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A note on AdS/CFT dual of $SL(2, Z)$ action on 3D conformal field theories with $U(1)$ symmetry

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Abstract

In this Letter, we elaborate on the $SL(2, Z)$ action on three-dimensional conformal field theories with $U(1)$ symmetry introduced by Witten, by trying to give an explicit verification of the claim regarding holographic dual of the S operation in AdS/CFT correspondence. A consistency check with the recently proposed prescription on boundary condition of bulk fields when we deform the boundary CFT in a non-standard manner is also discussed.

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1. Introduction

Mirror symmetry found in three-dimensional theories with extended supersymmetry [7] gives us much insight about non-trivial duality in quantum field theory. For the cases with Abelian gauge groups, it was shown [8] that many aspects of duality may be derived by assuming a single ‘elementary’ duality, that is, the duality (in IR) between the $\mathcal{N} = 4$ SQED with single flavor hypermultiplet and the free theory of single hypermultiplet. The former has a global $U(1)$ symmetry that shifts the dual photon scalar of $U(1)$ gauge field. This symmetry is supposed to be the symmetry of $U(1)$ phase rotation in the latter. Because magnetic vortices break the shift symmetry of the dual photon, they can be identified to elementary excitations in the free theory side.

Recently, it was observed in Ref. [1] that the above simplest duality between vortex and particle may be seen as an invariance under certain transformation on three-dimensional CFTs. Specifically, given a CFT with global $U(1)$ symmetry, this transformation is defined by gauging the $U(1)$ symmetry without introducing gauge kinetic term. Although the above example is in the context of supersymmetric version of this transformation, there is no problem in defining this transformation in non-supersymmetric cases, in

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general. The intriguing fact shown in Ref. [1] is the possibility of extending this transformation into a set of transformations forming the group $SL(2, Z)$. The above transformation corresponds to S with $S^2 = -1$, while the transformation T with $(ST)^3 = 1$ was introduced.

The meaning of this $SL(2, Z)$ in the space of 3D CFTs has been studied in Refs. [1,4–6] for theories in which Gaussian approximation is valid in calculating correlation functions [12,13]. (See [10] for implications on QHE.) These analysis identified the $SL(2, Z)$ as certain transformations of basic correlation functions of the theory. While we may be almost convinced that the transformations of correlation functions found in these analysis hold true in general, its proof is currently limited to the theories with Gaussian approximation.

As suggested in Ref. [1], another way of interpreting the $SL(2, Z)$ transformations may be provided by AdS/CFT correspondence [9]. According to AdS/CFT, a global $U(1)$ symmetry in the CFT corresponds to having a $U(1)$ gauge theory in the bulk, whose asymptotic value on the boundary couples to the $U(1)$ current of the CFT. The $U(1)$ gauge theory in the bulk has a natural $SL(2, Z)$ duality [2]. While it is easy to identify the T operation in the CFT as the usual 2π shift of the bulk θ parameter [1], describing holographic dual of the S operation turns out to be much more subtle. It was suggested that the S-transformed CFT is dual to the same gauge theory in the bulk, but its $U(1)$ current couples to the S-dualized gauge field. Note that the resulting CFT with different coupling to the bulk field is not equivalent to the original CFT [14,18].

Although a compelling discussion on holographic dual of the S operation was provided in Ref. [1] using various aspects of AdS/CFT [17–19], and was further supported in Ref. [6] by explicitly calculating certain correlation functions, a rigorous verification of the claim is missing. In this Letter, we propose a rigorous argument that fills this gap.

2. Setting up the stage

This section is intended to give a brief review of relevant facts in Ref. [1] on $SL(2, Z)$ transformations of 3D CFTs, as a necessary preparation for the discussion in next section.

