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Hölder estimates of p-harmonic extension operators

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Abstract

It is now a well-known fact that for $1 < p < \infty$ the p-harmonic functions on domains in metric measure spaces equipped with a doubling measure supporting a $(1, p)$-Poincaré inequality are locally Hölder continuous. In this note we provide a characterization of domains in such metric spaces for which p-harmonic extensions of Hölder continuous boundary data are globally Hölder continuous. We also provide a link between this regularity property of the domain and the uniform p-fatness of the complement of the domain.

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1. Introduction

Given a non-empty bounded open set $\Omega \subset \mathbb{R}^n$ and a function f on $\partial\Omega$, we denote by $P_\Omega f$ the (Perron–Wiener–Brelot) Dirichlet solution of f over Ω. A boundary point

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$\xi \in \partial\Omega$ is called regular if $\lim_{x\to\xi} P_\Omega f(\xi) = f(x)$ for every continuous function f on $\partial\Omega$. We say that Ω is regular if every boundary point is regular. Thus, if Ω is regular, then P_Ω maps $C(\partial\Omega)$ to $\mathcal{H}(\Omega)\cap C(\overline{\Omega})$, where $\mathcal{H}(\Omega)$ is the family of harmonic functions on Ω. It is natural to raise the following question:

Question 1.1. Does the better continuity of a boundary function f guarantee the better continuity of $P_\Omega f$?

In [1] the first named author studied this question in the context of Hölder continuous functions on Euclidean domains. The purpose of this paper is to study the same problem for p-harmonic functions in a general metric measure space for $1 < p < \infty$. In this context we can raise the same question as above. Even in the setting of Euclidean domains (with the standard Lebesgue measures as well as p-admissible measures), the results of this paper for the non-linear problem are new.

Throughout the paper we let $X = (X, d, \mu)$ be a complete connected metric space endowed with a metric d and a positive complete Borel measure μ such that $0 < \mu(U) < \infty$ for all bounded open sets U. Let $B(x, r) = \{y \in X : d(x, y) < r\}$ denote the open ball centered at x with radius r. For simplicity we sometimes abbreviate it to B and write $\lambda B = B(x, \lambda r)$. We assume that μ is doubling, i.e., $\mu(2B) \leqslant C\mu(B)$ for all balls B. The doubling property yields positive constants C and Q such that

$$\mu(B(x, r)) \leqslant Cr^Q. \tag{1.1}$$

We assume $Q > 1$ and fix $1 < p \leqslant Q$ for which X supports a $(1, p)$-Poincaré inequality. Then X supports a $(1, q)$-Poincaré inequality for some $q < p$ by the results of Keith–Zhong [11]. Therefore the notions of p-harmonicity, p-Dirichlet problem, p-Perron solution, p-regularity, p-capacity, and p-Wiener criterion studied by A. Björn, J. Björn, MacManus, and Shanmugalingam [6,3,4,2] can be used in our setting. These notions will be described in the next section. Now letting $P_\Omega f$ denote the p-Perron solution of a function f on the boundary $\partial\Omega$, we can raise the same question posed in Question 1.1. In this note we study this question in the context of Hölder continuous functions. Let $0 < \beta \leqslant \alpha \leqslant 1$. Consider the family $\Lambda_\alpha(E)$ of all bounded α-Hölder continuous functions u on E with norm

$$\|u\|_{\Lambda_\alpha(E)} := \sup_{x\in E} |u(x)| + \sup_{\substack{x,y\in E\\ x\neq y}} \frac{|u(x) - u(y)|}{d(x, y)^\alpha} < \infty.$$

We are concerned about the finiteness of the operator norm:

$$\|P_\Omega\|_{\alpha\to\beta} := \sup_{\substack{f\in\Lambda_\alpha(\partial\Omega)\\ \|f\|_{\Lambda_\alpha(\partial\Omega)}\neq 0}} \frac{\|P_\Omega f\|_{\Lambda_\beta(\Omega)}}{\|f\|_{\Lambda_\alpha(\partial\Omega)}}.$$

In Euclidean domains with weighted measure this problem with respect to p-harmonic functions was first treated by Heinonen et al. [9, Theorem 6.44]. Using the Wiener criterion [17,12], [9, Theorem 6.18], they proved that if $X \setminus \Omega$ satisfies p-capacity density condition or is uniformly p-fat (see the definition in the next section), then for $0 < \alpha \leqslant 1$ there exists $\beta > 0$ such that $\|P_\Omega\|_{\alpha\to\beta} < \infty$. The exponent β is less than α and depends not only on α but also on p, the structure constants of p-harmonicity and uniform p-fatness. For sufficiently small α we may take $\beta = \alpha/2$. The case $\alpha = \beta$ does not seem to be deduced from their arguments.

The case $\alpha = \beta$ was studied by the first named author [1] for the classical setting, i.e. for harmonic functions in Euclidean domains. The crucial parts were based on the comparison of the local and the global harmonic measure decay properties. In the present setting, a p-harmonic *measure* can be defined as an upper Perron solution of the indicator function of a set on the boundary. However, the p-harmonic *measure* is no longer a measure because of the non-linear nature of p-harmonicity. Even in the case $p = 2$ we are guaranteed that 2-harmonic measure is a measure only if we adopt the Cheeger 2-harmonicity rather than the 2-harmonicity defined by upper gradient minimizers (see Section 3). We shall get around this difficulty by some non-linear techniques in Section 3 and give the characterizations of domains Ω for which $\|P_\Omega\|_{\alpha\to\alpha} < \infty$ (Theorem 2.2). We shall demonstrate that the property $\|P_\Omega\|_{\alpha\to\alpha} < \infty$ becomes stronger as α becomes larger (Corollary 2.3). The precise formulation will be given in the next section.

2. Statements of results

By the symbol C we denote an absolute positive constant whose value is unimportant and may change even in the same line. The integral mean of u over the measurable set E is denoted

$$⨍_E u\, d\mu = \frac{1}{\mu(E)} \int_E u\, d\mu.$$

Definition. We say that a Borel function g on X is an *upper gradient* of a real-valued function u on X if

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \leqslant \int_\gamma g\, ds$$

for all non-constant rectifiable paths $\gamma : [0, l_\gamma] \to X$ parameterized by arc length. If the above inequality fails only for a curve family with zero p-modulus (see e.g. [10, Section 2.3] for a discussion on modulus of curve families), then g is referred to as a *p-weak upper gradient* of u. Should u have a p-weak upper gradient from the class $L^p(X)$, then the *minimal p-weak upper gradient* of u is the p-weak upper gradient of u in $L^p(X)$ that is pointwise the smallest almost everywhere among the class of all p-weak upper gradients of u that are in $L^p(X)$; this smallest weak gradient is denoted g_u.

Definition. We say that X supports a $(1, p)$-*Poincaré inequality* if there are constants $\kappa \geqslant 1$ and $C_p \geqslant 1$ such that for all balls $B(x,r) \subset X$, all measurable functions u on X, and all p-weak upper gradients g of u,

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leqslant C_p r \left(\fint_{B(x,\kappa r)} g^p \, d\mu \right)^{1/p}$$

with $u_{B(x,r)} = \fint_{B(x,r)} u \, d\mu$. The constant κ is called the *scaling constant* for the Poincaré inequality.

A consequence of the $(1, p)$-Poincaré inequality is the following p-Sobolev inequality (see [14, Lemma 2.1]): if $0 < \gamma < 1$ and $\mu(\{z \in B(x, R) : |u(z)| > 0\}) \leqslant \gamma \mu(B(x, R))$, then there exists a positive constant C_γ depending on γ such that

$$\left(\fint_{B(x,R)} |u|^p \, d\mu \right)^{1/p} \leqslant C_\gamma R \left(\fint_{B(x,\kappa R)} g_u^p \, d\mu \right)^{1/p}. \tag{2.1}$$

We fix $1 < p \leqslant Q$, where Q is as in the upper mass bound inequality (1.1), and hereafter assume that X supports a $(1, p)$-Poincaré inequality. By Hölder's inequality $(1, p)$-Poincaré inequality implies $(1, q)$-Poincaré inequality for every $q \geqslant p$. It is a remarkable result of Keith and Zhong [11] that the Poincaré inequality is self-improving, i.e., if X is proper (that is, closed and bounded subsets of X are compact) and supports a $(1, p)$-Poincaré inequality, then X supports a $(1, q)$-Poincaré inequality for some $q < p$. Note that a complete metric space equipped with a doubling measure is necessarily proper. In this paper, we rely on this result. Following [19], we consider a version of Sobolev spaces on X.

Definition. Let

$$\|u\|_{N^{1,p}(X)} = \left(\int_X |u|^p \, d\mu \right)^{1/p} + \inf_g \left(\int_X g^p \, d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients g of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$. The space $N^{1,p}(X)$ equipped with the norm $\| \cdot \|_{N^{1,p}(X)}$ is a Banach space and a lattice [19]. We say that a property holds p-q.e. if it holds outside a set E with $\mathrm{Cap}_p(E) = 0$, where $\mathrm{Cap}_p(E) = \inf \|u\|_{N^{1,p}(X)}^p$ with the infimum being taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on E. We let

$$N_0^{1,p}(\Omega) = \{u \in N^{1,p}(X) : u = 0 \ p\text{-q.e. on } X \setminus \Omega\}.$$

Hereafter, let $\Omega \subset X$ be a bounded domain (connected open set) with $\mathrm{Cap}_p(X \setminus \Omega) > 0$. We now introduce the notion of p-harmonicity and p-Dirichlet solutions on Ω.

Definition. We call a function u on Ω a *p-minimizer* in Ω if $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and

$$\int_U g_u^p \, d\mu \leqslant \int_U g_{u+\varphi}^p \, d\mu \tag{2.2}$$

for all relatively compact subsets U of Ω and for every function $\varphi \in N_0^{1,p}(U)$. A *p-harmonic* function is a continuous p-minimizer (every p-minimizer is equal p-q.e. to a p-harmonic function; see [14]).

