There is a ring automorphism of $R$ which takes $e$ to $e'$, and this is the main result of Section 3.

Azumaya's generalization of the Krull–Schmidt theorem for modules plays a crucial role above, as it does in Section 4 where we prove a Krull–Schmidt theorem for ring-direct sum decomposition of left $A$ rings.

In Part II, "Reduction of $A$ not $N$ to $A$ and $N$," we show how to build an arbitrary left $A$ not $N$ ring $R = (S, D, \delta)$ from a left $A$ and $N$ ring $S$, a divisible torsion group $D$, and an additive map $\delta : S \otimes_S S \to D$. The additive group of $R$ is $S \oplus D$ and the multiplication in $R$ is given by

$$(s_1 + d_1) \cdot (s_2 + d_2) = (s_1s_2) + \delta(s_1 \otimes s_2)$$
where \( s_1s_2 \) denotes the product in \( S \). Using properties of injective modules over the ring \( \mathbb{Z} \) of integers, we show that the structure of \( R \) is completely determined by the behavior of \( \delta \) on the multiplication kernel, \( \ker(\mu_S : S \otimes_S S \rightarrow S^2) \).

This kernel, which is obviously zero when \( S \) has an identity, turns out to be a finite subgroup of \( S \otimes_S S \) (see Sect. 6) and its finiteness is critical in developing most of the important properties of \( R = (S, D, \delta) \).

Finally, in Part III, "A Not \( N \) Indecomposable Rings," we use the results of Parts I and II to determine which left \( A \) and \( N \) rings \( S \) can have the form \( S(R) \) for some indecomposable left \( A \) not \( N \) ring \( R \).

**Notations.** \( \mathbb{Z} \) always denotes the integers.

### I. PPD AND KRULL-SCHMIDT

#### 2. THE PPD AND CHAIN CONDITIONS

A **principal idempotent** of a ring \( R \) is an idempotent element \( e \) of \( R \) such that \( e + \text{rad } R \) is the identity element of \( R/\text{rad } R \). Since idempotents can be lifted modulo nil ideals [J, Chap. 3, Sect. 8] every left \( A \) ring has at least one principal idempotent. (If \( R = \text{rad } R \), then 0 is its principal idempotent.)

By the PPD (Principal Peirce Decomposition) of a ring \( R \) with respect to the principal idempotent \( e \) we mean the decomposition

\[
(R, +) = (1 - e) eRe \oplus eRe \oplus eR(1 - e) \oplus (1 - e) R(1 - e).
\]

(1)

Here \( 1 - e \) is used symbolically only; that is, \( (1 - e)x \) means \( x - ex \) whether or not \( R \) has an identity element. The multiplication in \( R \) is determined by the following ring and module multiplications.

\[ \]

1. \( U \) is a ring with identity \( 1_U = e \); \( A_U \) and \( eB \) are unitary modules; 
2. \( N \) is a ring, \( N = \text{rad } N \); \( N \) and \( B_N \) are modules; 
3. \( AB \subseteq N \) and \( BA \subseteq \text{rad } U \); 
4. and all other products are zero, that is, 
\[
0 = A^2 = AN = UA = UN = BU = B^2 = NU = NB.
\]

(5)

To verify that \( BA \subseteq \text{rad } U \): Note that, since \( e \) becomes the identity modulo \( \text{rad } R \), both \( B \) and \( A \) are contained in \( \text{rad } R \), and hence \( BA \subseteq (\text{rad } R) \cap eRe = \text{rad}(eRe) \). To see that \( N = \text{rad } N \): First note, as above, \( N \subseteq \text{rad } R \). So, for each \( n \) in \( N \), \( 1 + n \) has an inverse \( 1 + a + u + b + n' \). One readily verifies that \((1 + n')(1 + n) = 1\).
The following converse to the above observations is used later, to construct examples of left \( A \) rings with various prescribed properties.

**Proposition 2.0.** Given a ring \( R \) and an additive decomposition

\[
(R, +) = A \oplus U \oplus B \oplus N,
\]

in which (2) through (5) hold, then (6) is the PPD of \( R \) with respect to \( e = 1_U \).

**Proof.** The nontrivial fact to be checked is that \( e \) becomes the identity modulo \( \text{rad} \ R \). So it suffices to show

\[
\text{rad} \ R = A \oplus \text{rad} \ U \oplus B \oplus N \quad \text{(additive \( \oplus \)).}
\]

Since the right side of (7) is a two-sided ideal \( I \) of \( R \), the inclusion \( \text{rad} \ R \supseteq I \) will follow if we show that, for each \( i \in I, 1 + i \) is invertible (in any ring with identity containing \( R \) [J, Chap. 1]. It is straightforward to check that there is a factorization

\[
1 + i = (1 + n)(1 + b)(1 + u)(1 + a) \quad (u \in \text{rad} \ U).
\]

\( 1 + n \) and \( 1 + u \) have inverses since \( N = \text{rad} \ N \) and \( u \in \text{rad} \ U \); and \( B^2 - 0 = A^2 \) shows that \( (1 - b)(1 + b) = 1 \) and \( (1 - a)(1 + a) = 1 \).

Now that \( \text{rad} \ R \supseteq I \), equality follows from the fact that \( R/I \cong U/\text{rad} \ U \) which has radical zero.

**Mnemonic.** To remember this notation, it may help to call \( U \) the unitary subring of the PPD, and note that \( U \) operates "between" \( A \) and \( B \) (i.e., on the right of \( A \) and the left of \( B \)).

The following result completes Hopkins' description [H] of the relation between chain condition in \( R \) and in any PPD of \( R \), and is used often. A divisible group is called finitely decomposable if it is the direct sum of finitely many indecomposable groups (each necessarily \( \cong \mathbb{Z}(p^\infty) \) or the additive group of rational numbers). Recall that \( D(R) \) is the maximal divisible, torsion subgroup of \( R \).

**Theorem 2.1.** Let

\[
(R, +) = A \oplus U \oplus B \oplus N
\]

be a PPD of a left \( A \) ring \( R \). Then \( D(R) \subseteq N \). Furthermore,

\[
A \text{ and } N/D(R) \text{ are finite sets and } D(R) \text{ is a finitely decomposable group}
\]

\[
u U \text{ and } v B \text{ are Artinian and Noetherian}.
\]

Conversely, let a ring \( R \) have a PPD in which \( D(R) \subseteq N \) and (2) and (3) hold. Then \( R \) is left \( A \).
Proof. Since $D(R)$ annihilates $R$ on the left and right (by (1.1)), we get

$$D(R) = (1 - e) D(R)(1 - e) \subseteq (1 - e) R(1 - e) = N.$$ 

Next we make the following observations.

1. $A' \subseteq A \Rightarrow A' \oplus BA'$ is a left ideal of $R.$ (4)
2. $U' \subseteq U \Rightarrow U' \oplus AU'$ is a left ideal of $R.$ (5)
3. $B' \subseteq B \Rightarrow B' \oplus AB'$ is a left ideal of $R.$ (6)
4. $N' \subseteq N \Rightarrow N' \oplus BN'$ is a left ideal of $R.$ (7)

For example, in (4), "$A' \subseteq A'$" should be read, "$A'$ is a left $N$-submodule of $A.$"

The facts (4) through (7) are easily verified by multiplying on the left by each of $A, U, B, N.$ A sample computation, for (4), is $A \cdot (BA') \subseteq NA' \subseteq A'.$

Proof of (2). Since $D(R)$ annihilates $R$, all subgroups of $D(R)$ are ideals of $R$; and this establishes finite decomposability of $D(R).$ By Eq. (7) and the fact that $N = \text{rad } N, N$ is a left Artinian nilpotent ring, so by Széle's theorem [Sz; or [F, (72-1)] $N$ satisfies the descending chain condition for subgroups. Therefore $N/D(R)$ is finite. By [H, (6.6)], $A$ is finite.

Proof of (3). By (5) and (6), $U, B$ are Artinian. Also $U$ is unitary. To see that $U$ is Noetherian we adapt Hopkins' proof that $U$ is Noetherian [H, (6.4)]. It suffices to show that the series

$$B \supseteq (\text{rad } U) B \supseteq (\text{rad } U)^2 B \supseteq (\text{rad } U)^3 B \supseteq \cdots$$

can be refined to a composition series for $U.$ For each $d, (\text{rad } U)^d B/(\text{rad } U)^{d+1} B$ is a semisimple module (since it is a unitary module over the semisimple Artinian ring $U/\text{rad } U$), so being Artinian forces it to have a composition series.

To see that our description of the PPD chain conditions is complete, we now prove the converse part of the theorem. To see that $A$ is Artinian, it suffices to check that $A$ is Artinian; and so it suffices to check that each of $A, U, B,$ and $N$ is Artinian. $A$ is Artinian because it is finite, and $U$ and $B$ are Artinian by (3). Since $UN = 0$, we want to prove $(N, \mid)$ is Artinian; and this is true since, by (2) both $N/D(R)$ and $D(R)$ are Artinian. (This uses the fact that a finitely decomposable divisible torsion group is Artinian.)

Note that when $D(R) = 0$ (equivalently, $N$ is finite), the above proof that $R$'s left $A$ also shows that $R$ is left $N$. Thus we have also proved the nontrivial half of:

**Corollary 2.2** (Fuchs-Széle [F, (73.3)].) *Let $R$ be a left $A$ ring. Then*

$$R \text{ is left } N \iff D(R) = 0.$$
For the other half, i.e. (\(\Rightarrow\)), we merely have to recall that, because \(D(R)\) annihilates all of \(R\) (by (1.1)), every subgroup of \(D(R)\) is an ideal of \(R\); but \(Z(p^n)\) is not a Noetherian group.

Another immediate consequence of Theorem 2.1 is the following theorem of Szasz [K, p. 235, 10.17].

**Corollary 2.3.** Every left \(A\) ring whose additive group is torsionfree has a left identity element.

Another theorem of Szasz states that every left \(A\) ring is a direct sum of rings \(R_0 \oplus R_f\), where \((R_0, +)\) is torsion-free and \((R_f, +)\) is torsion [K, p. 239, 10.24]. By Corollary 2.2, \(R_0\) is left \(N\); and by the Chinese Remainder Theorem \(R_f\) is a direct sum of rings whose Abelian groups are primary. The form in which we use this is:

**Proposition 2.4.** If \(R\) is an indecomposable left \(A\) not \(N\) ring, then \((R, +)\) is a primary Abelian group. If \(R\) is an indecomposable left \(A\) and \(N\) ring, then \((R, +)\) is either torsion-free or primary.