A basic ingredient used in the discussion of Ref. [1] is the equation,

$$\begin{aligned}\int \mathcal{D}A \exp\bigl(i I(A, B)\bigr) &= \int \mathcal{D}A \exp\left(\frac{i}{2\pi} \int_Y d^3x\, \epsilon^{ijk} A_i \partial_j B_k\right) \\ &= \delta(B),\end{aligned} \tag{2.1}$$

where A and B are connections of line bundles on an oriented base three-manifold Y. The delta function on the right-hand side means that B is zero, that is, its field strength vanishes and there is no non-vanishing Wilson line. The path integral $\int \mathcal{D}A$ in the left-hand side includes summing over topologically distinct line bundles as well as integral over trivial connections. For topologically non-trivial connections, especially when a quantized magnetic flux on a 2-dimensional cycle Σ does not vanish,

$$\frac{1}{2\pi} \int_\Sigma F \neq 0, \tag{2.2}$$

it is not possible to define a global connection A such that $dA = F$. In this case, we need to understand $I(A, B)$ as follows. Pick up a compact-oriented four manifold X whose boundary is Y, and extend connections (and line bundles) A, B on Y to connections $\mathcal{A}$, $\mathcal{B}$ on X. Then $I(A, B)$ is defined to be

$$\frac{1}{2\pi} \int_X \mathcal{F}_\mathcal{A} \wedge \mathcal{F}_\mathcal{B}, \tag{2.3}$$

where $\mathcal{F}_\mathcal{A}$, $\mathcal{F}_\mathcal{B}$ are the field strengths of $\mathcal{A}$, $\mathcal{B}$. Because for any *closed* four manifold $\bar{X}$, $\frac{1}{4\pi^2} \int_{\bar{X}} \mathcal{F}_\mathcal{A} \wedge \mathcal{F}_\mathcal{B}$ is an integer Chern number, the above definition of $I(A, B)$ is easily shown to be independent of extensions modulo 2π. This is fine as long as we are concerned only with $e^{iI(A,B)}$.

In Ref. [1], several ways of showing (2.1) were given. In simple terms, we split $A = A_{\text{triv}} + A'$, where A_{triv} is a globally defined trivial connection, and A' is a representative of a given topologically non-trivial line bundle (which does not have a global definition). Note that we can write

$$I(A, B) = \frac{1}{2\pi} \int_Y d^3x\, \epsilon^{ijk} A_i^{\text{triv}} \partial_j B_k + I(A', B), \tag{2.4}$$

because $\epsilon^{ijk}\partial_j B_k = \frac{1}{2}\epsilon^{ijk}F_{jk}$ is well-defined on Y. The path integral over $A_{\rm triv}$ gives us delta function setting the field strength of B zero. Then, only remaining component of B is its possible Wilson line in $H^1(Y, U(1))$. With vanishing field strength of B, it is clear as in (2.4) that $I(A', B)$ is invariant under adding trivial connection to A', that is, $I(A', B)$ depends only on the cohomology of the field strength of A'. This cohomology (characteristic class) belongs to $H^2(Y, \mathbf{Z})$ as a consequence of Dirac quantization (a Chern's theorem), or more specifically,

$$\frac{1}{2\pi}\int_\Sigma F \in \mathbf{Z}, \tag{2.5}$$

where Σ is any integer coefficient 2-cycle. Thus, we see that $I(A', B)$ is a kind of bilinear form,

$$H^2(Y, \mathbf{Z}) \times H^1\big(Y, U(1)\big) \to \mathbf{R}. \tag{2.6}$$

In Ref. [1], this bilinear form was identified and summing over $H^2(Y, \mathbf{Z})$ was shown to give the remaining delta function setting Wilson line of B zero. A possible intuitive picture on this may be the following. Consider a non-zero 1-cycle γ on which there is a Wilson line $e^{i\int_\gamma B} \in U(1)$. We roughly consider Y as a product of γ and two-dimensional transverse space Σ, and write $I(A', B)$ as

$$\begin{aligned} I(A', B) &\sim \frac{1}{2\pi}\int_Y B \wedge F_A \\ &\sim \frac{1}{2\pi}\int_\Sigma F_A \cdot \int_\gamma B \sim n \cdot \int_\gamma B, \end{aligned} \tag{2.7}$$

where $n \in \mathbf{Z}$. Hence, summing over $F_A \in H^2(Y, \mathbf{Z})$ involves something like

$$\sum_{n\in\mathbf{Z}} e^{in\cdot\int_\gamma B}, \tag{2.8}$$

which imposes vanishing Wilson line, $e^{i\int_\gamma B} = 1$. This argument is intended to be just illustrative, and we refer to Ref. [1] for rigorous derivation.