By $H_\Omega^p f$ we denote the solution to the p-Dirichlet problem on Ω with boundary data $f \in N^{1,p}(\Omega)$, i.e., $H_\Omega^p f$ is a function on $\overline{\Omega}$ that is p-harmonic in Ω with $f - H_\Omega^p f \in N_0^{1,p}(\Omega)$. For every $f \in \mathrm{Lip}(\partial\Omega)$ there is a function $Ef \in \mathrm{Lip}(\overline{\Omega})$ such that $f = Ef$ on $\partial\Omega$. Therefore we can define $H_\Omega^p f$ by the function $H_\Omega^p Ef$; this is independent of the extension *Ef*. We say that a lower semicontinuous function u on Ω is *p-superharmonic* in Ω if $-\infty < u \leqslant \infty$, u is not identically ∞ in any component of Ω, and $H_{\Omega'}^p v \leqslant u$ in Ω' for every non-empty open set $\Omega' \Subset \Omega$ and all functions $v \in \mathrm{Lip}(\partial\Omega')$ such that $v \leqslant u$ on $\partial\Omega'$. If $-u$ is p-superharmonic, then we say u is p-subharmonic.

Definition. Given a function f on $\partial\Omega$ we let $\mathcal{U}_f$ be the set of all p-superharmonic functions u on Ω bounded below such that $\liminf_{\Omega \ni x \to \xi} u(x) \geqslant f(\xi)$ for each $\xi \in \partial\Omega$. The upper Perron solution of f is defined by

$$\overline{P}_\Omega^p f(x) = \inf_{u \in \mathcal{U}_f} u(x) \quad \text{for } x \in \Omega.$$

Similarly, we define the lower Perron solution by

$$\underline{P}_\Omega^p f(x) = \sup_{u \in \mathcal{L}_f} u(x) \quad \text{for } x \in \Omega,$$

where $\mathcal{L}_f = -\mathcal{U}_{-f}$ is the set of all p-subharmonic functions u on Ω bounded above such that $\limsup_{\Omega \ni x \to \xi} u(x) \leqslant f(\xi)$ for each $\xi \in \partial\Omega$. Since in this paper p is fixed, henceforth we drop the reference to p in the notation of the Perron solutions; $\overline{P}_\Omega f = \overline{P}_\Omega^p f$ and $\underline{P}_\Omega f = \underline{P}_\Omega^p f$. If $\overline{P}_\Omega f = \underline{P}_\Omega f$, then we say f is resolutive and write $P_\Omega f$ for this common function.

It is known that every continuous function on $\partial\Omega$ is resolutive and that $H_\Omega^p f = P_\Omega f$ in Ω for every $f \in N^{1,p}(X)$. We say that $\xi \in \partial\Omega$ is *p-regular* if

$$\lim_{\Omega \ni x \to \xi} P_\Omega f(x) = f(\xi) \quad \text{for all } f \in C(\partial\Omega).$$

If $\xi \in \partial\Omega$ is a p-regular point and f is a bounded function on $\partial\Omega$ which is continuous at ξ, then

$$\lim_{\Omega\ni x\to\xi} \underline{P}_\Omega f(x) = \lim_{\Omega\ni x\to\xi} \overline{P}_\Omega f(x) = f(\xi).$$

The validity of the *Kellogg property* is known: the set of all p-irregular points on $\partial\Omega$ is of p-capacity zero. See [3,4,2] for these accounts. A domain Ω with no p-irregular boundary point is called a *p-regular* domain.

By $\mathcal{H}^p(\Omega)$ we denote the family of all p-harmonic functions on Ω. The counterpart of the classical result mentioned at the beginning is the following: if Ω is p-regular, then P_Ω maps $C(\partial\Omega)$ to $\mathcal{H}^p(\Omega)\cap C(\overline{\Omega})$. Now, as in Question 1.1, we may ask whether the Hölder continuity of the boundary function f results in a better regularity of $P_\Omega f$. Heuristically one might think that the finiteness of $\|P_\Omega\|_{\alpha\to\beta}$ with $0<\beta\leqslant\alpha$ implies the p-regularity of the domain Ω. This is not the case, as observed by an example in [1] for the linear case. Indeed, it is easy to see that every singleton set has zero p-capacity for $p\leqslant Q$, and it can be seen that removing a single point yields a p-irregular domain for which $\|P_\Omega\|_{\alpha\to\beta}<\infty$. To avoid such a pathological example we consider the following notion. We say that $a\in\partial\Omega$ is a *p-trivial boundary point* if there is $r>0$ such that $\mathrm{Cap}_p(\partial\Omega\cap B(a,r))=0$. We rule out p-trivial boundary points as we have the following proposition.

Proposition 2.1. *Suppose* $\|P_\Omega\|_{\alpha\to\beta}<\infty$ *for some* $0<\beta\leqslant\alpha$. *Then* Ω *is a p-regular domain if and only if* $\partial\Omega$ *has no p-trivial points.*

The proof can be carried out in the same way as in [1, Theorem 1] with the aid of the Kellogg property [4]. For the reader's convenience it will be given in Section 7. A p-trivial boundary point can be regarded as an interior point from the point of view of potential theory. Adding all p-trivial boundary points to the domain, we obtain a domain with no p-trivial boundary point; the potential theoretical property of the resulting domain is the same as that of the original domain. In light of Proposition 2.1, we may assume that Ω is p-regular in the sequel.

In this paper, we concentrate mostly on the case $\alpha=\beta$. In particular we study several conditions for $\|P_\Omega\|_{\alpha\to\alpha}<\infty$ to be true. The following local or interior Hölder continuity of p-harmonic functions is proved in [14, Theorem 5.2]: there exists $\alpha_0>0$ such that every p-harmonic function in any domain Ω is locally α_0-Hölder continuous in Ω (see Lemma 3.4 in Section 3 for the precise formulation). This constant α_0 depends only on p and the constants associated with the doubling property of μ and the Poincaré inequality, but not on Ω. In general, $\alpha_0<1$. It should be noted that in the setting of general metric measure spaces, even if $p=2$ one cannot hope to obtain local Lipschitz regularity for p-harmonic functions. Indeed, the example discussed at the beginning of [15, p. 4] demonstrates that the largest possible value of α for the questions above is the index α_0 given by Kinnunen and Shanmugalingam [14]. This is one difference between the classical case and the present case. In order to have $\|P_\Omega\|_{\alpha\to\alpha}<\infty$, we restrict ourselves to $\alpha\leqslant\alpha_0$.

From the point of view of the classical results, the conditions for $\|P_\Omega\|_{\alpha\to\alpha} < \infty$ involve the *p-harmonic measure* and the *exterior conditions* of the domain Ω such as the relative capacity:

$$\mathrm{Cap}_p(E,U) := \inf\left\{\int_U g_u^p\,d\mu : u \in N_0^{1,p}(U) \text{ and } u \geqslant 1 \text{ on } E\right\}.$$

Definition. Given an open set U in X and a Borel set $E \subset \partial U$, by the p-harmonic measure $\omega_p(E;U)$ we mean the upper Perron solution $\overline{P}_U\,\chi_E$ of the boundary function χ_E in U; see [4].

Note that $\omega_p(E;U)$ need not be a measure unless $p=2$ because of the non-linear nature of p-harmonicity. Even in the case $p=2$ we are guaranteed that $\omega_p(E;U)$ is a measure only if we adopt the Cheeger 2-harmonicity rather than the 2-harmonicity defined above by upper gradient minimizers (see Section 3).

We use $\varphi_{a,\alpha}(x) = \min\{d(x,a)^\alpha, 1\}$ for $a \in \partial\Omega$ as a test boundary function with respect to α-Hölder continuity. Let $S(x,r) = \{y \in X : d(x,y) = r\}$ be the *sphere* with center at x and radius r; it should be noted that while $\partial B(x,r) \subset S(x,r)$, the sphere can be a larger set than $\partial B(x,r)$. The following is the main theorem of this paper.

Theorem 2.2. *Let Ω be a p-regular domain. Suppose $0 < \alpha \leqslant \alpha_0$, where α_0 is a positive constant such that every p-harmonic function in Ω is locally α_0-Hölder continuous in Ω as explained above* [14, Theorem 5.2]. *Consider the following four conditions*:

(i) $\|P_\Omega\|_{\alpha\to\alpha} < \infty$.

(ii) *There exists a constant C_1 such that whenever $a \in \partial\Omega$,*

$$P_\Omega\varphi_{a,\alpha}(x) \leqslant C_1 d(x,a)^\alpha \quad \textit{for every } x \in \Omega. \tag{2.3}$$

(iii) *Global harmonic measure decay property* (*abbreviated to* GHMD(α)). *There exist constants $C_2 \geqslant 1$ and $r_0 > 0$ such that whenever $a \in \partial\Omega$ and $0 < r < r_0$,*

$$\omega_p(x; \partial\Omega\setminus B(a,r), \Omega) \leqslant C_2\left(\frac{d(x,a)}{r}\right)^\alpha \quad \textit{for every } x \in \Omega\cap B(a,r).$$

(iv) *Local harmonic measure decay property* (*abbreviated to* LHMD(α)). *There exist constants $C_3 \geqslant 1$ and $r_0 > 0$ such that whenever $a \in \partial\Omega$ and $0 < r < r_0$,*

$$\omega_p(x; \Omega\cap S(a,r), \Omega\cap B(a,r)) \leqslant C_3\left(\frac{d(x,a)}{r}\right)^\alpha \quad \textit{for every } x \in \Omega\cap B(a,r).$$

Then we have

$$\text{(i)} \iff \text{(ii)} \Rightarrow \text{(iii)} \Leftarrow \text{(iv)}.$$

If (iv) *holds for some $\alpha' > \alpha$, then* (i) *and* (ii) *hold.*

Moreover, if X is Ahlfors Q-regular, i.e.,

$$C^{-1}r^Q \leqslant \mu(B(x,r)) \leqslant Cr^Q \quad \textit{for every ball } B(x,r), \tag{2.4}$$

then (iii) $\Longleftrightarrow$ (iv).

As an immediate corollary, we observe that the larger α is the stronger the property $\|P_\Omega\|_{\alpha\to\alpha} < \infty$ is.

Corollary 2.3. *Assume that X is Ahlfors Q-regular. If* $0 < \beta \leqslant \alpha \leqslant \alpha_0$ *and* $\|P_\Omega\|_{\alpha\to\alpha} < \infty$, *then* $\|P_\Omega\|_{\beta\to\beta} < \infty$.