For later reference, it will be useful to have a slight variant of Hopkins' result [H, (6.8)].

**Lemma 2.5.** Let \(R = A \oplus U \oplus B \oplus N\) be a PPD of a left \(A\) ring \(R\). Then \(R = R^2 \Leftrightarrow AB = N\).

Finally, we need the following extension of Hopkins' theorem that a left \(A\) ring with identity is also left \(N\).

**Proposition 2.6.** If \(R\) is a left \(A\) ring and \(R = R^2\), then \(R\) is also left \(N\).

**Proof.** We want to show \(D(R) = 0\) (by Corollary 2.2). Since nonzero divisible groups are infinite, it will suffice to show that \(N\) (in Theorem 2.1), which contains \(D(R)\), is finite. By (3) of Theorem 2.1, \(B\) is a finitely generated (and unitary) \(U\)-module; say \(B = \sum_{i=1}^{n} Ub_i\). Then, by Lemma 2.5, \(N = AB = \sum Ab_i\); and every \(Ab_i\) is finite because \(A\) is.

3. PPD Uniqueness

Given a ring \(R\), "ring(\(Z + R\))" means the ring with identity whose additive group is \(Z \oplus R\) and whose multiplication is \((x_1 + r_1)(x_2 + r_2) = (x_1x_2) + (r_1x_2 + z_1r_2 + r_1r_2)\).

**Theorem 3.1.** Let \(d\) and \(e\) be principal idempotents of a left \(A\) ring \(R\). Then some ring automorphism of \(R\) carries \(d\) to \(e\). In fact, \((1 + r)^{-1} d(1 + r) = e\) for some unit \(1 + r\) in ring \((Z + R)\).
The proof requires the following known lemma whose proof is included because I could not locate it anywhere.

**Lemma 3.2.** Let $d$ and $e$ be idempotents in a ring $R^1$ with identity, and suppose
\[ R^1 d \cong R^1 e \quad \text{and} \quad R^1(1 - d) \cong R^1(1 - e) \quad (R^1\text{-module } \cong). \] (1)

Then there is a unit $u$ of $R^1$ such that $d = u^{-1}e$.  

**Proof.** First verify the following simple fact. Let $M$ be a module with two direct sum decompositions
\[ M = X_1 \oplus X_2 \quad \text{with projection maps } x_i : M \to X_i \]
\[ = Y_1 \oplus Y_2 \quad \text{with projection maps } y_i : M \to Y_i , \]
and suppose there is an automorphism $\varphi$ of $M$ such that $\varphi(Y_i) = X_i (i = 1, 2)$. Then $x_i = \varphi y_i \varphi^{-1}$ (functions being written as left operators).

Apply this to the decompositions
\[ R^1 = R^1 d \oplus R^1(1 - d) = R^1 e \oplus R^1(1 - e) \] (2)
with an automorphism $\varphi$ of $R^1$ (as a left $R^1$-module) given by (1). Then $\varphi$ equals right multiplication by $u = \varphi(1)$, which is a unit because its inverse is $u^{-1} = \varphi^{-1}(1)$. Since the projections in Eq. (2) are $x_1 =$ right multiplication by $d$, $y_1 =$ right multiplication by $e$, we have
\[ d = x_1(1) = \varphi y_1 \varphi^{-1}(1) = \varphi y_1(u^{-1}) = \varphi(u^{-1} e) = u^{-1} e u. \]

**Proof of Theorem 3.1.** By Szasz's theorem 2.4, $R = R_0 \oplus \cdots \oplus R_n$ (ring $\oplus$) where $(R_0, +)$ is torsion-free and each other $(R_i, +)$ is primary for a distinct prime $p_i$. We claim that it suffices to prove the theorem under the additional hypothesis that $(R, +)$ is either torsion-free or primary.

To prove the claim, we show that if $R = S_0 \oplus \cdots \oplus S_n$ is another such decomposition ($S_0$ torsion-free, $S_i p_i$-primary when $i \geq 1$), then every $R_i = S_i$. When $i \neq 0$ this is true by purely group-theoretic considerations. Also, when $i \neq 0$, $R_0 S_i$ is contained in both $R_0$ (which is torsion-free) and $S_i$ (which is torsion), so $R_0 S_i = 0$. Since $R_0$ is torsion-free, $R_0 = R_0^p$ by Corollary 2.3. Therefore
\[ R_0 = R_0 R = R_0 S_0 \subseteq S_0 \]
and similarly, $S_0 \subseteq R_0$ so equality holds.

Now let $(R, +)$ be $p$-primary.

Since $(R, +)$ is $p$-primary, its elements can be uniquely "divided" by each integer prime to $p$. Hence $R$ can be made into a unitary module over the ring $\mathbb{Z}_p$. 

of \(p\)-adic fractions \(a/b\), where \(a\) and \(b\) are integers and \(b\) is prime to \(p\). Define the additive group of \(R^1\) by

\[(R^1, +) = \mathbb{Z}_p \oplus R,\]

and make \(R^1\) into a ring with identity by using the module action of \(\mathbb{Z}_p\) on \(R\) to multiply an element of \(\mathbb{Z}_p\) by an element of \(R\).

Note that the left ideals of \(R\), being \(\mathbb{Z}_p\)-modules, are also left ideals of \(R^1\). However, \(R^1\) is not left \(A\), and might not even be left \(N\).

We now proceed to the verification of the isomorphisms (1) of Lemma 3.2. For the first of these, let \(R = R/\text{rad} R\). Since \(d = e = 1\), we have \(RD \cong RE\) as \(R\)-modules. Since \(d\) and \(e\) are idempotents, this isomorphism can be lifted modulo \(\text{rad} R\) to an \(R\)-module isomorphism \(RD \cong RE\) [J, III 8, Proposition 1]. Then \(R^1 d \cong R^1 e\) follows from the fact that \(RD = R^1 d\) and \(RE = R^1 e\).

We have to work harder to obtain the second isomorphism in (1) because \(R(1 - d) \neq R^1(1 - d)\).

Write the isomorphic modules \(R^1 d\) and \(R^1 e\) as direct sums of indecomposable modules \(D_i\) and \(E_i\) with \(D_i \cong E_i\), as in (3).

\[
R^1 = D_1 \oplus \cdots \oplus D_n \oplus R^1(1 - d) = E_1 \oplus \cdots \oplus E_n \oplus R^1(1 - e). \tag{3}
\]

We claim that \(RE\) (and \(RD\)) has finite composition length as an \(R^1\)-module; equivalently, as an \(R\)-module. To see this, write the PPD \(R = A \oplus U \oplus B \oplus N\) with respect to \(e = 1_U\). Then \(RE = R1_U = A \oplus U\) which has finite length as a left \(U\)-module by Theorem 2.1, hence also as a left \(R\)-module.

Since each \(D_i\) and \(E_i\) in (3) is now indecomposable and of finite length, its endomorphism ring is local, that is, the set of all nonunits forms a two-sided ideal \([L, p. 23, Corollaries 1 and 2]\). If we can show that the endomorphism ring of \(R^1(1 - e)(\text{and } R^1(1 - d))\) is local, too, then Azumaya’s generalization of the Krull–Schmidt theorem [A, or L, p. 78] will apply to the decompositions in (3), yielding the desired isomorphism \(R^1(1 - d) \cong R^1(1 - e)\).

Since \((1 - e)^2 = (1 - e)\), the endomorphism ring of the left \(R^1\)-module \(R^1(1 - e)\) is \(E = (1 - e)R^1(1 - e)\). Now,

\[
E = (1 - e)(\mathbb{Z}_p + R)(1 - e) = \mathbb{Z}_p(1 - e) \oplus (1 - e)R(1 - e) \quad (\text{additive } \oplus).
\]

Since \(e\) is a principal idempotent of \(R\), \((1 - e)R(1 - e)\) equals zero modulo \(\text{rad} R\) and hence is nilpotent; hence it is contained in \(\text{rad} E\). Consequently

\[
\text{rad} E = \mathbb{Z}_p(1 - e) \oplus (1 - e)R(1 - e) \quad (\text{additive } \oplus),
\]

so \(E/\text{rad} E\) is the field \(\mathbb{Z}_p/ p\mathbb{Z}_p\). Thus \(E\) is local ring, so, by Lemma 3.2, there is a unit \(u = v + r\) in \(R^1 (v \in \mathbb{Z}_p, r \in R)\) such that \(u^{-1} du = e\). Since \(v\) must be a
unit in \( \mathbb{Z}_p \), conjugation by \( v + r \) equals conjugation by \( 1 + v^{-1}r \), which belongs to ring \( (\mathbb{Z} + R) \). The theorem is now proved when \((R, +)\) is primary.

Now let \((R, +)\) be torsion-free. Then by, Theorem 2.1, the PPD of \( R \) with respect to \( e = 1_U \) takes the form \( R = U \oplus B \). Then \( 1_U \) is a left identity for \( R \), and the principal idempotents of \( R \) are immediately seen to be the elements \( 1_U + b_0 \) where \( b_0 \) ranges through \( B \).

Given a particular \( b_0 \), we have \((1 - b_0)(1 + b_0) = 1\) in ring \( (\mathbb{Z} + R) \) since \( B^2 = 0 \); and \((1 - b_0)(1_U)(1 + b_0) = 1_U + b_0 \). \( \square \)

**Example 3.3.** Let \( R = A \oplus U \oplus B \oplus N \) be the PPD of a left \( A \) ring with respect to a principal idempotent \( e = 1_U \). Hopkins proved that, when \((\text{rad} \, R)^3 = 0\), the subring \( N \) is independent of the choice of principal idempotent \( e \); and he asked whether this is true in general \([H, \, p. \, 729]\)? We answer this in the negative, and begin by showing:

If \( AB \neq 0 \) and \( NA \neq 0 \), then none of the sets \( A, U, B, N \) is independent of the choice of \( e \).

Imbed \( R \) as a two-sided ideal in a ring \( R^1 \) with identity \( 1 \); for example, take the additive group of \( R^1 \) to be \( \mathbb{Z} \oplus R \), \( \mathbb{Z} \) the integers. For any \( a_0 \in A \), \((1 - a_0)(1 + a_0) = 1\) because \( A^2 = 0 \), so conjugation by \( 1 + a_0 \) is an automorphism of \( R \). Now, for \( n \in N \),

\[
(1 - a_0) n(1 + a_0) = n(1 + a_0) = n + na_0 \in N \oplus A
\]

so a suitable choice of \( a_0 \), together with (1), shows that \( N \) can be "moved" by an automorphism of \( R \) (hence is not independent of the choice of \( e \)). To see that \( U \) can be moved, note that

\[
(1 - a_0) 1_U(1 + a_0) = (1 - a_0) 1_U = 1_U - a_0 \in U \oplus A.
\]

Similarly, \( B \) can be moved, too. Conjugation by a suitable \( 1 + b_0 \) then shows that \( A \) can be moved.