Now, we are ready to describe the $SL(2, Z)$ actions defined in Ref. [1] on three-dimensional conformal field theories with global $U(1)$ symmetry. The definition of a conformal field theory here means to specify the global $U(1)$ current J^i and introduce a background gauge field A_i without kinetic term that couples to J^i. A theory is thus specified by

$$\left\langle \exp\left(i\int_Y d^3x\, A_i J^i\right)\right\rangle, \tag{2.9}$$

where $\langle\cdots\rangle$ means to evaluate expectation value in the given CFT. The above generating functional can produce all correlation functions of $U(1)$ current J^i. The S operation is defined by letting A_i be dynamical and introducing a background gauge field B_i with a coupling

$$I(A, B) = \frac{1}{2\pi}\int_Y d^3x\, \epsilon^{ijk} A_i \partial_j B_k, \tag{2.10}$$

that is, the transformed theory is now specified by

$$\int \mathcal{D}A \left\langle \exp\left(i\int_Y d^3x\, A_i J^i\right)\right\rangle \exp\big(iI(A, B)\big), \tag{2.11}$$

where $\langle\cdots\rangle$ means expectation value in the original conformal field theory. Noting that $I(A, B) \sim \int_Y B \wedge F_A$, we see that the $U(1)$ current of the S-transformed theory that B couples is $\tilde{J}^i = \frac{1}{2\pi}(\star F_A)^i = \frac{1}{4\pi}\epsilon^{ijk}(F_A)_{jk}$. The $U(1)$ symmetry corresponding to this current is the shift symmetry of dual photon scalar of A_i.

The definition of T operation is a little subtle, because it involves modifying a theory in a way which is not manifest in low energy action that is supposed to define the theory. Concretely, the T operation is defined to shift the 2-point function of J^i by a contact term,

$$\begin{aligned} \big\langle J^i(x) J^j(y)\big\rangle \to &\big\langle J^i(x) J^j(y)\big\rangle \\ &+ \frac{i}{2\pi}\epsilon^{ijk}\frac{\partial}{\partial x^k}\delta^3(x-y). \end{aligned} \tag{2.12}$$

Because the above contact term has mass dimension 4, which is the right dimension of JJ correlation, this term does not introduce any dimensionful coupling. Moreover, it does not conflict with any symmetry of the theory (in some cases [16], we need this term to preserve gauge invariance). In fact, whenever there is freedom to add local contact terms that are consistent with the symmetry of a theory, this signals the intrinsic inability of our low energy action in predicting them, and we have to *renormalize* them. In other words, they must be treated as input parameters rather than outputs. Note that this is not an unusual thing; it is an essential concept of renormalization in quantum field theory. The effect of the modification (2.12) on our generating functional (2.9) is

$$\left\langle \exp\left(i \int_Y d^3x\, A_i J^i \right) \right\rangle \to \left\langle \exp\left(i \int_Y d^3x\, A_i J^i \right) \right\rangle \times \exp\left(\frac{i}{4\pi} \int_Y d^3x\, \epsilon^{ijk} A_i \partial_j A_k \right), \tag{2.13}$$

which can be shown by first expanding the exponent in series of J and re-exponentiating the effects of T operation on J correlation functions.

Another fact in Ref. [1], which is needed to show the $SL(2, Z)$ group structure of the above transformations is,

$$\int \mathcal{D}\mathcal{A} \exp\big(i I(A)\big) = 1, \tag{2.14}$$

up to possible phase factor [2,3]. This equation should be understood as a statement that the theory has only one physical state and trivial [15]. Here,

$$I(A) = \frac{1}{4\pi} \int_Y d^3x\, \epsilon^{ijk} A_i \partial_j A_k \equiv \frac{1}{16\pi} \int_X d^4x\, \epsilon^{ijkl} F_{ij} F_{kl} = \frac{1}{4\pi} \int_X F \wedge F, \tag{2.15}$$

defined with some extension over X similarly as before [11]. This is well-defined modulo 2π for a spin manifold Y. Using (2.1) and (2.14), it is readily shown that S and T satisfy the $SL(2, Z)$ generating algebra, $(ST)^3 = 1$ and $S^2 = -1$, where -1 is the transformation $J^i \to -J^i$ commuting with everything.