Remark 2.4. There is a domain Ω for which the LHMD(α) holds and yet $\|P_\Omega\|_{\alpha\to\alpha} = \infty$. In fact, let $\Omega = \{z \in \mathbb{C} : |z| < 1, |\arg z| < \pi/(2\alpha)\}$ for $0 < \alpha \leqslant 1$. Then it is easy to see that LHMD(α) holds with respect to the classical harmonic measure. Define $\varphi(z) = |z|^\alpha$ for $\partial\Omega$. Then $\|\varphi\|_{\Lambda_\alpha(\partial\Omega)} < \infty$, whereas the classical Dirichlet solution $P_\Omega\varphi$ satisfies $\|P_\Omega\varphi\|_{\Lambda_\alpha(\Omega)} = \infty$ since $P_\Omega\varphi(x) \approx x^\alpha \log(1/x)$ as $x \downarrow 0$ on the positive real axis. Thus the statement (iv) $\Rightarrow$ (i) with the same exponent α does not necessarily hold true in the above theorem.

Definition. We say that E is *uniformly p-fat* or satisfies the *p-capacity density condition* if there exist constants $C_4 > 0$ and $r_0 > 0$ such that

$$\frac{\mathrm{Cap}_p(E \cap B(a,r), B(a,2r))}{\mathrm{Cap}_p(B(a,r), B(a,2r))} \geqslant C_4 \tag{2.5}$$

whenever $a \in E$ and $0 < r < r_0$.

See [16,18] for more on uniform fatness in the Euclidean setting, and [6] for the metric space setting. If we ignore the exact Hölder exponent, we obtain the following characterization.

Theorem 2.5. *Assume that X is Ahlfors Q-regular. Let* Ω *be a p-regular domain. Then the following five conditions are equivalent*:

(i) $\|P_\Omega\|_{\alpha\to\alpha} < \infty$ *for some* $\alpha > 0$.
(ii) (2.3) *holds for some* $\alpha > 0$.
(iii) GHMD(α) *holds for some* $\alpha > 0$.
(iv) LHMD(α) *holds for some* $\alpha > 0$.
(v) $X \setminus \Omega$ *is uniformly p-fat.*

We say that a measurable set E satisfies the *volume density condition* if there exist constants $C_5 > 0$ and $r_0 > 0$ such that

$$\frac{\mu(E \cap B(a,r))}{\mu(B(a,r))} \geqslant C_5 \tag{2.6}$$

whenever $a \in E$ and $0 < r < r_0$. The volume density condition is stronger than the p-capacity density condition, and hence we obtain the following.

Corollary 2.6. *If* $X \setminus \Omega$ *satisfies the volume density condition, then it is p-uniformly fat, and hence* $\|P_\Omega\|_{\alpha\to\alpha} < \infty$ *for some* $\alpha > 0$.

The arguments of the paper are based mostly on the comparison theorem of p-harmonic functions and on the properties of the De Giorgi class [14], which includes the p-harmonic functions in its membership. Therefore, our results are applicable not only to p-harmonic functions but also to Cheeger p-harmonic functions as well as the $\mathcal{A}$-harmonic functions in the Euclidean setting with the usual uniform ellipticity assumptions on $\mathcal{A}$. We shall give precise definitions of Cheeger p-harmonic functions and related functions as well as several properties of the De Giorgi class in the next section.

The proof of Theorem 2.2 will be given as a series of lemmas. The crucial part is GHMD $\Rightarrow$ LHMD (Lemma 5.1), for which we need the Ahlfors Q-regularity of μ. This part will be proved in Section 5. Other parts of the theorem remain true under a weaker hypothesis that μ is doubling and supports a $(1, p)$-Poincaré inequality. Section 4 will be devoted to the proof of these parts. The proof of Theorem 2.5 will be given in Section 6. The final section deals with conditions for $\|P_\Omega\|_{\alpha\to\beta} < \infty$ when $0 < \beta \leqslant \alpha$, and includes the proof of Proposition 2.1. In the case $\beta < \alpha$, the characterization for $\|P_\Omega\|_{\alpha\to\beta} < \infty$ is far from complete. Nevertheless, we shall show that some parts of Theorem 2.2 holds true.

3. Quasiminimizers and De Giorgi class

Definition. We call a function u on X a *p-superminimizer* in Ω if $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and the energy minimizing inequality (2.2) holds for all relatively compact subsets U of Ω and for every non-negative function $\varphi \in N^{1,p}_0(U)$.

Remark 3.1. Let u be a p-superminimizer in Ω. Then the lower regularization $\operatorname{ess\,lim\,inf}_{y\to x} u(y)$ is a lower semicontinuous representative [13, Theorem 5.1] and it is a *p-superharmonic* function [13, Proposition 7.4]. Conversely, a bounded p-superharmonic (resp. p-subharmonic) function is a p-superminimizer (resp. p-subminimizer) [13, Corollary 7.8]. An unbounded p-superharmonic function need not to be a p-superminimizer; the truncation of such a p-superharmonic function is a p-superminimizer.

Cheeger [7] introduced the partial derivatives du and gave an alternative definition of Sobolev spaces. As long as $1 < p < \infty$, the Cheeger Sobolev space and $N^{1,p}(X)$ coincide. Moreover, the minimal p-weak upper gradient and the Cheeger derivative are comparable, i.e.,

$$C^{-1}|du(x)| \leqslant g_u(x) \leqslant C|du(x)|. \tag{3.1}$$

See [19, Theorem 4.10], [20, Corollary 3.7] for these accounts.

Definition. We call a function u on X a *Cheeger p-minimizer* in Ω if $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and

$$\int_U |du|^p \, d\mu \leqslant \int_U |d(u+\varphi)|^p \, d\mu \tag{3.2}$$

for all relatively compact subsets U of Ω and for every function $\varphi \in N^{1,p}_0(U)$. A *Cheeger p-harmonic* function is a continuous Cheeger p-minimizer. We call a function u on X a *Cheeger p-superminimizer* in Ω if $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and (3.2) holds for all relatively compact subsets U of Ω and for every non-negative function $\varphi \in N^{1,p}_0(U)$. A lower semicontinuous p-superminimizer is a *Cheeger p-superharmonic* function. If $-u$ is Cheeger p-superharmonic, then u is said to be *Cheeger p-subharmonic*.

Definition. We say that a function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *p-quasiminimizer* in Ω if there is a constant $C_6 \geqslant 1$ such that

$$\int_U g_u^p \, d\mu \leqslant C_6 \int_U g_{u+\varphi}^p \, d\mu \tag{3.3}$$

for all relatively compact subsets U of Ω and for every function $\varphi \in N^{1,p}_0(U)$. We call a function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ a *p-quasisuperminimizer* in Ω if (3.3) holds for all relatively compact subsets U of Ω and for every non-negative function $\varphi \in N^{1,p}_0(U)$. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is said to be a *p-quasisubminimizer* in Ω if (3.3) holds for all relatively compact subsets U of Ω and for every non-positive function $\varphi \in N^{1,p}_0(U)$.

Clearly, p-harmonic and Cheeger p-harmonic functions are p-quasiminimizers; p-superharmonic and Cheeger p-superharmonic functions are p-quasisuperminimizers, while p-subharmonic and Cheeger p-subharmonic functions are p-quasisubminimizers.

Definition. Given an open set Ω, a function $u \in N^{1,p}_{loc}(\Omega)$ is said to belong to the De Giorgi class $\mathrm{DG}_p(\Omega)$ if there are constants $C > 0$ and $\kappa \geqslant 1$ such that

$$\int_{B(z,\rho)} g^p_{(u-k)_+}\, d\mu \leqslant \frac{C}{(R-\rho)^p} \int_{B(z,R)} (u-k)^p_+\, d\mu$$

whenever $k \in \mathbb{R}$, $0 < \rho < R < \operatorname{diam}(X)/3$, and $B(z, \kappa R) \subset \Omega$.

In what follows let κ be the scaling constant from the $(1, q)$-Poincaré inequality. Then we have the following [14, Proposition 3.3].

Lemma 3.2. *If u is a quasisubminimizer on Ω, then $u \in \mathrm{DG}_p(\Omega)$. If u is a quasiminimizer on Ω, then both u and $-u$ belong to $\mathrm{DG}_p(\Omega)$.*

In light of the above lemma, our results hold true if p-harmonicity is replaced by Cheeger p-harmonicity. We now collect together some properties of the De Giorgi class. The following lemma is from [14, Theorem 4.2].

Lemma 3.3. *There is a constant $C_7 > 1$ such that whenever $0 < R < \operatorname{diam}(X)/3$ and $u \in \mathrm{DG}_p(B(z, \kappa R))$,*

$$\sup_{B(z,R/2)} u \leqslant k_0 + C_7 \left(\fint_{B(z,R)} (u - k_0)^p_+\, d\mu \right)^{1/p} \quad \textit{for every } k_0 \in \mathbb{R}.$$

This estimate yields the local Hölder continuity of p-quasiminimizers [14, Theorem 5.2]. The next result gives control over the oscillation of a function in the De Giorgi class. Here, by $\operatorname{osc}_E u$ we denote the oscillation $\sup_E u - \inf_E u$.

Lemma 3.4. *Suppose that both u and $-u$ belong to $\mathrm{DG}_p(B(x, 2\kappa R))$. Then*

$$\operatorname*{osc}_{B(x,r)} u \leqslant C \left(\frac{r}{R} \right)^{\alpha_0} \operatorname*{osc}_{B(x,R)} u \quad \textit{for } 0 < r \leqslant R$$

for some $0 < \alpha_0 \leqslant 1$ and $C \geqslant 1$ independent of u, x and R.

The above lemma is deduced from a certain measure estimate [14, Proposition 5.1]. We shall need its precise form in the proof of Theorem 2.2 (iii) $\Rightarrow$ Theorem 2.2 (iv).