Rings \( R \) satisfying (1) are easy to construct: Let \( U \) be any finite ring with identity such that \((\text{rad} \, U)^2 \neq 0\), for example the integers modulo 8. Then form the direct sum

\[
(R, +) = A \oplus U \oplus B \oplus N
\]

with \( A = B = N = \text{rad} \, U \). Define a multiplication which makes (2) into a PPD by identifying \( R \) with the 2 \( \times \) 2 matrix ring

\[
R = \begin{bmatrix} U & B \\ A & N \end{bmatrix}.
\]

(By Proposition 2.0, Eq. (2) above is indeed a PPD of \( R \).)
Then $R$ satisfies (1) above and we have the desired negative answer to Hopkins' question.

The following application of the Uniqueness Theorem (3.1) gives an instance of a property of $R$ which is inherited from its PPD, and is needed in Construction 8.3.

**Corollary 3.4.** Let $(R, +) = A \oplus U \oplus B \oplus N$ be a PPD of a left $A$ ring $R$ such that $R = R^2$. If $U$ is an indecomposable ring, then so is $R$.

**Proof.** Suppose $R = X \oplus Y$ (ring $\oplus$). Then we can take a PPD of each of $X$ and $Y$, and their direct sum is a PPD of $R$. By uniqueness, $U$, which we now call $U(R)$, is isomorphic to $U(X) \oplus U(Y)$; so indecomposability of $U$ forces, say, $U(Y)$ to be zero. But then the unitary $U(Y)$-modules $A(Y)$ and $B(Y)$ must be zero, too. Since $R = R^2$, we also have $Y = Y^2$. Hence, by Lemma 2.5, $N(Y) = A(Y) \cdot B(Y) = 0$. So $Y = 0$, and $R$ is indecomposable.

4. **KruI-Schmidt Uniqueness**

If a left $A$ ring $R$ is written as a direct sum of indecomposable rings $R = R_1 \oplus \cdots \oplus R_n$, what uniqueness properties do the summands $R_i$ have? At one extreme we have the case where $R$ has an identity element. Here the $R_i$ are unique subsets of $R$, namely those ideals of the form $Re$ where $e$ is a primitive central idempotent. At the other extreme is the case $R^2 = 0$, where one is limited to what can be said about direct sum decompositions of Abelian groups. The main result proved in this section is:

**Krull-Schmidt Theorem 4.1.** Let $R = \bigoplus_{i=1}^{m} R_i = \bigoplus_{i=1}^{n} T_i$ be two ring-direct sum decompositions of a left $A$ ring $R$, with each $R_i$ and $T_i$ indecomposable. Then $m = n$ and the $T_i$ can be renumbered so that

\begin{align*}
R_i &\cong T_i \text{ for all } i \text{ (as rings and } R-R \text{ bimodules)}, \quad \text{(1)} \\
\text{and} \\
R_i &= T_i \text{ whenever } R_i = R_i^2. \text{ (In fact, } R_i^2 = T_i^2 \text{ for every } i.) \quad \text{(2)}
\end{align*}

The following portion of a lemma of Warfield [W, p. 463] produces a reduction to the left $A$ and $N$ case.

**Lemma 4.2.** Let $M$ be an indecomposable module (over some ring) and suppose that $M$ is the union of an increasing infinite sequence $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ of fully invariant submodules each of which has both chain conditions. Then $\text{end } (M)$ is a local ring.
Here a local ring means a ring with 1 in which the set of all nonunits forms a two-sided ideal. (Note a misprint in Warfield's original statement and proof: the letter "R" appears with two distinct meanings.)

**Lemma 4.3.** Let R be a left A ring which is indecomposable (as a ring). Then E = \text{end}(RRR) is a local ring.

**Proof.** By Proposition 2.4, (R, +) is either primary or torsion-free. We consider the p-primary case first. It will suffice to express (R, +) as the union of an increasing sequence of fully invariant subgroups \( M_i \) each of which is a left \( A \) and \( N \) subring of R. Then R and all the \( M_i \)'s are left \( R \otimes R \) opposite modules and Warfield's lemma (Lemma 4.2) applies. For each \( i \) let

\[
M_i = \{ r \in R | p^i \cdot r = 0 \}.
\]

Then the \( M_i \) clearly form an increasing sequence of fully invariant subgroups of \((R, +)\) whose union is R. Hence it will suffice to prove that there is an integer \( n \) such that

\[
M_i \text{ is a left } A \text{ and } N \text{ ring whenever } i \geq n.
\]

Since the Abelian group \( S(R) = R/D(R) \) is the direct sum of (possibly infinitely many) cyclic \( p \)-groups of bounded orders [F, (72.2)], there is an \( n \) such that \( p^n \cdot S(R) = 0 \). Since a divisible Abelian group is a direct summand of every Abelian group containing it, we have

\[
(R, +) = S \oplus D(R)
\]

for some subgroup \( S \cong S(R) \). So \( p^n S = 0 \) and hence \( S \subseteq M_i \) whenever \( i \geq n \). This, together with \( D(R) \cdot R = 0 \) (by (1.1)), shows that every left ideal \( L \) of \( M_i \) is a left ideal of \( R \):

\[
R \cdot L = (S + D(R)) \cdot L = S \cdot L \subseteq M_i \cdot L \subseteq L.
\]

Therefore the ring \( M_i \) is left \( A \). Since \( p^i M_i = 0 \), \( M_i \) contains no divisible torsion subgroups \( \neq 0 \); so (by Corollary 2.2) \( M_i \) is also left \( N \) and the lemma is proved when \( (R, +) \) is primary.

When \( (R, +) \) is torsion-free, the ring \( R \) is left \( A \) and \( N \) (by Proposition 2.4) so the well-known case of Warfield's lemma where \( M_1 = M_2 = \cdots = M \) applies.

**Proof of 4.1.** Let \( R = \bigoplus_{i=1}^n R_i = \bigoplus_{i=1}^n T_i \) (ring \( \bigoplus \)) with each \( R_i \) and \( T_i \) an indecomposable ring (hence also an indecomposable \( R-R \) bimodule). Since each bimodule endomorphism ring \( \text{end}(R_i) \) is local (Lemma 4.3) it follows from the proof of the Krull–Schmidt theorem for modules given in [A, especially
p. 120, lines 9–12] that \( m = n \) and the \( T_i \) can be renumbered such that, for each \( i \),
\[
R_i \oplus \cdots \oplus R_n = R_1 \oplus \cdots \oplus R_i \oplus T_{i+1} \oplus \cdots \oplus T_n
\]
and the projection map: \( R \rightarrow T_i \) given by the right side of \((3)_i\) carries \( R_i \) isomorphically onto \( T_i \). Call this isomorphism \( \varphi_i : R_i \rightarrow T_i \) and note that \( \varphi_i \), being a bimodule coordinate projection, is both an isomorphism of \( R-R \) bimodules and of rings. Thus assertion (1) is established.

To establish (2), note that since \( \varphi_i \) is a ring isomorphism, \( \varphi_i(R_i^2) = T_i^2 \). Since \( \varphi_i \) is also an \( R-R \) bimodule isomorphism, we get
\[
T_i^2 = \varphi_i(R_i^2) = R_i \varphi_i(R_i) = \varphi_i^{-1}(T_i) T_i = \varphi_i^{-1}(T_i^2) = R_i^2.
\]
This completes the proof.

**COROLLARY 4.4.** Let \( S = S_1 \oplus \cdots \oplus S_n \) be a decomposition of a left \( A \) ring \( S \), with each \( S_i \) an indecomposable ring; and let \( \varphi \) be any ring automorphism of \( S \). Then \( \varphi \) permutes the elements of the set \( \{ S_i \mid S_i = S_i^2 \} \).

The total annihilator of a ring \( S \) means the set of all elements \( x \) of \( S \) such that \( Sx = 0 = xS \).

**LEMMA 4.5.** Let \( I \) be a ring such that \( I = I^2 \). If \( y \) belongs to the total annihilator of \( I \), then \( x \otimes y = 0 \) in \( I \otimes I \).

**Proof.** \( x \otimes y \in I \otimes y = I^2 \otimes y = I \otimes I y = 0 \).

Let a left \( A \) ring \( S \) be written as the direct sum of indecomposable rings, and then group the summands so that \( S = T \oplus I \), where \( T \) is the sum of all the nonidempotent summands and \( I = I^2 \). By the Krull–Schmidt (Theorem 4.1), \( I \) is a unique subset of \( S \), while \( T \) is only unique up to isomorphism. The main point of the next corollary (which is needed in Sect. 7) is to show that, in \( S \otimes_S S \), \( T \otimes T \) becomes absolutely unique.

For subgroups \( A \) and \( B \) of a ring \( S \), we denote the subgroup of \( S \otimes_S S \) generated by all elements \( a \otimes b \) by unsubscripted \( A \otimes B \).

**COROLLARY 4.6.** Let \( S = T_1 \oplus \cdots \oplus T_m \oplus I_1 \oplus \cdots \oplus I_n \) be a ring-direct sum decomposition of a left \( A \) ring \( S \). Suppose every \( T_i \) and \( I_j \) is an indecomposable ring, that each \( T_i \supset T_i^2 \), and \( I_j = I_j^2 \). Then for any ring automorphism \( \varphi \) of \( S \), the automorphism \( \varphi \otimes \varphi \) of the group \( S \otimes_S S \) satisfies:
\[
\varphi \otimes \varphi \text{ permutes the elements of the set } \{ I_j \otimes I_j \mid j = 1, \ldots, n \};
\]
\[
(\varphi \otimes \varphi) \left( \bigoplus_{i,j=1}^{m} T_i \otimes T_j \right) = \bigoplus_{i,j=1}^{m} (T_i \otimes T_j).
\]
Proof. Assertion (1) follows from Corollary 4.4. To obtain (2) let $T = \bigoplus T_j$ and $I = \bigoplus I_j$. Then $I = I^2$. Let $T' = \varphi(T)$. By Corollary 4.4, $\varphi(I) = I$. Therefore

$$S = T \oplus I = T' \oplus I \quad \text{(ring \oplus)}.$$ 

Take $t'_j$ in $T'$ and write $t'_j = t_j + i_j (j = 1, 2)$. Then $i_j$ belongs to the total annihilator of $I$, because both $T$ and $T'$ annihilate $I$. Now,

$$(t_1 + i_1) \otimes (t_2 + i_2) = (t_1 \otimes t_2) + (i_1 \otimes t_2) + (t_1 \otimes i_2) + (i_1 \otimes i_2) \quad \text{in } S \otimes S. \quad \text{(3)}$$

By Lemma 4.5, $i_1 \otimes i_2 = 0$ in $I \otimes I$, hence in $S \otimes S$. Also, $I = I^2$ shows

$$i_1 \otimes t_2 \in I \otimes T = I^2 \otimes T = I \otimes IT = 0$$

and similarly $t_1 \otimes i_2 = 0$. Thus (3) now reads $t'_1 \otimes t'_2 = t_1 \otimes t_2$, and the corollary is proved.  