3. Holographic dual of the S operation in AdS/CFT

We now try to elaborate on the claim in Ref. [1] and to give an explicit proof that the S operation on CFTs is dual to the Abelian S-duality in the bulk AdS in AdS/CFT correspondence.

Let X denote the bulk AdS, and $\partial X = Y$ be our space–time. Let $\mathcal{A}$ be the $U(1)$ gauge field in the bulk whose boundary value couples to the global $U(1)$ current J^i in the CFT side. According to AdS/CFT, we have

$$\left\langle \exp\left(i \int_Y d^3x\, A_i J^i \right) \right\rangle = \int_{\mathcal{A}_i \to A_i} \mathcal{D}\mathcal{A} \exp\big(i S(\mathcal{A})\big), \tag{3.16}$$

where $S(\mathcal{A}) = \frac{1}{e^2} \int_X \mathcal{F}_\mathcal{A} \wedge *\mathcal{F}_\mathcal{A} + \cdots$ is the action of the bulk gauge field and we omitted other bulk fields for simplicity. Before considering holographic dual of S operation, it is easy to identify from (3.16) the holographic dual of T operation as in Ref. [1]. The T operation simply multiplies $e^{iI(A)}$ in both sides of (3.16). But, note that $I(A) = \frac{1}{4\pi} \int_X \mathcal{F}_\mathcal{A} \wedge \mathcal{F}_\mathcal{A}$ modulo 2π irrespective of the bulk extension $\mathcal{A}$ as long as its boundary value is fixed, hence in the right-hand side, multiplying $e^{iI(A)}$ is equivalent to shifting the bulk θ term,

$$S(\mathcal{A}) \supset \frac{\theta}{8\pi^2} \int_X \mathcal{F}_\mathcal{A} \wedge \mathcal{F}_\mathcal{A}, \tag{3.17}$$

by $\theta \to \theta + 2\pi$.

Now, using (3.16), we want to show that (2.11) is nothing but the bulk path integral of the same bulk theory, but with the boundary condition that the 'dual' field $\mathcal{B}$ has the specified boundary value B_i. In terms of the original field $\mathcal{A}$, this corresponds to specifying electric field on the boundary, instead of specifying magnetic field. (When $B_i = 0$, the boundary condition in terms of $\mathcal{A}$ is that the electric field vanishes on the boundary, as given in Ref. [1].)

Using AdS/CFT and the fact that $I(A, B) = \frac{1}{2\pi} \times \int_Y d^3x\, \epsilon^{ijk} A_i \partial_j B_k$ can be written as a bulk integral (up to mod 2π)

$$I(A, B) = \frac{1}{2\pi} \int_X \mathcal{F}_\mathcal{B} \wedge \mathcal{F}_\mathcal{A}, \tag{3.18}$$

where $\mathcal{B}$ and $\mathcal{A}$ are 'arbitrary' extensions of B_i and A_i, we have

$$\left\langle \exp\left(i \int_Y d^3x\, A_i J^i \right) \right\rangle \exp\left(\frac{i}{2\pi} \int_Y d^3x\, \epsilon^{ijk} A_i \partial_j B_k \right)$$

$$= \int_{\mathcal{A}_i \to A_i} \mathcal{D}\mathcal{A} \exp\left(iS(\mathcal{A}) + \frac{i}{2\pi} \int_X \mathcal{F}_{\mathcal{B}} \wedge \mathcal{F}_{\mathcal{A}}\right), \tag{3.19}$$

where $\mathcal{B}$ is some fixed extension of B_i. (2.11) is the integral of this quantity over the boundary value A_i, hence (2.11) is equal to the r.h.s. of (3.19) without any boundary conditions on $\mathcal{A}$,

$$\begin{aligned} &\int \mathcal{D}A \left\langle \exp\left(i \int d^3x\, A_i J^i\right)\right\rangle \\ &\quad \times \exp\left(\frac{i}{2\pi} \int d^3x\, \epsilon^{ijk} A_i \partial_j B_k\right) \\ &\qquad = \int \mathcal{D}\mathcal{A} \exp\left(iS(\mathcal{A}) + \frac{i}{2\pi} \int_X \mathcal{F}_{\mathcal{B}} \wedge \mathcal{F}_{\mathcal{A}}\right). \end{aligned} \tag{3.20}$$