Lemma 3.5. *Let $0 < R < \operatorname{diam}(X)/(6\kappa)$ and $u \in \mathrm{DG}_p(B(z, 2\kappa R))$. Suppose $0 \leqslant u \leqslant M$ on $B(z, 2\kappa R)$ and*

$$\frac{\mu(\{x \in B(z,R) : u(x) > M - s\})}{\mu(B(z,R))} \leqslant \gamma < 1$$

for some $0 < s < M$. *Then for any* $\delta > 0$ *there is* $\eta = \eta(p, q, \gamma, \delta) > 0$ *such that*

$$\frac{\mu(\{x \in B(z, R) : u(x) > M - \eta s\})}{\mu(B(z, R))} \leqslant \delta.$$

Though the proof is similar to [14, Proposition 5.1], for the reader's convenience it is given here.

Proof. In this proof we fix u and z and write $A(h, R) = \{x \in B(z, R) : u(x) > h\}$. Let us recall that a $(1, q)$-Poincaré inequality with $q < p$ is assumed to hold in X. For the moment let $M - s \leqslant h < k < M$. We claim

$$\left(\frac{(k-h)\mu(A(k, R))}{(M-h)\mu(B(z, R))}\right)^{pq/(p-q)} \leqslant C\,\frac{\mu(A(h, \kappa R)) - \mu(A(k, \kappa R))}{\mu(B(z, R))} \tag{3.4}$$

whenever $\mu(A(h, R)) \leqslant \gamma\mu(B(z, R))$. To prove this, let

$$v(x) = \min\{u(x), k\} - \min\{u(x), h\} = \begin{cases} k - h & \text{if } u(x) \geqslant k, \\ u(x) - h & \text{if } h < u(x) < k, \\ 0 & \text{if } u(x) \leqslant h. \end{cases}$$

Then we have $g_v = g_u \cdot \chi_{\{h<u<k\}}$ and $\mu(\{x \in B(z, R) : v(x) > 0\}) \leqslant \gamma\mu(B(z, R))$. Hence the q-Sobolev inequality together with the doubling property of μ implies

$$\left(\fint_{B(z,R)} v^q \, d\mu\right)^{1/q} \leqslant CR\left(\fint_{B(z,\kappa R)} g_v^q \, d\mu\right)^{1/q},$$

where C depends on γ. Hence

$$\begin{aligned}(k-h)\mu(A(k, R)) = \int_{A(k,R)} v \, d\mu &\leqslant \int_{B(z,R)} v \, d\mu \\ &\leqslant \mu(B(z, R))^{1-1/q}\left(\int_{B(z,R)} v^q \, d\mu\right)^{1/q} \\ &\leqslant CR\,\mu(B(z, R))^{1-1/q}\left(\int_{B(z,\kappa R)} g_v^q \, d\mu\right)^{1/q} \\ &= CR\,\mu(B(z, R))^{1-1/q}\left(\int_{A(h,\kappa R)\setminus A(k,\kappa R)} g_u^q \, d\mu\right)^{1/q}.\end{aligned}$$

Since $q < p$, it follows from Hölder's inequality and the definition of $\mathrm{DG}_p(B(z, 2\kappa R))$ that

$$\begin{aligned}&\int_{A(h,\kappa R)\setminus A(k,\kappa R)} g_u^q\, d\mu \\ &\leqslant \Big(\int_{A(h,\kappa R)\setminus A(k,\kappa R)} g_u^p\, d\mu\Big)^{q/p} (\mu(A(h, \kappa R)) - \mu(A(k, \kappa R)))^{1-q/p} \\ &\leqslant C\Big(\frac{1}{R^p}\int_{A(h,2\kappa R)} (u-h)^p\, d\mu\Big)^{q/p} (\mu(A(h, \kappa R)) - \mu(A(k, \kappa R)))^{1-q/p}\,.\end{aligned}$$

Hence $(k - h)\mu(A(k, R))$ is bounded by

$$\begin{aligned}&C\mu(B(z, R))^{1-1/q}\Big(\int_{A(h,2\kappa R)} (u-h)^p\, d\mu\Big)^{1/p} (\mu(A(h, \kappa R)) - \mu(A(k, \kappa R)))^{1/q-1/p} \\ &\leqslant C\mu(B(z, R))^{1-1/q}(M-h)\,(\mu(B(z, 2\kappa R))^{1/p}\,(\mu(A(h, \kappa R)) - \mu(A(k, \kappa R)))^{1/q-1/p} \\ &\leqslant C(M-h)\,(\mu(B(z, R))^{1-(1/q-1/p)}\,(\mu(A(h, \kappa R)) - \mu(A(k, \kappa R)))^{1/q-1/p}\,.\end{aligned}$$

Therefore (3.4) follows and the claim is proved.

Now we let $k_i = M - 2^{-i}s$ and apply (3.4) with $k = k_i$ and $h = k_{i-1}$. Note that if $i \geqslant 1$, then

$$\mu(A(M - 2^{1-i}s, R)) \leqslant \mu(A(M - s, R)) \leqslant \gamma\mu(B(z, R)).$$

Hence (3.4) becomes

$$\left(\frac{2^{-i}s\mu(A(M-2^{-i}s, R))}{2^{1-i}s\mu(B(z, R))}\right)^{pq/(p-q)} \leqslant C\,\frac{\mu(A(M - 2^{1-i}s, \kappa R)) - \mu(A(M - 2^{-i}s, \kappa R))}{\mu(B(z, R))}.$$

Adding the above inequality for $i = 1, \dots, v$ and using the monotonicity, we obtain

$$v\left(\frac{\mu(A(M - 2^{-v}s, R))}{\mu(B(z, R))}\right)^{pq/(p-q)} \leqslant C\,\frac{\mu(A(M - s, \kappa R))}{\mu(B(z, R))} \leqslant C\,\frac{\mu(B(z, \kappa R))}{\mu(B(z, R))} \leqslant C.$$

Hence, for arbitrary $\delta > 0$, choosing $v > C\,\delta^{-pq/(p-q)}$ and setting $\eta = 2^{-v}$ we see that

$$\frac{\mu(A(M - \eta s, R))}{\mu(B(z, R))} < \delta.$$

Thus the lemma is proved. $\square$

Combining the above lemmas, we obtain the following.

Lemma 3.6. *Let* $0 < R < \mathrm{diam}(X)/(6\kappa)$ *and* $u \in \mathrm{DG}_p(B(z, 2\kappa R))$. *Suppose* $0 \leqslant u \leqslant 1$ *on* $B(z, 2\kappa R)$ *and*

$$\frac{\mu(\{x \in B(z,R) : u(x) > 1-s\})}{\mu(B(z,R))} \leqslant \gamma < 1$$

for some $0 < s < 1$. *Then there exists* $t = t(p, q, \gamma, s) > 0$ *such that*

$$u \leqslant 1 - t \ \text{ on } \ B(z, R/2).$$

Proof. Consider $\delta > 0$ such that $C_7 \delta^{1/p} < 1/2$, where C_7 is the constant from Lemma 3.3. By Lemma 3.5 we find η with $0 < \eta < 1$ such that

$$\frac{\mu(\{x \in B(z,R) : u(x) > 1-\eta s\})}{\mu(B(z,R))} \leqslant \delta.$$

As we have $0 \leqslant (u - (1-\eta s/2))_+ \leqslant \eta s/2$, applying Lemma 3.3 with $k_0 = 1 - \eta s/2$ we obtain

$$\begin{aligned}
\sup_{B(z,R/2)} u &\leqslant 1 - \frac{\eta s}{2} + C_7 \left(\fint_{B(z,R)} \left(u - \left(1 - \frac{\eta s}{2}\right)\right)_+^p d\mu \right)^{1/p} \\
&\leqslant 1 - \frac{\eta s}{2} + C_7 \frac{\eta s}{2} \left(\frac{\mu(\{B(z,R) : u(x) > 1 - \eta s/2\})}{\mu(B(z,R))} \right)^{1/p} \\
&\leqslant 1 - \frac{\eta s}{2} + C_7 \frac{\eta s}{2} \delta^{1/p} \leqslant 1 - \frac{\eta s}{4}.
\end{aligned}$$

Thus the lemma follows with $t = \eta s/4$. □

Corollary 3.7. *Let* $0 < R < \mathrm{diam}(X)/(6\kappa)$ *and* $B(z_1, R/2) \cap B(z_2, R/2) \neq \emptyset$. *Suppose* $u \in \mathrm{DG}_p(B(z_2, 2\kappa R))$ *with* $0 \leqslant u \leqslant 1$ *in* $B(z_2, 2\kappa R)$. *If* $u \leqslant 1 - \varepsilon_1$ *on* $B(z_1, R/2)$ *for some* $\varepsilon_1 > 0$, *then there is a positive constant* $\varepsilon_2 = \varepsilon_2(\varepsilon_1) < 1$ *such that* $u \leqslant 1 - \varepsilon_2$ *on* $B(z_2, R/2)$.

4. Proof of Theorem 2.2

The proof of Theorem 2.2 is given as a series of lemmas. In this section we shall prove the parts of Theorem 2.2 that do not need the Ahlfors regularity of μ. Throughout this section, the standing assumption is that the $(1, p)$-Poincaré inequality is supported on X and that μ is a doubling measure with the exponent Q from the upper volume condition (1.1) satisfying $Q \geqslant p$.

4.1. Condition (ii) implies Condition (iii)

Lemma 4.1. *Condition* (ii) $\Rightarrow$ *Condition* (iii).

The proof of this lemma follows verbatim the proof of the analogous result in [1], and is therefore left to the reader to verify. The only tool needed is the comparison theorem.

4.2. Condition (i) is equivalent to Condition (ii)

Let us recall the following geometric property [8, Proposition 4.4].

Lemma 4.2. *The space X is quasiconvex, i.e., there is a constant $C_8 \geqslant 1$ such that every pair of points $x, y \in X$ can be joined by a curve of length at most $C_8 d(x, y)$. Hence if $x \in E \subsetneq X$, then*

$$\operatorname{dist}(x, X \setminus E) \leqslant \operatorname{dist}(x, \partial E) \leqslant C_8 \operatorname{dist}(x, X \setminus E) \quad \textit{for } x \in E.$$

Proof. See [8, Proposition 4.4] for a proof of the first assertion. For the second assertion it suffices to show the last inequality with $x \in E \subsetneq X$ and $y \in X \setminus E$. There is a curve γ joining x and y with length no more than $C_8 d(x, y)$. Since $x \in E$ and $y \in X \setminus E$, there exists a point $z \in \gamma \cap \partial E$. Hence

$$\operatorname{dist}(x, \partial E) \leqslant d(x, z) \leqslant C_8 d(x, y).$$

Since $y \in X \setminus E$ is arbitrary, we obtain the required inequality. $\square$

Lemma 4.3. *Condition* (i) $\iff$ *Condition* (ii).