II. REDUCTION OF A NOT N TO A AND N

5. $R = (S, D, \delta)$

Let $S$ be a ring, its multiplication written $s_1s_2$, let $D$ be a divisible torsion group, and $\delta: S \otimes_S S \rightarrow D$ any additive map. Define the additive group of $R = (S, D, \delta)$ to be $S \oplus D$, and define a multiplication in $R$ by

$$(s_1 + d_1) \cdot (s_2 + d_2) = s_1 \cdot s_2 = (s_1s_2) + \delta(s_1 \otimes s_2).$$

It is easy to check that $R$ is an associative ring in which $D \cdot R = 0 = R \cdot D$.

Recall that a group is finitely decomposable if it is the direct sum of finitely many indecomposable groups.

**Theorem 5.1.** Every left A not N ring $R$ equals some $(S, D, \delta)$ with $D = D(R)$ and $S$ left A and N, and $D$ nonzero and finitely decomposable. \quad (1)

Conversely, if (1) holds, then $R = (S, D, \delta)$ is left A not N.

**Proof.** Let $R$ be left A not N. Then $D = D(R)$, being a divisible group, is a direct summand of $(R, +)$. Say $(R, +) = S \oplus D$. Since $D$ is a two-sided ideal of $R$, $R/D$ is a ring whose additive group is $\cong S$. Therefore the natural isomorphism $S \rightarrow (R/D, +)$ can be used to make $S$ into a ring. But, since $S$ will not be a subring of $R$, we must distinguish between the multiplication in $S$ and that in
So let $s_1 s_2$ denote multiplication in the ring $S(R/D)$, and let $r_1 \cdot r_2$ denote the multiplication in $R$. These two multiplications define a function $\delta_1 : S \times S \to D$ by

$$s_1 \cdot s_2 = (s_1 s_2) + \delta_1(s_1, s_2). \quad (2)$$

To see that $\delta_1$ can be factored through $S \otimes S$, recall that $D \cdot R = 0 = R \cdot D$ (by (1.1)) and hence $(s_1 \cdot s_2) \cdot s_3 = (s_1 s_2) \cdot s_3$. So

$$\delta_1(s_1 s_2, s_3) = (s_1 s_2) \cdot s_3 - (s_1 s_2) s_3$$

$$= (s_1 \cdot s_2) \cdot s_3 - (s_1 s_2) s_3$$

$$= (by \text{ associativity in } R \text{ and } S) \delta_1(s_1, s_2 s_3).$$

If we denote, by $\delta$, the additive homomorphism: $S \otimes S \to D$ induced by $\delta_1$, we get

$$s_1 \cdot s_2 = (s_1 s_2) + \delta(s_1 \otimes s_2) \quad (3)$$

so $R = (S, D, \delta)$.

Since $R$ is left $A$, so is $S \cong R/D$; and since $D(S) = 0$ we see from Corollary 2.2 that $S$ is also left $N$. Finally, $D = D(R)$ is finitely decomposable by Theorem 2.1; and $D(R) \neq 0$ since $R$ is not left $N$.

Conversely, let $R = (S, D, \delta)$ and suppose (1) holds. Then $R$ is left $A$ because both $D$ and $S \cong R/D$ are. And since $D(R) = D \neq 0$, $R$ is not left $N$. \[\square\]

If $(S, D, \delta) \cong R \cong (S', D', \delta')$ then $D \cong D(R) \cong D'$ as groups and $S \cong S(R) \cong S'$ as rings. To state the uniqueness of $\delta$, recall that $\mu_S : S \otimes S \to S^2$ is the map defined by $s \otimes t \to st$. We note that injectivity of divisible groups plays a crucial role in the following proof.

**Theorem 5.2 ("Uniqueness of $\delta$")**. The following assertions are equivalent for a left $A$ and $N$ ring $R$ and a divisible torsion group $D$.

$$(S, D, \delta_1) \cong (S, D, \delta_2) \quad (\text{ring } \cong). \quad (1)$$

There exist a group automorphism $\theta$ of $D$ and a ring automorphism $\psi$ of $S$ such that $\delta_1 = \theta \delta_2 (\psi \otimes \psi)$ on $\ker \mu_S$. \quad (2)

**Proof.** Since we must distinguish among three multiplications here, we denote the multiplication in $(S, D, \delta_1)$ by $x \cdot y$, that in $(S, D, \delta_2)$ by $x \dd y$, and that in $S$ by $xy$.

Now assume (1) and let $\theta$ be a ring isomorphism of $(S, D, \delta_1)$ onto $(S, D, \delta_2)$. Since $S$ is left $A$ and $N$, $D(S) = 0$ by Corollary 2.2; so $D$ is a fully invariant subgroup of $(R, +) = S \oplus D$. Therefore $\theta(D) = D$. Define $\varphi : S \to S$ and $\Delta : S \to D$ by $\theta(s) = \varphi(s) + \Delta(s)$. Then $\varphi$ and $\Delta$ are additive homomorphisms,
and directness of the sum \( S \oplus D \) shows that \( \varphi \) is both one-to-one and onto. Since \( \theta \) is a ring homomorphism, we get (for \( s, t \in S, d, e \in D \))

\[
\begin{align*}
(s + d) \cdot (t + e) &\xrightarrow{\theta} [(qs) + (At) + (\theta d)] \cdot [(qt) + (At) + (\theta e)], \\
= (qs)(qt) + \delta_2(qs \otimes qt), \\
= (st) + \delta_1(s \otimes t) \xrightarrow{\eta} \varphi(st) + \Delta(st) + \theta \delta_1(s \otimes t).
\end{align*}
\]

(3)

Comparing \( S \)-coordinates shows \( \varphi(st) = (qs)(qt) \), so \( \varphi \) is a ring automorphism of \( S \). Comparing \( D \)-coordinates and solving for \( \delta_1 \) shows

\[
\delta_1(s \otimes t) = \theta^{-1} \delta_2(\varphi \otimes \varphi)(s \otimes t) - \theta^{-1} \Delta \mu_S(s \otimes t)
\]

which implies

\[
\delta_1 = \theta^{-1} \delta_2(\varphi \otimes \varphi) \text{ on ker } \mu_S
\]

(4)

which is (2) except that the automorphism of \( D \) is written \( \theta^{-1} \).

Conversely, suppose that (5) holds, where \( \theta \) is a group automorphism of \( B \) and \( \varphi \) a ring automorphism of \( S \). Then (5) implies the existence of an additive map \( \alpha \) in (6) making \( \text{"triangle 1"} \) commute.

\[
\begin{array}{c}
\begin{array}{c}
S \otimes S \\
\xrightarrow{\mu_S(s \otimes s)}
\end{array}
\xrightarrow{\delta_1 = \theta^{-1} \delta_2(\varphi \otimes \varphi)}
\begin{array}{c}
D
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\triangle 1
\end{array}
\]

(6)

Injectivity of the divisible \( \mathbb{Z} \)-module \( D \) [L, p. 89] shows that \( \alpha \) can be extended to an additive homomorphism \( \beta : S \to D \), and we define \( \Delta : S \to D \) by \( \beta = -\theta^{-1} \Delta \) as shown. Then (4) holds; and hence so does (3), provided we define \( \theta \) on \( S \) to be \( \varphi(st) = (qs) + \Delta(s) \). The conclusion of (3) is that \( \theta \) is a ring homomorphism, hence isomorphism. Therefore (1) holds. \( \square \)

Remark. Given \( S \) and \( D \), the "Uniqueness of \( \delta \)" shows that the isomorphism class of \( R = (S, D, \delta) \) is completely determined by the action of \( \delta \) on the multiplication kernel, ker \( \mu_S \). We show, in the next section, that ker \( \mu_S \) is a finite subgroup of \( S \), and \( \delta \) can always be chosen to be zero on a subgroup of \( S \otimes S \) of finite index.

Here is one application of the structure and uniqueness theorems.

**Corollary 5.3.** Let \( R \) be an indecomposable left \( A \) not \( N \) ring such that \( S(R) \neq 0 \). Then \( S(R) \) cannot have a left or right identity element.

*Proof.* We first prove: In any ring \( S \) with a one- or two-sided identity element, ker \( \mu_S = 0 \). Let \( e \) be a left identity of \( S \). Then every element of \( S \otimes S \) has the form \( e \otimes s \). So \( 0 = \mu_S(e \otimes s) = es = s \) implies \( e \otimes s = 0 \).
Now let $R$ be an indecomposable left $A$ not $N$ ring. By Theorem 5.1 we can write $R = (S, D, \delta)$ with $D \neq 0$. If $S$ has a left or right identity, the preceding paragraph shows that $\delta = 0$ on $\ker \mu_S$, so "Uniqueness of $\delta$" shows $(S, D, \delta) \cong (S, D, 0) = \text{the ring} \text{ direct sum} S \oplus D$. Indecomposability of $R$ then forces $S(R) \subseteq S = 0$.

In discussing the PPD of a ring, we freely make use of the notation and assertions of Section 2, (1) through (5).