The proof in [1] of the result analogous to the above lemma uses the Poisson integral representation of harmonic functions on balls. Since we are dealing with more general (non-linear) values of p, we do not have the Poisson representation. We instead use the local Hölder continuity (Lemma 3.4).

Proof. First suppose that Condition (i) holds. By the definition of $\varphi_{a,\alpha}$ we see that $\varphi_{a,\alpha} \in \Lambda_\alpha(\partial\Omega)$ with $\|\varphi_{a,\alpha}\|_{\Lambda_\alpha(\partial\Omega)} \leqslant 2$. Hence

$$|P_\Omega \varphi_{a,\alpha}(x) - P_\Omega \varphi_{a,\alpha}(y)| \leqslant 2\|P_\Omega\|_{\alpha\to\alpha} d(x, y)^\alpha \quad \text{for } x, y \in \Omega.$$

Since a is a p-regular point by assumption, we obtain Condition (ii) with $C_1 = 2\|P_\Omega\|_{\alpha\to\alpha}$ by letting $y \to a$.

Next suppose that Condition (ii) holds. Let $f \in \Lambda_\alpha(\partial\Omega)$. By the maximum principle

$$\sup_{x\in\Omega} |P_\Omega f(x)| \leqslant \sup_{\xi\in\partial\Omega} |f(\xi)| \leqslant \|f\|_{\Lambda_\alpha(\partial\Omega)}.$$

As Ω is bounded, it now suffices to show that

$$|P_\Omega f(x) - P_\Omega f(y)| \leqslant C\|f\|_{\Lambda_\alpha(\partial\Omega)} d(x,y)^\alpha \quad \text{for } x, y \in \Omega \text{ with } d(x,y) \leqslant 1. \tag{4.1}$$

To this end, let $x, y \in \Omega$ such that $d(x,y) \leqslant 1$. Without loss of generality we may assume that $\operatorname{dist}(x, X\setminus\Omega) \geqslant \operatorname{dist}(y, X\setminus\Omega)$. Let $R = \operatorname{dist}(x, X\setminus\Omega)/(2\kappa)$ with $\kappa \geqslant 1$ from the q-Poincaré inequality. Since $\partial\Omega$ is compact, there is a point $x^* \in \partial\Omega$ such that $\operatorname{dist}(x, \partial\Omega) = d(x, x^*)$. By Lemma 4.2 we have

$$2\kappa R \leqslant d(x, x^*) \leqslant 2\kappa C_8 R. \tag{4.2}$$

Set $f_0 : \partial\Omega \to \mathbb{R}$ by $f_0(\xi) = f(\xi) - f(x^*)$ for $\xi \in \partial\Omega$. Since

$$|f_0(\xi)| \leqslant \begin{cases} |f(\xi) - f(x^*)| \leqslant \|f\|_{\Lambda_\alpha(\partial\Omega)} d(\xi, x^*)^\alpha \leqslant \|f\|_{\Lambda_\alpha(\partial\Omega)} \varphi_{x^*,\alpha}(\xi) & \text{if } d(\xi, x^*) \leqslant 1, \\ |f(\xi)| + |f(x^*)| \leqslant 2\|f\|_{\Lambda_\alpha(\partial\Omega)} \leqslant 2\|f\|_{\Lambda_\alpha(\partial\Omega)} \varphi_{x^*,\alpha}(\xi) & \text{if } d(\xi, x^*) > 1, \end{cases}$$

it follows from Condition (ii) that

$$|P_\Omega f_0(z)| \leqslant 2C_1 \|f\|_{\Lambda_\alpha(\partial\Omega)} d(z, x^*)^\alpha \quad \text{for } z \in \Omega. \tag{4.3}$$

The rest of the proof is split into two cases.

Case 1: $d(x,y) \leqslant d(x,x^*)/(2\kappa C_8)$. Then $d(x,y) \leqslant R = d(x, X\setminus\Omega)/(2\kappa)$ by (4.2). Since $P_\Omega f_0$ is p-harmonic in $\mathrm{DG}_p(B(x, 2\kappa R))$, it follows from Lemma 3.4 that

$$\operatorname*{osc}_{B(x,r)} P_\Omega f_0 \leqslant C\left(\frac{r}{R}\right)^{\alpha_0} \operatorname*{osc}_{B(x,R)} P_\Omega f_0 \quad \text{for } 0 < r \leqslant R.$$

We observe from (4.2) that $d(z, x^*) \leqslant d(x,z) + d(x,x^*) \leqslant (1 + 2\kappa C_8)R$ when $z \in B(x,R)$. Thus by (4.3) we have $\operatorname{osc}_{B(x,R)} P_\Omega f_0 \leqslant C\|f\|_{\Lambda_\alpha(\partial\Omega)} R^\alpha$. Hence

$$\begin{aligned} |P_\Omega f(x) - P_\Omega f(y)| = |P_\Omega f_0(x) - P_\Omega f_0(y)| &\leqslant C\left(\frac{d(x,y)}{R}\right)^{\alpha_0} \|f\|_{\Lambda_\alpha(\partial\Omega)} R^\alpha \\ &\leqslant C\|f\|_{\Lambda_\alpha(\partial\Omega)} d(x,y)^\alpha. \end{aligned}$$

In the last inequality, we have used the facts that $\alpha \leqslant \alpha_0$ and $d(x,y)/R \leqslant 1$.

Case 2: $d(x, y) \geqslant d(x, x^*)/(2\kappa C_8)$. Then $d(y, x^*) \leqslant d(x, y) + d(x, x^*) \leqslant (1 + 2\kappa C_8) d(x, y)$. Therefore we have from (4.3) that

$$\begin{aligned}|P_\Omega f(x) - P_\Omega f(y)| &= |P_\Omega f_0(x) - P_\Omega f_0(y)| \leqslant |P_\Omega f_0(x)| + |P_\Omega f_0(y)| \\ &\leqslant 2C_1 \|f\|_{\Lambda_\alpha(\partial\Omega)} (d(x, x^*)^\alpha + d(y, x^*)^\alpha) \\ &\leqslant 2C_1 \|f\|_{\Lambda_\alpha(\partial\Omega)} ((2\kappa C_8)^\alpha + (1 + 2\kappa C_8)^\alpha) d(x, y)^\alpha.\end{aligned}$$

Now combining both cases we obtain (4.1). The proof is complete. □

4.3. Condition (iv) implies Condition (iii)

Lemma 4.4. *Condition* (iv) $\Rightarrow$ *Condition* (iii).

Proof. Let $a \in \partial\Omega$ and $0 < r < r_0$ with r_0 as in the statement of LHMD(α). Since $\chi_{\Omega\cap S(a,r)} \geqslant \omega_p(\partial\Omega \setminus B(a, r); \Omega)$ on $\partial(\Omega \cap B(a, r))$, it follows from the comparison theorem that

$$\omega_p(\Omega \cap S(a, r); \Omega \cap B(a, r)) = \overline{P}_{\Omega\cap B(a,r)}[\chi_{\Omega\cap S(a,r)}] \geqslant \overline{P}_{\Omega\cap B(a,r)}[\omega_p(\partial\Omega \setminus B(a, r); \Omega)]$$

on $\Omega \cap B(a, r)$. As Ω is a p-regular domain, every point on $\partial\Omega \cap \overline{B(a, r)}$ is a p-regular boundary point for $\Omega \cap B(a, r)$ (see [2]). Since the upper Perron solution is the largest p-harmonic solution to a given boundary data problem (see [4]), we have

$$\omega_p(\Omega \cap S(a, r); \Omega \cap B(a, r)) \geqslant \omega_p(\partial\Omega \setminus B(a, r); \Omega) \quad \text{on } \Omega \cap B(a, r).$$

Now it is clear that Condition (iv) implies Condition (iii). □

4.4. Condition (iv) with $\alpha' > \alpha$ yields Condition (ii)

The counterpart of the following lemma was given in [1]. The proof given there heavily relied on the linearity. Here, we shall employ a simple iteration argument, applicable to the non-linear situation as well.

Lemma 4.5. LHMD(α') *for some* $\alpha' > \alpha \Rightarrow$ *Condition* (ii).

Proof. Let $a \in \partial\Omega$ and $u = P_\Omega \varphi_{a,\alpha}$. We will show that $u(x) \leqslant C \operatorname{dist}(x, a)^\alpha$. Set

$$\psi(\rho) = \sup_{\Omega\cap S(a,\rho)} u(x).$$

It suffices to show that $\psi(\rho) \leqslant C\rho^\alpha$ for small $\rho > 0$. Let $0 < \rho < r < 1$. Then by the definition of $\varphi_{a,\alpha}$ we see that $u \leqslant r^\alpha + \psi(r)\chi_{S(a,r)\cap\Omega}$ on the boundary of $\Omega \cap B(a,r)$. The comparison theorem yields

$$u(x) \leqslant r^\alpha + \psi(r)\omega_p(x; \Omega \cap S(a,r), \Omega \cap B(a,r)) \quad \text{for } x \in \Omega \cap B(a,r).$$

Hence, LHMD (α') implies

$$\psi(\rho) \leqslant r^\alpha + C_3 \left(\frac{\rho}{r}\right)^{\alpha'} \psi(r).$$

Let $\tau = (2C_3)^{1/(\alpha'-\alpha)} > 1$. If $\tau\rho \leqslant r$, then $C_3(\rho/r)^{\alpha'-\alpha} \leqslant 1/2$. Thus

$$\psi(\rho) \leqslant r^\alpha + \frac{1}{2}\left(\frac{\rho}{r}\right)^\alpha \psi(r),$$

whenever $0 < \tau\rho \leqslant r < 1$. Let $\rho_j = \tau^j\rho$ and let $k \geqslant 1$ be the integer such that $\tau^k\rho \leqslant 1 < \tau^{k+1}\rho$. Then we obtain

$$\psi(\rho_j) \leqslant \rho_{j+1}^\alpha + \frac{1}{2\tau^\alpha}\psi(\rho_{j+1}) \quad \text{for } j = 0, \ldots, k-1.$$

Hence

$$\begin{aligned}
\psi(\rho) = \psi(\rho_0) &\leqslant \rho_1^\alpha + \frac{1}{2\tau^\alpha}\psi(\rho_1) \\
&\leqslant \rho_1^\alpha + \frac{1}{2\tau^\alpha}\left(\rho_2^\alpha + \frac{1}{2\tau^\alpha}\psi(\rho_2)\right) = \rho_1^\alpha + \frac{\rho_2^\alpha}{2\tau^\alpha} + \frac{1}{(2\tau^\alpha)^2}\psi(\rho_2) \\
&\leqslant \tau^\alpha\rho^\alpha + \frac{(\tau^2\rho)^\alpha}{2\tau^\alpha} + \cdots + \frac{(\tau^k\rho)^\alpha}{(2\tau^\alpha)^{k-1}} + \frac{1}{(2\tau^\alpha)^k} \\
&\leqslant \tau^\alpha\rho^\alpha\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{k-1}}\right) + \left(\frac{1}{\tau^k}\right)^\alpha \leqslant 3\tau^\alpha\rho^\alpha.
\end{aligned}$$

Here we have used the facts that $\psi(\rho_k) \leqslant 1$ and $\tau^{k+1}\rho > 1$. Thus the desired inequality follows. □

5. Proof of Theorem 2.2 continued

What remains to be proved is the most challenging part of Theorem 2.2.