6. Finiteness

**Lemma 6.1.** Let

$$(S, \cdot) = A \oplus U \oplus B \oplus N$$

be the PPD of a ring $S$ with respect to the principal idempotent $e = 1_U$. Then

$$S \otimes S = A \oplus U \oplus B \oplus \frac{(A \otimes U \oplus B) \oplus (N \otimes N \oplus N)}{K}. \quad (2)$$

More precisely, (2) is a canonical isomorphism of Abelian groups, in which $a \otimes b \leftrightarrow ab$, $u_1 \otimes u_2 \leftrightarrow u_1u_2$, $u \otimes b \leftrightarrow ub$, $a \otimes b \leftrightarrow (a \otimes b) + K$, $n_1 \otimes n_2 \leftrightarrow (n_1 \otimes n_2) + K$, where $K$ is (additively) generated by the elements $(na \otimes b) - (n \otimes ab)$

and $(a \otimes bn) - (ab \otimes n)$. \quad (3)

With respect to this identification, $\mu_S$ equals the identity map on $A \oplus U \oplus B$, and $\ker \mu_S \subseteq [(A \otimes U \oplus B) \oplus (N \otimes N \oplus N)]/K$.

Proof. Since the right-hand side of (1) is a direct sum of Abelian groups, $S \otimes S$ is the direct sum of the 16 groups $X \otimes Y$ where $X$ and $Y$ range independently through the set \{A, U, B, N\}. From this we obtain $S \otimes S$ by imposing the additional relations $rs \otimes t = r \otimes st$; that is, by factoring out the subgroup generated by the elements

$$(r, s, t) = (rs \otimes t) - (r \otimes st) \quad (4)$$

with $r$, $s$, and $t$ in $S$. In fact, $r$, $s$, and $t$ can be restricted to lie in any additive generating set for $S$. The set we use is $A \cup U \cup B \cup N$. This results in $4^3 = 64$ types of relations being imposed on $\oplus (X \otimes Y)$, and yields (2). Here is one way of accounting for all this.
Construct the commutative diagram (5) below.

$$S \otimes \mathbb{Z} S = \bigoplus (X \otimes \mathbb{Z} Y) \xrightarrow{\tau} S \otimes S S \xrightarrow{\mu} S^2$$

$$T = A \oplus U \oplus B \oplus (A \otimes U B) \oplus (N \otimes N N)$$

Here $\alpha$ is defined on $A \otimes \mathbb{Z} B$ and $N \otimes \mathbb{Z} N$ to be the "tensor map" $a \otimes b \mapsto a \otimes b$ and $n_1 \otimes n_2 \mapsto n_1 \otimes n_2$, and on the other $14$ groups $X \otimes \mathbb{Z} Y$, $\alpha$ is the multiplication map $x \otimes y \mapsto xy$.

Next, recall that $e = 1_U$, and then define $\beta$ on the summands of $T$ by $a \mapsto a \otimes e$, $u \mapsto u \otimes e = e \otimes u$, $b \mapsto e \otimes b$, $a \otimes b \mapsto a \otimes b$, and $n_1 \otimes n_2 \mapsto n_1 \otimes n_2$. Clearly the diagram commutes, so $\beta$ is "onto." To complete the proof it therefore suffices to check that $\ker \beta$ is the group $K$ described in (3).

Take $a + u + b + x \in \ker \beta$, with $x \in (A \otimes_U B) \oplus (N \otimes_N N)$. Applying $\mu_S \beta$ shows $0 = a + u + b + \mu_S \beta(x)$, so directness of the sum shows $0 = a = u = b$ and hence $\ker \beta \subseteq (A \otimes_U B) \oplus (N \otimes_N N)$.

By (5), $\ker \beta = \alpha(\ker \tau)$. Now we use the description (4) of the generators of $\ker \tau$, where $r, s, t$ belong to $A \cup U \cup B \cup N$. However, in view of the previous paragraph, we can further restrict $r, s,$ and $t$ to be elements such that $\alpha$ sends $(r, s, t)$ to nonzero elements of $(A \otimes_U B) \oplus (N \otimes_N N)$; and these are easily seen to furnish the generators (3) of $K$.

We can now prove that $\ker \mu_S$ is a finite subgroup of $S$, and at the same time find a slightly "more canonical" form for $(S, D, S)$.

**Theorem 6.2.** Let $S$ be a left $A$ and $N$ ring. Then there is an additive decomposition $S \otimes_S S = S_0 \oplus S_f$ where

$S_f$ is finite and contains $\ker \mu_S$; and

(1) $S_f$ is finite and contains $\ker \mu_S$; and

(2) Every left $A$ not $N$ ring supported by $S$ is isomorphic to a ring of the form $R = (S, D, \delta)$ where $\delta(S_0) = 0$.

**Proof.** In the notation of Lemma 6.1, let $S_0 = A \oplus U \oplus B$ and let $S_f = ([A \otimes_U B] \oplus (N \otimes_N N))/K$. Then, "$S_f \supseteq \ker \mu_S"$, is part of the conclusion of the lemma. We prove next that $S_f$ is finite.

It suffices to prove that $A \otimes_U B$ and $N \otimes_N N$ are finite. Since $S$ is left $A$ and $N$, Theorem 2.1 and Corollary 2.2 show that $N$ is finite, hence so is $N \otimes_N N$.

Also, by the same theorem, $B$ is a finitely generated left $U$ module, and both $A$ and $B$ are unitary $U$ modules. If we let $b_1, \ldots, b_n$ be a set of generators, we then
have \( A \otimes B = \sum A \otimes b_i \). Since \( A \) is finite (Theorem 2.1 again), so is each of its homomorphic images \( A \otimes b_i \), and this completes the proof of (1).

To obtain (2), start with an arbitrary \((S, D, \delta_1)\). Define \( \delta_2 \) by \( \delta_2(S_0) = 0 \) and \( \delta_2 = \delta_1 \) on \( S_f \). Then, by (1), \( \delta_1 = \delta_2 \) on ker \( \mu_S \). "Uniqueness of \( \delta \)" Theorem 5.2 then shows that \((S, D, \delta_1) \cong (S, D, \delta_2)\) as desired.

As an easy consequence, we now obtain the finiteness property mentioned in the Introduction.

**Corollary 6.3.** Let \( S \) be a left \( A \) and \( N \) ring. Then:

1. \( S \) can support only a finite number of indecomposable nonisomorphic left \( A \) not \( N \) rings \( R \).

2. For any given finitely decomposable divisible torsion group \( D \), there are only finitely many nonisomorphic left \( A \) not \( N \) rings \( R = (S, D, \delta) \), indecomposable or not.

**Proof.** We prove (2) first. By "Uniqueness of \( \delta \)" (Theorem 5.2) it suffices to show there exist only finitely many homomorphisms \( \delta_1 : \ker \mu_S \rightarrow D \). Since \( \ker \mu_S \) is finite, it suffices to find a finite subgroup \( D' \) of \( D \) such that \( D' \) contains the image of every such \( \delta_1 \). Let \( e \) be the exponent of \( \ker \mu_S \). Then, since \( D \) is the direct sum of finitely many \( \mathbb{Z} \left( p^a \right) \) groups (\( p \) can vary here), the set of elements of \( D \) whose order divides \( e \) will do for \( D' \).

To prove (1), write \( R = (S, D, \delta) \) and recall that \((R, +) = S \oplus D\) must be \( p \)-primary for some prime \( p \) (Proposition 2.4); so \( D \) will be the direct sum of, say, \( \#(D) \) copies of \( \mathbb{Z}(p^a) \). For a finite \( p \)-primary Abelian group \( G \), we define \( \#(G) \) by, "\( G \) is the direct sum of \( \#(G) \) nonzero cyclic groups."

In view of statement (2) it will suffice to show that, given \( S \), there exist only finitely many nonisomorphic \( D \) such that \((S, D, \delta)\) is indecomposable. This is an immediate consequence of:

If \((S, D, \delta)\) is indecomposable, then \( \#(\ker \mu_S) \geq \#(D) \). (3)

To establish (3), suppose that the opposite inequality holds. Then \( \#(\ker \mu_S) \leq \#(\ker \mu_S) < \#(D) \). Let \( D_1 \) be a \( \mathbb{Z} \)-injective hull of \( \delta(\ker \mu_S) \) contained in \( D \) [L, Sect. 4.2]. Then \( D = D_1 \oplus D_2 \) for some subgroup \( D_2 \). Since \( \#(D_1) = \#(\ker \mu_S) < \#(D) \), we conclude that \( D_2 \neq 0 \) and therefore

\[ R = (S, D, \delta) = (S, D_1, \delta) \oplus (0, D_2, 0) \]

is the desired decomposability of \( R \). \( \blacksquare \)
III. A NOT N INDECOMPOSABLE RINGS

7. Decomposition of the Support Ring

We now answer the question, "Does a given left A and N ring S support some indecomposable A not N ring R?" in terms of the indecomposable ring-direct summands of S. By Proposition 2.4, \((R, +)\) must be primary, hence so is \((S, +)\); so we confine our attention to this situation. The main result is:

**Theorem 7.1.** Let a left A and N ring S, with \((S, +)\) primary, have a decomposition

\[
S = \left( \bigoplus_{i=1}^{m} T_i \right) \oplus \left( \bigoplus_{j=1}^{n} I_j \right) \quad \text{(ring \(\oplus\))}
\]

with each \(T_i\) and \(I_j\) an indecomposable ring, each \(I_j\) idempotent \((I_j = I_j^2)\), and each \(T_i \neq T_i^2\). Then S supports some indecomposable left A not N ring R if and only if:

1. every idempotent summand \(I_j\) has nonzero multiplication kernel; and
2. either \(m = 0\) or \(m \geq 2\); or else \(m = 1\) and \(T_1\) has nonzero multiplication kernel.

**Remark.** The theorem immediately produces one large family of rings \(S = S(R)\) which support an indecomposable left A not N ring R: \(S\) can be any direct sum of two or more nonidempotent (e.g. nilpotent) left A and N rings (all primary for the same prime).

To determine which idempotent indecomposable rings \(I\) can serve as direct summands of \(S\) we have to know when \(\ker \mu_I\) is nonzero. In the next section we show that there are many such \(I\) whose multiplication kernel is and is not zero.

Finally, to complete the picture we must deal with the case \(m = 1\). Thus the question becomes: "If \(T\) is indecomposable left A and N and \(T \triangleleft T^2\), is \(\ker \mu_T \neq 0\)?" I know of no example where \(\ker \mu_T = 0\), and in Section 9 we show that \(\ker \mu_T \neq 0\) whenever \(T\) is nilpotent and \(pT = 0\). However, a complete disposition of the case \(T^2 \neq T\) = indecomposable remains elusive.

The remainder of this section is devoted to a proof of Theorem 7.1. We begin with two lemmas.

**Lemma 7.2 (Cross Products).** Let \(A \oplus B\) be the direct sum of rings \(A\) and \(B\). Then

\[
(A \oplus B) \otimes_{A \oplus B} (A \oplus B) = (A \otimes_{A} A) \oplus (R \otimes_{B} R) \oplus \left( \frac{A}{A\bar{z}} \otimes_{\bar{z} \bar{z}} \frac{B}{B\bar{z}} \right) \oplus \left( \frac{B}{B\bar{z}} \otimes_{\bar{z}} \frac{A}{A\bar{z}} \right).
\]
More precisely, (1) is a canonical isomorphism of Abelian groups given by \( a_1 \otimes a_2 \mapsto a_1 \otimes a_2, b_1 \otimes b_2 \mapsto b_1 \otimes b_2, a \otimes b \mapsto (a + A^2) \otimes (b + B^2), b \otimes a \mapsto (b + B^2) \otimes (a + A^2)\). Moreover, if \( A \oplus B \) is left \( A \) and \( N \), and \((A \oplus B, +)\) is primary, then

\[
(A/A^2) \otimes_{\mathbb{Z}} (B/B^2) \cong 0 \Leftrightarrow A = A^2 \text{ or } B = B^2. \tag{2}
\]

**Proof.** Let \( S = A \oplus B \), and note that \( S \otimes_{\mathbb{Z}} S \) is the direct sum of the four groups \( A \otimes_{\mathbb{Z}} A, B \otimes_{\mathbb{Z}} B, A \otimes_{\mathbb{Z}} B, \) and \( B \otimes_{\mathbb{Z}} A \). The kernel of \( S \otimes_{\mathbb{Z}} S \to S \otimes_{\mathbb{Z}} S \) is generated by all elements of the form \((xy \otimes z) - (x \otimes yz)\) where \( x, y, \) and \( z \) range through some additive generating set of \( S \). So they can be restricted to lie in \( A \oplus B \).

Taking \( x, y, \) and \( z \) in \( A \) gives the term \( A \otimes_{\mathbb{Z}} A \), and taking them in \( B \) gives \( B \otimes_{\mathbb{Z}} B \). The terms of the form \((a_1 a_2 \otimes b) - (a_1 \otimes a_2 b) = (a_1 a_2 \otimes b)\) (since \( AB = 0 \)) and \((ab_1 \otimes b_2) - (a \otimes b_1 b_2) = -(a \otimes b_1 b_2)\) give a set of generators of the kernel of \( A \otimes_{\mathbb{Z}} B \to A/A^2 \otimes_{\mathbb{Z}} B/B^2 \); and similarly we obtain the last term in (1).

Note that the implication \( \Leftarrow \) in (2) is trivial. So suppose \((A/A^2) \otimes_{\mathbb{Z}} (B/B^2) = 0\). Now, \( A/A^2 \) and \( B/B^2 \) are left \( A \) and \( N \) rings with trivial multiplication; that is, they are \( A \) and \( N \) as Abelian groups; and this makes them finite. Since they are both primary for the same prime \( p \) this makes them each the direct sum of, say, \( m \) and \( n \) nonzero cyclic \( p \)-groups, respectively. So their tensor product is the direct sum of \( mn \) nonzero cyclic \( p \)-groups. In particular it is nonzero, unless \( A \) or \( B \) is zero.

The next lemma is an immediate consequence of the preceding one.

**Lemma 7.3.** Let \( S = A \oplus B \) (ring \( \oplus \)). Then, in the notation of Lemma 7.2,

\[
\ker \mu_S = \ker \mu_A \oplus \ker \mu_B \oplus \left( \frac{A}{A^2} \otimes_{\mathbb{Z}} \frac{B}{B^2} \right) \oplus \left( \frac{B}{B^2} \otimes_{\mathbb{Z}} \frac{A}{A^2} \right). \tag{1}
\]

\[(A, E, \alpha) \oplus (B, F, \beta) = (S, E \oplus F, \gamma), \text{ where } \gamma = \alpha \text{ on } A \otimes_A A, \gamma = \beta \text{ on } (B \otimes_B B), \text{ and } \gamma = 0 \text{ on both } (A/A^2) \otimes_{\mathbb{Z}} (B/B^2) \tag{2}\]

and \((B/B^2) \otimes_{\mathbb{Z}} (A/A^2)\).

**Proof of Theorem 7.1.** Let \( p \) be the prime such that \((S, +)\) is \( p \)-primary. Write \( S = T \oplus I \) with \( T = \bigoplus_{j=1}^{m} T_j \) and \( I = \bigoplus_{j=1}^{n} I_j \).

We first construct \( R \) supposing that (2) and (3) (in the statement of the theorem) hold and \( m \geq 1 \). A separate, simpler argument covers the case \( m = 0 \).

Let \( D \) be the \( \mathbb{Z} \)-injective hull of \( \ker \mu_T \). Since \( \ker \mu_T \) is a finite group (Theorem 6.2) hence the direct sum of some finite number of cyclic \( p \)-groups, \( D \) is the direct sum of the same number of \( \mathbb{Z}(p^\infty) \) groups; hence \( D \) is finitely decomposable. To see that \( D \) is nonzero, we must show that \( \ker \mu_T \neq 0 \) when \( m \geq 2 \). Writing \( T = T_1 \oplus U \) (with \( U \neq U^2 \) because \( m \geq 2 \)) we see from Lemmas 7.2 and 7.3 that \( \ker \mu_T \) contains the nonzero subgroup \( T_1 \otimes_{\mathbb{Z}} U/U^2 \).
In order to define $\delta$, note that by Lemma 7.2,

$$S \otimes S S = (T \otimes T T) \oplus (I \otimes I I) \quad \text{(additive $\oplus$).} \quad (4)$$

Define $\delta$ on $\ker \mu_T$ to be the inclusion map: $\ker \mu_T \rightarrow D$. Since each $I_j = I_j^2$,

$$I \otimes I I = \bigoplus_{j=1}^{n} (I_j \otimes I I_j) \quad \text{so} \quad \ker \mu_I = \bigoplus_{j=1}^{n} \ker \mu_{I_j}. \quad (5)$$

By hypothesis each $\ker \mu_{I_j}$ is a nonzero group (also finite, by Theorem 6.2, and $p$-primary), so there is a nonzero homomorphism $\delta: \ker \mu_{I_j} \rightarrow D$ for each $I_j$. Thus $\delta$ is defined on all of $\ker \mu_S$, and we extend it to an additive homomorphism $\delta: S \otimes S S \rightarrow D$ by injectivity of $D$ as a $\mathbb{Z}$-module. There will be many such extensions of $\delta$. However, by "Uniqueness of $\delta$" (Theorem 5.2) they will all produce isomorphic rings $(S, D, \delta)$.

We prove that the ring $R = (S, D, \delta)$ thus obtained is indecomposable. If not, then we claim it must have a decomposition of the form

$$R \cong (A, E, \alpha) \oplus (B, F, \beta) \quad \text{(ring $\cong$ and $\oplus$), where} \quad (6)$$

$$A \oplus B = S \quad \text{(ring $\oplus$), and} \quad E \oplus F = D$$

(note the placement of isomorphism and equal signs) with both terms $(A, E, \alpha)$ and $(B, F, \beta)$ nonzero. To see this, let $R = X_1 \oplus X_2$ (ring $\oplus$) with both terms $\neq 0$, and set $X_i = (S_i, D_i, \delta_i)$ (Theorem 5.1). Then $D = D(R) = \bigoplus_{i=1}^{S} D(X_i) = D_1 \oplus D_2$, so $D_1$ and $D_2$ will do for $E$ and $F$, respectively. Also,

$$S \cong S(R) \cong S(X_1) \oplus S(X_2) \cong S_1 \oplus S_2 \quad \text{(ring $\cong$ and $\oplus$).}$$

Therefore $S$ has a decomposition $S = A \oplus B$ (ring $\oplus$) with $A \cong S_1$ and $B \cong S_2$ (ring $\cong$). Since $A \cong S_1$ and $E = D_1$, we can choose $\alpha$ such that $(A, E, \alpha) \cong (S_1, D_1, \delta_1)$; and a similar statement applies to $B$ and $\beta$. Thus we obtain (6).

The conditions at the right of (6) imply that, for some $\gamma$,

$$(A, E, \alpha) \oplus (B, F, \beta) = (S, D, \gamma) \cong (S, D, \delta) \quad \text{(ring $\oplus$ and $\cong$).} \quad (7)$$

By "Uniqueness of $\delta$" (Theorem 5.2) there is a ring automorphism $\varphi$ of $S$ and a group automorphism $\theta$ of $D$ such that

$$\gamma = \theta \delta(\varphi \otimes \varphi) \text{ on } \ker \mu_S. \quad (8)$$

We can write each of $A$ and $B$ as a direct sum of indecomposable rings. Call the nonidempotent rings which occur in this way $T_j'$ and the idempotent ones $I_j'$. Note that our notation does not distinguish whether a given indecomposable summand is contained in $A$ or $B$. By the Krull–Schmidt theorem for left $A$ rings
(Theorem 4.1), applied to $S$, we can choose the numbering so that each $T_j' \cong T_j$ and $I_j'$ equals $I_j$.

After all these preliminaries, we claim: One of the rings $A$ and $B$ must be idempotent. So suppose not. Then some of the $T_j'$ are summands of $A$ and some are summands of $B$. Let $T' = \bigoplus_{j=1}^{m} T_j'$ (the direct sum of all the $T_j'$). We get our contradiction by computing $\gamma(T' \otimes T')$ in two different ways. In the remainder of this proof an unsubscripted $\otimes$ of the form $P \otimes Q$ will mean the subgroup of $S \otimes_{S} S$ generated by the elements $p \otimes q$.