Lemma 5.1. *Suppose μ satisfies the Ahlfors Q-regularity. Then the* GHMD(α) *and the* LHMD(α) *conditions are equivalent*, *i.e.*, *Condition* (iii) $\Longleftrightarrow$ *Condition* (iv).

We have already seen that the LHMD(α) implies the GHMD(α). It is sufficient to show the converse part. The proof consists of a series of lemmas. In the rest of this section we assume the Ahlfors Q-regularity of μ. We begin with some geometric properties. By $A(x, r, R)$ we denote the annulus $B(x, R) \setminus B(x, r)$ with center at x and radii r and R.

Definition. Given a set $E \subset X$, we say that E is *uniformly perfect* if there are constants $0 < C_9 < 1$ and $r_0 > 0$ such that $A(x, C_9 r, r) \cap E \neq \emptyset$ for every $x \in E$ and all $0 < r < r_0$.

Definition. We say that X is *linearly locally connected* (abbreviated to LLC) if there are constants $C_{10} > 1$ and $r_0 > 0$ such that for every $a \in X$ and $0 < r < r_0$ each pair of points $x, y \in S(a, r)$ can be connected by a curve lying in $A(a, r/C_{10}, C_{10}r)$.

The LLC property was introduced by Heinonen–Koskela [10]. It is known that the Ahlfors Q-regularity and p-Poincaré inequality with $p \leqslant Q$ together yield the LLC property of X [10, Corollary 5.8], [8, Proposition 4.5].

5.1. Condition (iii) implies uniform perfectness

In this section we shall prove the following.

Lemma 5.2. *If Ω satisfies the* GHMD *for some α, then $\partial\Omega$ is uniformly perfect.*

To prove the above lemma we need the following capacitary estimates for condensers. This estimate holds true even for general doubling measures with (1.1), not necessarily Ahlfors regular.

Lemma 5.3. *If* $0 < 2r \leqslant R < \operatorname{diam}(\Omega)/2$, *then*

$$\operatorname{Cap}_p(\overline{B(a,r)}, B(a,R)) \leqslant \begin{cases} Cr^{Q-p} & \text{if } p < Q, \\ C\left(\log \dfrac{R}{r}\right)^{1-p} & \text{if } p = Q. \end{cases}$$

Proof. It is easy to find $u \in N_0^{1,p}(B(a, 2r))$ such that $u = 1$ on $\overline{B(a, r)}$ and $g_u \leqslant C/r$. Hence $\operatorname{Cap}_p(\overline{B(a,r)}, B(a,R)) \leqslant Cr^{-p}\mu(B(a,r)) \leqslant Cr^{Q-p}$ by (1.1). If $p = Q$, then the better estimate can be proved as follows. Let $k \geqslant 1$ be the unique positive integer such that $2^k r \leqslant R < 2^{k+1}r$, and let $\psi(t)$ be a piecewise linear function on $[0, \infty)$ such that $\psi(t) = 1$ for $0 \leqslant t \leqslant r$, $\psi(2^i r) = 1 - i/k$ for $i = 0, \ldots, k$, and $\psi(t) = 0$ for $t \geqslant 2^k r$.

Then $u(x) = \psi(d(x,a)) \in N_0^{1,p}(B(a,R))$ with

$$\int_{2^i r \leqslant d(x,a) < 2^{i+1} r} g_u^p \, d\mu \leqslant \left(\frac{1}{k2^i r}\right)^p \mu(B(a, 2^{i+1} r)) \leqslant Ck^{-p}(2^i r)^{Q-p}$$

for $i = 0, \ldots, k$. Summing up the above inequalities, we obtain the required estimates. $\square$

Proof of Lemma 5.2. Let $a \in \partial\Omega$ and $0 < \rho_1 < \rho_2 < \operatorname{diam}(\Omega)/2$. Suppose $A(a, \rho_1, \rho_2)$ does not intersect $\partial\Omega$. We will prove that ρ_1/ρ_2 cannot be too close to 0. Without loss of generality, we may assume that $\rho_1 \leqslant \rho_2/(2C_{10}^2)$. By the LLC property we see that $A(a, C_{10}\rho_1, \rho_2/C_{10}) \subset \Omega$. For simplicity we let $r = C_{10}\rho_1$ and $R = \rho_2/C_{10}$. Then

$$A(a, r, R) \subset \Omega. \tag{5.1}$$

Letting ρ_2 be larger if necessary, we may assume that $S(a, C_{10}R)$ has a point $b \in \partial\Omega$. Let $K = \overline{B(a,r)} \setminus \Omega$. Observe from (5.1) that $K = B(a, R) \setminus \Omega$. By Lemma 5.3,

$$\operatorname{Cap}_p(K, \Omega \cup K) \leqslant \operatorname{Cap}_p(\overline{B(a,r)}, B(a,R)) \leqslant \begin{cases} Cr^{Q-p} & \text{if } p < Q, \\ C\left(\log \dfrac{R}{r}\right)^{1-p} & \text{if } p = Q. \end{cases} \tag{5.2}$$

Let u_K be the p-potential for the condenser $(K, \Omega \cup K)$, i.e. $u_K = 1$ p-q.e. on K, $u_K = 0$ p-q.e. on $X \setminus (\Omega \cup K)$ and

$$\operatorname{Cap}_p(K, \Omega \cup K) = \int_X g_{u_K}^p \, d\mu.$$

Since $r \leqslant R/2$ and $A(a, r, R)$ does not intersect $\partial\Omega$, we have $u_K \leqslant \omega_p(\partial\Omega \setminus B(b, R/2); \Omega)$ on Ω. Hence by the GHMD(α) condition,

$$u_K(x) \leqslant C_2 \Big(\frac{d(x,b)}{R/2}\Big)^\alpha \quad \text{for } x \in \Omega \cap B(b, R/2).$$

Setting $\beta = (2(3C_2)^{1/\alpha})^{-1}$ and noting that $u_K = 0$ on $B(b, R/2) \setminus \Omega$, we obtain $u_K \leqslant 1/3$ on $B(b, \beta R)$. It follows from (5.1) and the comparison principle that

$$u_K = 1 - \omega_p(\partial\Omega \setminus B(a, R); \Omega) \quad \text{ on } \Omega,$$

so that GHMD(α) together with the fact that $u_K = 1$ on $B(a, \beta R) \setminus \Omega \subset B(a, R) \setminus \Omega$ yields again $u_K \geqslant 2/3$ on $B(a, \beta R)$. Let $v = \max\{u_K, 1/3\} - 1/3 \geqslant 0$. Then

$$\frac{\mu(\{x \in B(a, 2C_{10}R) : v(x) = 0\})}{\mu(B(a, 2C_{10}R))} \geqslant \frac{\mu(B(b, \beta R))}{\mu(B(a, 2C_{10}R))} \geqslant \gamma,$$

where $\gamma > 0$ depends only on β. Hence the p-Sobolev inequality (2.1) implies

$$\left(\fint_{B(a,2C_{10}R)} v^p \, d\mu\right)^{1/p} \leqslant CR \left(\fint_{B(a,2C_{10}\kappa R)} g_v^p \, d\mu\right)^{1/p}.$$

Since by the doubling property of μ we have

$$\int_{B(a,2C_{10}R)} v^p \, d\mu \geqslant \int_{B(a,\beta R)} (1/3)^p \, d\mu \geqslant C\mu(B(a, 2C_{10}R)),$$

we obtain

$$\mathrm{Cap}_p(K, \Omega \cup K) = \int g_{u_K}^p \, d\mu \geqslant \int_{B(a,2C_{10}\kappa R)} g_v^p \, d\mu \geqslant CR^{-p}\mu(B(a, 2C_{10}R)) \geqslant CR^{Q-p}.$$

Here, the Ahlfors Q-regularity is used in the last inequality. This, together with (5.2), implies that R/r is bounded and therefore so is ρ_2/ρ_1. The lemma is proved. □

5.2. Condition (iii) implies Condition (iv)

In this section we shall prove Lemma 5.1 and thus complete the proof of Theorem 2.2.