Expand $S \otimes_{S} S$ by using $S = A \oplus B$ and Lemma 7.2. We get a nonzero term

$$0 \neq (A/A^2) \otimes_{Z} (B/B^2) \subseteq \ker \mu_S \cap \ker \gamma$$

where $\ker \gamma$ appears because of Lemma 7.3 (2). Writing $A$ and $B$ as direct sums of the rings $T_j'$ and $I_j$ in (9), we see that there is an element

$$0 \neq k \in (T' \otimes T') \cap \ker \mu_S \cap \ker \gamma$$

because each $I_j/I_j^2 = 0$. But then

$$0 = \gamma(k) = \theta \delta(\varphi \otimes \varphi)(k)$$

with

$$(\varphi \otimes \varphi)(k) \in (\varphi \otimes \varphi)(T' \otimes T') = T' \otimes T' = T \otimes T$$

where the first equality in (10) follows from Corollary 4.6; and so does the second one, as soon as one realizes that $T$ is the image of $T'$ under some ring automorphism of $S$. Now, since $\varphi$ is a ring automorphism, $(\varphi \otimes \varphi) \ker \mu_S = \ker \mu_S$. Also, $\varphi \otimes \varphi$ and $\theta$ are one-to-one. We conclude that

$$0 \neq (\varphi \otimes \varphi)(k) \in (T \otimes T) \cap \ker \mu_S \cap \ker \delta$$

$$= \ker \mu_T \cap \ker \delta \quad \text{(Lemmas 7.3, 7.2).}$$

But this contradicts the fact that, by construction, $\delta$ is one-to-one on $\ker \mu_T$. So the first claim is proved.

We can now suppose that $B = B^0$. Then $B$ is the direct sum of a subset of the $I_j$; and $A$ is the direct sum of the remaining $I_j$ together with $T'$. The next claim is: $\gamma(A \otimes A)$ is an essential $\mathbb{Z}$-submodule of $D$.

First note that $\gamma = \alpha$ on $A \otimes A$ (Lemma 7.3). Also, as in (10), $(\varphi T') \otimes (\varphi T') = T \otimes T$. Therefore

$$\gamma(A \otimes A) \supseteq \theta \delta(\varphi \otimes \varphi)(T' \otimes T') \supseteq \theta \delta(\ker \mu_T).$$

But by construction, $\delta(\ker \mu_T)$ is an essential $\mathbb{Z}$-submodule of $D$. Hence the second claim is proved.

Now we can show that $B = 0$. By directness in (7), $\gamma(B \otimes B) = \beta(B \otimes B)$ has zero intersection with $\gamma(A \otimes A) = \alpha(A \otimes A)$, an essential $\mathbb{Z}$-submodule of
This forces $\gamma(B \otimes B) = 0$. But $B$ is the direct sum of a subset of the $I_j$, and each $\varphi(I_j) = \text{some } I_k$ (Corollary 4.4). Therefore, if there were an index $j$ such that $I_j \subseteq B$ we would have

$$0 = \gamma(B \otimes B) \supseteq \theta \delta(\varphi \otimes \varphi)(I_j \otimes I_j) = \theta \delta(I_k \otimes I_k) \supseteq \theta \delta(\ker \mu_A) \neq 0.$$ 

Thus $B = 0$. Also, in (7), $\gamma(A \otimes A) = \alpha(A \otimes A)$ and thus $\gamma(A \otimes A) \cap F = 0$. Since $\gamma(A \otimes A)$ is essential in $D$, we get $F = 0$ and hence $(B, F, \beta) = 0$, our desired contradiction. Thus $R = (S, D, \delta)$ must be indecomposable.

Now we do the case $m = 0$, that is, $S = I = \bigoplus_{i=1}^n I_i$. Let $D = \mathbb{Z}(p^m)$ and let $d_0$ be an element of order $p$ in $D$. Since $(S, +)$ is primary there is an element $m_i$ of order $p$ in each nonzero subgroup $\ker \mu_{I_i}$ of $S \otimes S$. There is an additive homomorphism $\delta: \bigoplus_{i=1}^n \mathbb{Z}m_i \rightarrow D$ such that each $\delta(m_i) = d_0$. By injectivity of $D$ we can extend $\delta$ to a homomorphism $\delta: S \otimes S \rightarrow D$.

We now show that $R = (S, D, \delta)$ is indecomposable.

As in (6), (7), and (8) above, decomposability of $R$ would imply a decomposition

$$(A, E, \alpha) \oplus (B, F, \beta) = (S, D, \theta \delta(\varphi \otimes \varphi)) \quad (\equiv R)$$

where $(A, E, \alpha)$ and $(B, F, \beta)$ are both nonzero, $A \oplus B = S$, $E \oplus F = D$, $\theta$ is a group automorphism of $D$ and $\varphi$ a ring automorphism of $S$. Since $D$ is an indecomposable group, either $E$ or $F$ must be zero; say $E = 0$. Since $S = S^2$, the stronger of the two conclusions of the Krull–Schmidt theorem for left $A$ rings holds: The indecomposable summands of $S$ are uniquely determined subrings of $S$ (Theorem 4.1). Therefore some $I_j \subseteq A$. But then $\varphi(I_j) = \text{some } I_k$ (Corollary 4.4) so

$$0 = \alpha(A \otimes A) \supseteq \theta \delta(\varphi \otimes \varphi)(I_j \otimes I_j) = \theta \delta(I_k \otimes I_k)$$

and the right side contains the element $\theta \delta(m_k) = \theta(d_0) \neq 0$. This contradiction completes the proof of the "if" part of the theorem.

For the "only if" part of the theorem, let $S = A \oplus B$ (ring $\oplus$) where $\ker \mu_A = 0$ and either $A = A^2$ or $B = B^2$ (or both). We show that $R = (S, D, \delta)$ can never be indecomposable if $A$ and $D$ are nonzero.

By Lemma 7.2, $S \otimes_S S = (A \otimes_A A) \oplus (B \otimes_B B)$. Let $\gamma = 0$ on $A \otimes_A A$ and $\gamma = \delta$ on $B \otimes_B B$. Then $\gamma = \delta$ on

$$\ker \mu_S = \ker \mu_A \oplus \ker \mu_B = 0 + \ker \mu_B$$

so "Uniqueness of $\delta$" (Theorem 5.2) shows

$$(S, D, \delta) \cong (A, 0, 0) \oplus (B, D, \gamma).$$

The proof of Theorem 7.1 is now complete.
8. Indecomposable, idempotent S

In this section we give examples of rings $S$ such that $\ker \mu_S = 0$ and $\ker \mu_S \neq 0$, where

$S$ is indecomposable, left $A$ and $N$, $S = S^2$, $S$ has no left or right identity element, and $(S, +)$ is primary. \hfill (8.1)

The lemma below reduces the problem to that of whether certain unitary modules over a ring with identity are flat.

**Lemma 8.2.** Let

$$(S, +) = A \oplus U \oplus B \oplus N$$

be the PPD of an idempotent left $A$ and $N$ ring $S = S^2$ with respect to the principal idempotent $e = 1_U$. Then

$$S \otimes S = A \oplus U \oplus B \oplus (A \otimes_U B).$$

More precisely, (2) is a canonical isomorphism of Abelian groups given by $a \otimes u \mapsto au, u_1 \otimes u_2 \mapsto u_1 u_2, u \otimes b \mapsto ub, a \otimes b \mapsto a \otimes b$. In particular

$$\ker \mu_S = \ker(A \otimes_U B \rightarrow AB).$$

**Proof.** We begin by showing (without supposing $S = S^2$) that there is an additive homomorphism

$$\varphi: AB \otimes_N AB \rightarrow A \otimes_U B$$

such that $ab \otimes a'b' \mapsto a \otimes ba'b' = aba' \otimes b'$. \hfill (4)

To do this, first recall that, given two "onto" module homomorphisms $f_i : H_i \rightarrow X_i (i = 1, 2)$ the kernel of $f_1 \otimes f_2 : H_1 \otimes_N H_2 \rightarrow X_1 \otimes_N X_2$ is generated by all elements of the form $k_1 \otimes h_2$ and $h_1 \otimes k_2$, where $h_1 \in H_1$ and $k_i \in \ker f_i$ [J, p. 98, Proposition 2]. Then consider the diagram

$\begin{array}{ccc}
A \otimes_U B \otimes_N A \otimes_U B & \rightarrow & A \otimes_U B \\
\downarrow & \nearrow \varphi & \\
AB \otimes_N AB & & AB \otimes_N AB
\end{array}$

where $a \otimes b \otimes a' \otimes b' \mapsto a \otimes ba'b'$. \hfill (5)

To see that the desired $\varphi$ exists making the diagram commute, we have to verify that $\ker \beta \subseteq \ker \alpha$. But by the theorem quoted above, $\ker \beta$ is generated by elements of the form $(\sum a_i \otimes b_i) \otimes (a' \otimes b')$ where $\sum a_i b_i = 0$, and those of
the form \((a \otimes b) \otimes (\sum a'_i \otimes b'_i)\) where \(\sum a'_i b'_i = 0\) and \(a\) takes these elements to zero; for example, since \(b_i a' \in BA \subseteq U\),

\[
\alpha \left[ \left( \sum a_i \otimes b_i \right) \otimes (a' \otimes b') \right] = \sum a_i \otimes b_i a'b' = \sum a_i b_i a' \otimes b' = 0.
\]

Thus \(\varphi\) exists.

We prove the lemma by specializing Lemma 6.1 to the case \(S = S^2\). Thus we have to prove that the map

\[
\theta: A \otimes_U B \rightarrow \frac{(A \otimes_U B) \oplus (N \otimes N)}{K}
\]

given by \(a \otimes b \rightarrow (a \otimes b) + K\) is an isomorphism of Abelian groups. Since \(S = S^2\), we have \(N = AB\) (see Lemma 2.5). Hence the form of \(K\) described in (3) of Lemma 6.1 shows that \(\theta\) is onto. Furthermore, by (4), \(K = (1 - \varphi)(N \otimes N)\). To see that \(\theta\) is one-to-one, suppose \(\sum a_i \otimes b_i \in K = (1 - \varphi)(N \otimes N)\); say \(\sum a_i \otimes b_i = (1 - \varphi)x\) with \(x \in N \otimes_N N\). Then, since \(\varphi(x) \in A \otimes B\), directness on the right of (6) shows that \(x\) (which belongs to \(N \otimes_N N\)) equals zero, hence \(\varphi(x) = 0\), hence \(\sum a_i \otimes b_i = 0\) as desired. 

**Construction 8.3.** Let

\[(S, +) = A \oplus_U U \oplus N,\]

where \(U\) is a left \(A\) and \(N\) ring with identity, \(A\) is a finite right ideal of \(U\), and \(B\) is a left ideal of \(U\),

- \(BA \subseteq \text{rad } U\),
- \(N = AB\).