Proof of Lemma 5.1. Let us assume the GHMD(α) property and prove the LHMD(α) property. Let $a \in \partial\Omega$ and $0 < r < r_0$. By the uniform perfectness of $\partial\Omega$ (Lemma 5.2), we find ρ such that $S(a, \rho) \cap \partial\Omega \neq \emptyset$ and $r/C_9 \leqslant \rho < r$. Let c be a small positive number to be determined later. By the LLC property and the doubling property of μ, we can find finitely many points $z_1, \ldots z_N \in A(a, \rho/C_{10}, C_{10}\rho)$ such that the union $\bigcup_{j=1}^N B(z_j, cr)$ is a covering of $S(a, \rho)$ that forms a chain, that is, for every $j, k \in \{1, \ldots, N\}$ there is a subcollection of balls $B_1, \ldots B_l$ such that $B(z_j, cr) = B_1$, $B(z_k, cr) = B_l$, and for $i \in \{1, \ldots, l-1\}$, $B_i \cap B_{i+1}$ is non-empty. Here N depends only on c and the space (X, d, μ). Observe that

$$\begin{aligned}\bigcup_{j=1}^N B(z_j, 4\kappa cr) &\subset A\Big(a, \frac{\rho}{C_{10}} - 4\kappa cr, C_{10}\rho + 4\kappa cr\Big) \\ &\subset A\Big(a, \Big(\frac{1}{C_9C_{10}} - 4\kappa c\Big)r, (C_{10} + 4\kappa c)r\Big). \qquad (5.3)\end{aligned}$$

Let $c > 0$ be small enough so that

$$4c\kappa \leqslant \frac{1}{2C_9C_{10}} =: \eta. \tag{5.4}$$

Consider

$$u = \begin{cases} \omega_p(\partial\Omega \cap B(a, \eta r); \Omega) & \text{on } \Omega, \\ 0 & \text{on } X \setminus \Omega. \end{cases}$$

Then $0 \leqslant u \leqslant 1$ on X and u is a p-subminimizer in $X \setminus \overline{B(a, \eta r)} \supset \bigcup_{j=1}^N B(z_j, 4\kappa cr)$ by (5.3) and (5.4). Hence from the discussion in the second section, $u \in \mathrm{DG}_p(\cup_{j=1}^N B(z_j, 4\kappa cr))$. Fix $z^* \in \partial\Omega \cap S(a, \rho)$. Without loss of generality we may assume that $z^* \in B(z_1, cr)$. Since

$$B(z^*, (4\kappa - 1)cr) \subset B(z_1, 4\kappa cr) \subset X \setminus \overline{B(a, \eta r)},$$

it follows from the comparison principle that $u \leqslant \omega_p(\partial\Omega \setminus B(z^*, (4\kappa - 1)cr); \Omega)$ on Ω. See Fig. 1.
Hence the GHMD property yields

$$u \leqslant \tfrac{1}{2} \quad \text{on } B(z^*, \beta r) \cap \Omega$$

for some $\beta > 0$ independent of a and r. Since $u = 0$ on $X \setminus \Omega$, it follows that $u \leqslant 1/2$ on $B(z^*, \beta r)$. Hence Lemma 3.6 with $R = 2cr$ yields that $u \leqslant 1 - \varepsilon_1$ on $B(z_1, cr)$ for some $\varepsilon_1 > 0$ independent of a and r. Since $\bigcup_{j=1}^N B(z_j, cr)$ is a chain, we find some ball, say $B(z_2, cr)$, intersecting $B(z_1, cr)$. Then Corollary 3.7 gives $u \leqslant 1 - \varepsilon_2$ on $B(z_2, cr)$ for some $\varepsilon_2 > 0$. We may repeat this argument finitely many times until, by the finiteness of the cover and by its chain property, we eventually obtain $u \leqslant 1 - \varepsilon_0$ on $\bigcup_{j=1}^N B(z_j, cr)$ for some $\varepsilon_0 > 0$ that is independent of a and r. In particular, $u \leqslant 1 - \varepsilon_0$ on $S(a, \rho)$. Since

$$\omega_p(\partial\Omega \cap B(a, \eta r); \Omega) + \omega_p(\partial\Omega \setminus B(a, \eta r); \Omega) = 1 \quad \text{on } \Omega,$$

it follows in particular that $\omega_p(\partial\Omega \setminus B(a, \eta r); \Omega) \geqslant \varepsilon_0$ on $\Omega \cap S(a, \rho)$. By the comparison principle we now have

$$\frac{1}{\varepsilon_0}\, \omega_p(\partial\Omega \setminus B(a, \eta r); \Omega) \geqslant \omega_p(\Omega \cap S(a, \rho); \Omega \cap B(a, \rho)) \quad \text{on } \Omega \cap B(a, \rho).$$

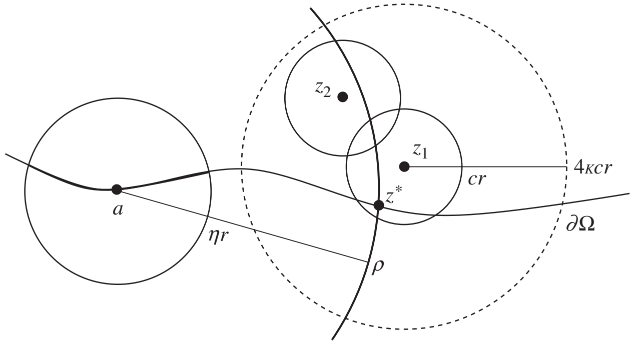

Fig. 1. $u \in \mathrm{DG}_p(\bigcup_{j=1}^{N} B(z_j, 4\kappa cr))$ and $u \leqslant \omega_p(\partial\Omega \setminus B(z^*, (4\kappa - 1)cr); \Omega)$.

Hence the GHMD(α) property yields

$$\omega_p(x; \Omega \cap S(a, r), \Omega \cap B(a, r)) \leqslant \omega_p(x; \Omega \cap S(a, \rho), \Omega \cap B(a, \rho)) \leqslant \frac{C_2}{\varepsilon_0}\left(\frac{d(x, a)}{\eta r}\right)^{\alpha}$$

for all $x \in \Omega \cap B(a, \rho)$. Because $\rho \geqslant r/C_9$, the required inequality holds also for points x in $\Omega \cap B(a, r) \setminus B(a, \rho)$. Therefore the LHMD($\alpha$) property follows. □

6. Proof of Theorem 2.5

For the proof of Theorem 2.5, it is sufficient to show the following.

Lemma 6.1. *The* LHMD(α) *property holds for some* $\alpha > 0$ *if and only if* $X \setminus \Omega$ *is uniformly p-fat.*

To this end, we shall use capacity estimates and the boundary regularity. Observe the following lemma from the results in [5], [6, Lemma 5.5].

Lemma 6.2. *Let* $a \in X$ *and* $0 < r < r_0$.

(i) *If* $0 < s \leqslant 1$, *then*

$$\mathrm{Cap}_p(\overline{B(a, sr)}, B(a, 2r)) \leqslant \mathrm{Cap}_p(\overline{B(a, r)}, B(a, 2r)) \leqslant C\, \mathrm{Cap}_p(\overline{B(a, sr)}, B(a, 2r)),$$

where C *depends only on* s.

(ii) *If* $E \subset \overline{B(a, r)}$ *and* $t \geqslant 1$, *then*

$$\mathrm{Cap}_p(E, B(a, 2tr)) \leqslant \mathrm{Cap}_p(E, B(a, 2r)) \leqslant C\, \mathrm{Cap}_p(E, B(a, 2tr)),$$

where C *depends only on* t.

For $a \in X$, $E \subset X$, and $r > 0$, we let

$$\varphi(a, E, r) = \frac{\mathrm{Cap}_p(E \cap B(a,r), B(a,2r))}{\mathrm{Cap}_p(B(a,r), B(a,2r))}.$$

Then the uniform p-fatness of E is restated as $\varphi(a, E, r) \geqslant C_4$ for $a \in E$ and $0 < r < r_0$. Let us observe that the validity of this inequality for $a \in \partial E$ is sufficient for us to conclude the uniform p-fatness of E.

Lemma 6.3. *If $\varphi(a, E, r) \geqslant C$ for every $a \in \partial E$ and $0 < r < r_0$, then E is uniformly p-fat.*

Proof. Let a be an arbitrary interior point of E. It is sufficient to show $\varphi(a, E, r) \geqslant C$. Let $R = d(a, X \setminus E) > 0$. By the quasiconvexity (Lemma 4.2) we find $b \in \partial E$ such that $R \leqslant d(a,b) \leqslant C_8 R$. We have the following two cases.

Case 1: $r \leqslant 2C_8 R$. Then $\overline{B(a, r/2C_8)} \subset E$. Hence Lemma 6.2 yields

$$\varphi(a, E, r) \geqslant \frac{\mathrm{Cap}_p(\overline{B(a, r/2C_8)}, B(a,2r))}{\mathrm{Cap}_p(\overline{B(a,r)}, B(a,2r))} \geqslant C.$$

Case 2: $r \geqslant 2C_8 R$. Then $\overline{B(b, r/2)} \subset \overline{B(a,r)} \subset \overline{B(b, 3r/2)}$ and $B(b,r) \subset B(a,2r) \subset B(b, 5r/2)$. Hence Lemma 6.2 yields

$$\begin{aligned}\mathrm{Cap}_p(E \cap \overline{B(a,r)}, B(a,2r)) &\geqslant \mathrm{Cap}_p(E \cap \overline{B(b,r/2)}, B(a,2r))\\ &\geqslant \mathrm{Cap}_p(E \cap \overline{B(b,r/2)}, B(b,5r/2))\\ &\geqslant C\,\mathrm{Cap}_p(E \cap \overline{B(b,r/2)}, B(b,r))\end{aligned}$$

and

$$\begin{aligned}\mathrm{Cap}_p(\overline{B(a,r)}, B(a,2r)) &\leqslant \mathrm{Cap}_p(\overline{B(b,3r/2)}, B(a,2r))\\ &\leqslant C\,\mathrm{Cap}_p(\overline{B(b,r/2)}, B(b,2r)) \leqslant C\,\mathrm{Cap}_p(\overline{B(b,r/2)}, B(b,r)).\end{aligned}$$

Therefore $\varphi(a, E, r) \geqslant C\varphi(b, E, r/2)$. Since $\varphi(b, E, r/2) \geqslant C$ for $b \in \partial E$ by assumption, we have $\varphi(a, E, r) \geqslant C$. The proof is now complete. $\square$

The following estimate plays an important role in the topic of modulus of continuity of the solution of the Dirichlet problem. See [17] for a version in the classical case and [6, Lemma 5.7] for a proof of the present version.

Lemma 6.4. *Let $a \in \partial\Omega$ and fix $r > 0$. Let u be the p-potential for $\overline{B(a,r)} \setminus \Omega$ with respect to $B(a,5r)$. Then*

$$1 - u(x) \leqslant \exp\left(-C \int_\rho^r \varphi(a, X \setminus \Omega, t)^{1/(p-1)} \frac{dt}{t}\right) \quad \textit{for } 0 < \rho \leqslant r \textit{ and } x \in B(a,\rho).$$

Proof of Lemma 6.1. First suppose that $X \setminus \Omega$ is uniformly p-fat. Let $a \in \partial\Omega$, $0 < r < r_0$, and let u be the p-potential for $\overline{B(a,r/5)} \setminus \Omega$ with respect to $B(a,r)$. By the comparison principle we have

$$\omega_p(\Omega \cap S(a,r); \Omega \cap B(a,r)) \leqslant 1 - u \quad \text{on } \Omega \cap B(a,r).$$

In view of Lemma 6.4 we have

$$\omega_p(x; \Omega \cap S(a,r), \Omega \cap B(a,r)) \leqslant 1 - u(x) \leqslant C\left(\frac{\rho}{r/5}\right)^\delta \quad \text{for } x \in B(a,\rho)$$
$$\text{and } 0 < \rho \leqslant r/5,$$

where $\delta > 0$ depends only on C_4 and p. Thus LHMD(δ) follows.

Conversely, suppose that LHMD(α) holds for some $\alpha > 0$. In light of Lemma 6.3, it is sufficient to show $\varphi(a, X \setminus \Omega, r) \geqslant C$ for every $a \in \partial\Omega$ and $0 < r < r_0$. Fix $a \in \partial\Omega$ and $0 < r < r_0$, and let v be the p-potential for $\overline{B(a,r)} \setminus \Omega$ with respect to $B(a,2r)$. Then the comparison principle yields

$$\omega_p(\Omega \cap S(a,r); \Omega \cap B(a,r)) \geqslant 1 - v \quad \text{on } \Omega \cap B(a,r).$$

In view of the LHMD(α) we find $C_{11} > 1$ such that

$$\omega_p(\Omega \cap S(a,r); \Omega \cap B(a,r)) \leqslant \tfrac{1}{2} \quad \text{on } \Omega \cap \overline{B(a,r/C_{11})}.$$

Hence, $v \geqslant 1/2$ on $\Omega \cap \overline{B(a,r/C_{11})}$. Since $v = 1$ p-q.e. on $\overline{B(a,r)} \setminus \Omega$, we have $v \geqslant 1/2$ p-q.e. on $\overline{B(a,r/C_{11})}$, so that $2v$ is an admissible function for computing the relative capacity $\mathrm{Cap}_p(\overline{B(a,r/C_{11})}, B(a,2r))$. Therefore

$$\mathrm{Cap}_p(\overline{B(a,r/C_{11})}, B(a,2r)) \leqslant \int_{B(a,2r)} (2g_v)^p \, d\mu = 2^p \, \mathrm{Cap}_p(\overline{B(a,r)} \setminus \Omega, B(a,2r)).$$

By Lemma 6.2 we have

$$\varphi(a, X \setminus \Omega, r) = \frac{\mathrm{Cap}_p(\overline{B(a,r)} \setminus \Omega, B(a,2r))}{\mathrm{Cap}_p(\overline{B(a,r)}, B(a,2r))} \geqslant 2^{-p} \frac{\mathrm{Cap}_p(\overline{B(a,r/C_{11})}, B(a,2r))}{\mathrm{Cap}_p(\overline{B(a,r)}, B(a,2r))} \geqslant C.$$

Thus the required inequality follows. $\square$

Proof of Corollary 2.6. Suppose that E satisfies the volume density condition (2.6). It is sufficient to show that E satisfies the capacity density condition (2.5) as well. Let $a \in E$ and let $r > 0$ be sufficiently small, say $r < \operatorname{diam}(X)/(4\kappa)$. Take a compact subset $K \subset E \cap B(a,r)$ such that $\mu(K) \geqslant C_5 \mu(B(a,r))/2$. Let u_K be the p-capacitary potential for the condenser $(K, B(a, 2\kappa r))$. Then $u_K = 1$ q.e. on K and hence μ-a.e. on K. Observe that $0 \leqslant 1 - u_K \leqslant 1$ on X and as $1 - u_K$ is a p-quasisubminimizer on $B(a, 2\kappa r)$, we have $1 - u_K \in \mathrm{DG}_p(B(a, 2\kappa r))$. In view of Lemma 3.6 we have

$$1 - u_K \leqslant 1 - \varepsilon \quad \text{on } B(a, r/2)$$

for some $\varepsilon > 0$. Hence

$$\begin{aligned}\mathrm{Cap}_p(B(a,r/2), B(a,2\kappa r)) &\leqslant \frac{1}{\varepsilon^p}\int g_{u_K}^p \, d\mu = \frac{\mathrm{Cap}_p(K, B(a, 2\kappa r))}{\varepsilon^p} \\ &\leqslant \frac{\mathrm{Cap}_p(K, B(a, 2r))}{\varepsilon^p}.\end{aligned}$$

Now by Lemma 6.2 and the monotonicity of the capacity we see that E satisfies the capacity density condition (2.5). □

7. Further generalizations

So far, we have regarded P_Ω as an operator from $\Lambda_\alpha(\partial\Omega)$ to $\Lambda_\alpha(\Omega)$ with the same exponent α. Let $0 < \beta \leqslant \alpha$. In this section, we regard P_Ω as an operator from $\Lambda_\alpha(\partial\Omega)$ to $\Lambda_\beta(\Omega)$. Let us begin with the proof of Proposition 2.1.

Proof of Proposition 2.1. It is clear that if Ω has a p-trivial point, then Ω is p-irregular. Conversely, suppose that Ω has no p-trivial point. For an arbitrary point $a \in \partial\Omega$ set $u = P_\Omega \varphi_{a,\alpha}$. We claim

$$\lim_{\Omega \ni x \to b} u(x) = \varphi_{a,\alpha}(b) \quad \text{for } b \in \partial\Omega. \tag{7.1}$$

Let $b \in \partial\Omega$ and $r > 0$. By assumption u is β-Hölder continuous, and hence

$$|u(x) - u(y)| \leqslant C r^\beta \quad \text{for } x, y \in B(b, r) \cap \Omega.$$

Since b is not p-trivial, we find a p-regular boundary point $b' \in \partial\Omega \cap B(b,r)$ by the Kellogg property [4]. Letting $y \to b'$, we obtain $|u(x) - \varphi_{a,\alpha}(b')| \leqslant C r^\beta$. By definition $|\varphi_{a,\alpha}(b) - \varphi_{a,\alpha}(b')| \leqslant d(b,b')^\alpha \leqslant (2r)^\alpha$, so that

$$|u(x) - \varphi_{a,\alpha}(b)| \leqslant C r^\beta + (2r)^\alpha \quad \text{for } x \in B(b,r).$$

Letting $r \to 0$, we obtain (7.1).

Since $\varphi_{a,\alpha}(a) = 0$ and $\varphi_{a,\alpha}(b) > 0$ for $b \in \partial\Omega \setminus \{a\}$ by (7.1), it follows that u is a barrier function at a and hence a is a p-regular boundary point. See [2] for a discussion on barriers and p-regularity. Hence Ω is a p-regular domain from the arbitrariness of $a \in \partial\Omega$. $\square$

Let us observe that some parts of Theorem 2.2 are extended in a straightforward manner.

Theorem 7.1. *Let $0 < \beta \leqslant \alpha \leqslant \alpha_0$ and let Ω be a p-regular domain. Consider the following four conditions*:

(i) $\|P_\Omega\|_{\alpha\to\beta} < \infty$.
(ii) *There exists a constant C_{12} such that whenever $a \in \partial\Omega$,*

$$P_\Omega \varphi_{a,\alpha}(x) \leqslant C_{12} d(x,a)^\beta \quad \text{for every } x \in \Omega. \tag{7.2}$$

(iii) GHMD(α, β). *There exist constants $C_{13} \geqslant 1$ and $r_0 > 0$ such that whenever $a \in \partial\Omega$ and $0 < r < r_0$,*

$$\omega_p(x; \partial\Omega \setminus B(a,r), \Omega) \leqslant C_{13} \frac{d(x,a)^\beta}{r^\alpha} \quad \text{for every } x \in \Omega \cap B(a,r).$$

(iv) LHMD(α, β). *There exist constants $C_{14} \geqslant 1$ and $r_0 > 0$ such that whenever $a \in \partial\Omega$ and $0 < r < r_0$,*

$$\omega_p(x; \Omega \cap S(a,r), \Omega \cap B(a,r)) \leqslant C_{14} \frac{d(x,a)^\beta}{r^\alpha} \quad \text{for every } x \in \Omega \cap B(a,r).$$

Then we have

$$\text{(i)} \iff \text{(ii)} \Rightarrow \text{(iii)} \Leftarrow \text{(iv)}.$$

Moreover, if (iii) *holds and $\gamma > 0$, then $\|P_\Omega\|_{\gamma\to\gamma'} < \infty$ with $\gamma' = \beta\gamma/(\alpha+\gamma)$.*

Proof. The proof of the assertion (i) $\iff$ (ii) $\Rightarrow$ (iii) $\Leftarrow$ (iv) can be obtained by an easy modification of the proof of Theorem 2.2. We leave the verification to the reader. Let us prove the last assertion. Suppose that (iii) holds. Let $a \in \partial\Omega$ and $0 < r < 1$. The comparison theorem yields

$$P_\Omega \varphi_{a,\gamma}(x) \leqslant r^\gamma + \omega_p(x; \partial\Omega \setminus B(a,r), \Omega) \leqslant r^\gamma + C_{13} \frac{d(x,a)^\beta}{r^\alpha} \quad \text{for } x \in \Omega \cap B(a,r).$$

Since $(\alpha+\gamma)/\beta > 1$, it follows in particular that

$$P_\Omega \varphi_{a,\gamma}(x) \leqslant (1+C_{13})r^\gamma = (1+C_{13})d(x,a)^{\beta\gamma/(\alpha+\gamma)} \quad \text{for } x \in \Omega \cap S(a, r^{(\alpha+\gamma)/\beta}).$$

Hence we have $\|P_\Omega\|_{\gamma\to\gamma'} < \infty$ with $\gamma' = \beta\gamma/(\alpha+\gamma)$ as (i) $\Longleftrightarrow$ (ii). $\square$

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