Then \(S\) can be made into a left \(A\) and \(N\) ring by declaring statement (1) to be a PPD; that is, \(S\) is the subring

\[
S = \begin{bmatrix}
U & B \\
A & N
\end{bmatrix}
\]

of the ring of all \(2 \times 2\) matrices over \(U\). Also, \(S = S^2\) and

\(U\) is an indecomposable ring \(\Rightarrow\) \(S\) is an indecomposable ring. (6)

**Proof.** Statement (1) is a PPD by Proposition 2.0. By Theorem 2.1, \(S\) is left \(A\); and \(S = S^2\) because \(N = AB\) (see Lemma 2.5). Hence \(S\) is left \(N\) by Proposition 2.6. Finally, (6) is a restatement of Corollary 3.4.

**Examples 8.4.** We show that there exist many examples of indecomposable, idempotent, left \(A\) and \(N\) rings \(S\) with \(\ker \mu_S = 0\) and many with \(\ker \mu_S \neq 0\).
Kernel $\mu_S \neq 0$. Here we use the following two facts about modules over an arbitrary ring $U$ with identity. (i) A unitary right $U$-module $A_U$ is flat if and only if

$$\ker(A \otimes_U B \to AB)$$

equals zero for every left ideal $B$ of $U$ [Fa, 11.20]; and (ii) finitely presented unitary flat modules are always projective [Fa, (11.30)].

Let $U$ be any indecomposable left $A$ and $N$ ring with identity, where $(U, +)$ is primary and $U$ has a finite right ideal $A \subseteq \text{rad } U$ such that $A_U$ is finitely presented but not projective. Then there is a left ideal $B$ of $U$ such that the kernel in (2) above is nonzero. Then the $S$ constructed by 8.3 has $\ker \mu_S \neq 0$ (Lemma 8.2).

For a finite such $S$, let $U$ be the integers modulo 8, $A = B = 2U$, and hence $N = 4U$.

For an infinite such $S$, let $F$ be an infinite field of characteristic $p \neq 0$, and let $U$ and $A$ be the collections of matrices

$$U = \begin{bmatrix} F & F \\ 0 & \mathbb{Z}/p^2\mathbb{Z} \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{Z}/p^2\mathbb{Z} \end{bmatrix}$$

where multiplication $(z + p^2\mathbb{Z}) \cdot f (z \in \mathbb{Z}, f \in F)$ is defined to be $zf$. Then $U$ is an indecomposable left $A$ and $N$ (but not right $A$ or $N$) ring with identity and $A$ is a finite right ideal $\subseteq \text{rad } U$. Moreover $A$ is finitely presented because $A \cong (e_{22}U)/A$ as $U$-modules. And this isomorphism shows that $A_U$ is not projective: If it were, then $A$ would be a direct summand of

$$e_{22}U = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{Z}/p^2\mathbb{Z} \end{bmatrix}$$

which it clearly is not.

Since $A_U$ is finitely presented but not projective, the discussion above shows that there is a left ideal $B$ of $U$ such that the $S$ constructed by Construction 8.3 has $\ker \mu_S \neq 0$.

Kernel $\mu_S = 0$. We have already observed, in the proof of Corollary 5.3, that the presence of a one- or two-sided identity in $S$ makes $\ker \mu_S = 0$. To see that $S$ need not have a one-sided identity proceed as follows.

Let $U$ be any indecomposable left $A$ and $N$ ring with identity such that $(U, +)$ is primary and $U$ has a finite right ideal $0 \neq A \subseteq \text{rad } U$. Apply Construction 8.3 with $B = U$. By Lemma 8.2 $\ker \mu_S$ can be identified with

$$\ker(A \otimes_U B \to AB) = \ker(A \otimes_U U \to AU)$$

and this equals zero because $U$ has an identity.
9. Nonidempotent $S$

In this section we give a partial answer to the question, "Which indecomposable, nonidempotent left $A$ and $N$ rings $S$ have $\ker \mu_S \neq 0$?"

The simplest examples are $S = p^m \mathbb{Z}/p^n \mathbb{Z}$ where $1 \leq m < n$. Another class of examples is described in the theorem below. And I do not know any counterexamples.

**Theorem 9.1.** Let $S$ be any nilpotent finite-dimensional algebra $\neq 0$ over a field $F$. Then

$$\ker \mu_S \neq 0.$$  

**Caution.** $S$ will only be a left $A$ and $N$ ring when $F$ is a finite field.

**Proof.** Let $S^e = 0$ but $S^{e-1} \neq 0$. There is an $F$-subspace $T$ of $S$ such that

$$S = T \oplus S^2 \quad (\oplus \text{ of vector spaces}). \quad (1)$$

Squaring both sides in (1) gives $S^2 = T^2 + S^3$; cubing gives $S^3 = T^3 + S^4$, and so on until $S^{e-1} = T^{e-1}$. Substituting each of these equations into the previous one gives

$$S = T \oplus (T + T^3 + \cdots + T^{e-1}). \quad (2)$$

Let $\tau: S \otimes_F S \rightarrow S \otimes S$ be the natural map $x \otimes y \rightarrow x \otimes y$. Then $\ker \tau$ is the $F$-subspace of $S \otimes_F S$ generated by the elements $x \otimes sy - xs \otimes y$ ($x, s, y \in S$).

Let $\mathcal{F}$ be any generating set for $\pi S$ and $\mathcal{G}$ any generating set for $\pi T$. We claim that

$$\ker \tau \text{ is generated by the elements } (x \otimes ty) - (xt \otimes y) \quad (3)$$

with $x$ and $y$ in $\mathcal{F}$ and $t$ in $\mathcal{G}$.

Since $(x \otimes sy) - (xs \otimes y)$ is $F$-linear in each of its three variables when the other two are held constant, it suffices to show that $s$ can be restricted to lie in $T$; and this follows from (2) and repeated use of

$$[x \otimes (t_1 t_2) y] - [x(t_1 t_2) \otimes y]$$

$$= (x \otimes t_1 t_2 y) - (xt_1 \otimes t_2 y) + (xt_2 \otimes t_2 y) - (xt_1 t_2 \otimes y).$$

Let $\beta$ be the composition

$$\beta: S \otimes_F S \rightarrow S \otimes S \rightarrow S^2.$$  

We now assume that $\mu$ is one-to-one, that is, $\ker \beta = \ker \tau$, and try to find a contradiction.
Specifically: we show successively, for \( n = 1, 2, \ldots \), that

\[
V_n = (T \otimes_F T) \oplus (T \otimes_F T^2) \oplus \cdots \oplus (T \otimes_F T^{n-1}) \
\cong T^2 + T^3 + \cdots + T^n \text{ via } x \otimes y \rightarrow xy. \tag{4\text{\textsubscript{n}}}
\]

When we reach \( n = e \) we have the contradiction \( T \otimes_F T^{e-1} \cong T^e = 0 \). So suppose \((4)_{n-1}\) holds. (When \( n = 1 \), we are thus not supposing anything.)

Part of assertion \((4)_{n-1}\) is that the sum \( T^2 + T^3 + \cdots + T^{n-1} \) is direct, hence

\[
S = T \oplus T^2 \oplus \cdots \oplus T^{n-1} \oplus U \text{ with } U \subseteq T^n + \cdots + T^{e-1}. \tag{5}
\]

Therefore

\[
S \otimes_F S = D_2 \oplus D_3 \oplus \cdots \oplus D_n \oplus W \tag{6}
\]

where

\[
D_i = \bigoplus_{a+b=i}(T^a \otimes_F T^b) = \text{the summand of "degree" } i \tag{7}
\]

and where \( W \) is the direct sum tensor products of the remaining pairs of terms in \((5)\). Decomposition \((6)\) will be useful because it gives a corresponding decomposition

\[
\ker \tau = (\ker \tau \cap D_2) \oplus \cdots \oplus (\ker \tau \cap D_n) \oplus (\ker \tau \cap W) \tag{8}
\]

as one can see by using the form \((3)\) for generators of \( \ker \tau \) with \( \mathcal{S} = T \cup T^2 \cup \cdots \cup T^{n-1} \cup U \) and \( \mathcal{F} = T \).

In view of \((6)\) and \((7)\) we can consider \( V_n \) to be a direct summand of \( S \otimes_F S \), and then the map in \((4)_{n}\) becomes \( \beta \mid V_n \). Clearly \( \beta(V_n) = T^2 + \cdots + T^n \).

So suppose \( v \in V_n \) and \( \beta(v) = 0 \). Then \( \tau(v) = 0 \), too. So, by \((8)\), \( v = d_2 + \cdots + d_n \) with \( d_i \in D_i \) and \( \tau(d_i) = 0 \). The induction hypothesis \((4)_{n-1}\) then implies that \( d_i = 0 \) except possibly for \( i = n \). Thus we are reduced to proving

\[
\beta: T \otimes_F T^{n-1} \rightarrow T^n \tag{9}
\]

is one-to-one.

Let \( \dim T = d \). Then \( \dim T^n \leq d^n \), so if we can prove that \( \dim T^n \geq d^n \), map \((9)\) will have to be one-to-one. Since \( T^n = \beta(D_n) \cong \tau(D_n) \) we have

\[
\dim T^n = \dim D_n - \dim \ker(\tau \mid D_n). \tag{10}
\]

By the isomorphism in \((4)_{n-1}\), \( \dim T^2 = \dim(T \otimes_F T) = d^2 \); so \( \dim T^3 = \dim(T \otimes_F T^2) = d^3 \); and so on until \( \dim T^{n-1} = d^{n-1} \). Then \((7)\) shows that \( \dim D_n = (n - 1) d^n \). To compute the second term on the right side of \((10)\), let \( \{t_i : i = 1, 2, \ldots, d\} \) be a basis for \( \mathcal{F} T \). To find a set of generators for the direct summand \( \ker(\tau \mid D_n) \) in \((8)\) we use the form \((3)\) for the generators of \( \ker \tau \).
For the set $\mathcal{S}$ of generators of $S$ we take the monomials of degree $\leq n$ in the $t_j$ together with $U$, that is,
$$\{t_i\} \cup \{t_it_j\} \cup \cdots \cup \{t_{i(1)} \cdot t_{i(2)} \cdots t_{i(n)}\} \cup U$$  \hfill (11)
and for the set of generators of $T$ we take $\{t_i\}$. The conclusion of (3) is that $\ker(\tau | D_n)$ is generated by elements of the form
$$x_a \otimes t_i x_b - x_a t_i \otimes x_b$$  \hfill (12)
where $a + b = n$ and $x_a$ and $x_b$ are monomials of degree $a$ and $b$, respectively; in (11). The number of such expressions (as opposed to elements) (12) is $(n - 2) d^n$ so (10) yields
$$\dim T^n \geq (n - 1) d^n - (n - 2) d^n = d^n$$
and this completes the proof.

References