We now perform a dualizing procedure in the bulk X, which is similar to the one in Ref. [2], but appropriately taking care of the fact that our space–time now has a boundary $\partial X = Y$. First we want to argue that, for a bulk 2-form field G, the integral

$$\int_{\mathcal{V} \to 0} \mathcal{D}\mathcal{V} \exp\left(\frac{i}{2\pi} \int_X \mathcal{F}_{\mathcal{V}} \wedge G\right) \tag{3.21}$$

over all possible connections $\mathcal{V}$ (and also sum over line bundles) in X with boundary condition that $\mathcal{V}$ vanishes on Y (up to gauge transformations), gives a delta function on G that precisely says G is a field strength of some connection of a line bundle. To show this, we consider a "closed" 4-manifold $\bar{X}$ which is obtained from X by attaching on $\partial X = Y$ a orientation reversed copy of X which we call X', as in Fig. 1. We also consider a 2-form field $\bar{G}$ on $\bar{X}$, whose value on X' is the identical copy of G on X. It is clear that G is a field strength of some connection on X if and only if $\bar{G}$ is a field strength of some connection on $\bar{X}$. As $\bar{X}$ is closed, we can use the well-known procedure of requiring $\bar{G}$ to be a field strength [2]; the integral

$$\int \mathcal{D}\bar{\mathcal{V}} \exp\left(\frac{i}{2\pi} \int_{\bar{X}} \mathcal{F}_{\bar{\mathcal{V}}} \wedge \bar{G}\right) \tag{3.22}$$

over connections $\bar{\mathcal{V}}$ on $\bar{X}$ gives a delta function imposing that $\bar{G}$ is a field strength of some connection on $\bar{X}$. Simply put, the integration over trivial part in $\bar{\mathcal{V}}$ imposes that $\bar{G}$ be a closed 2-form, while the remaining sum over line bundles requires $\bar{G}$ to satisfy Dirac quantization, $\bar{G} \in H^2(\bar{X}, \mathbf{Z})$.

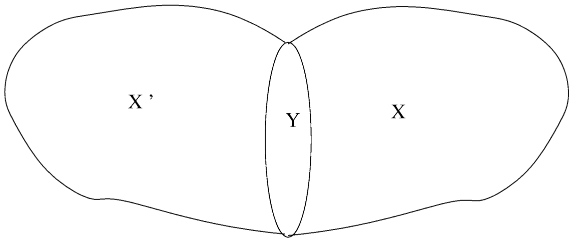

Fig. 1.

Thus, when expressed in terms of G, it gives the desired delta function (up to a constant factor) that says G should be a field strength on X. Now, we can split $\bar{\mathcal{V}}$ on $\bar{X}$ into a connection $\mathcal{V}$ on X and a connection $\mathcal{V}'$ on X', and we have

$$\begin{aligned} \frac{i}{2\pi} \int_{\bar{X}} \mathcal{F}_{\bar{\mathcal{V}}} \wedge \bar{G} &= \frac{i}{2\pi} \int_X \mathcal{F}_{\mathcal{V}} \wedge G + \frac{i}{2\pi} \int_{X'} \mathcal{F}_{\mathcal{V}'} \wedge G \\ &= \frac{i}{2\pi} \int_X \mathcal{F}_{\mathcal{V}} \wedge G - \frac{i}{2\pi} \int_X \mathcal{F}_{\mathcal{V}'} \wedge G \\ &= \frac{i}{2\pi} \int_X (\mathcal{F}_{\mathcal{V}} - \mathcal{F}_{\mathcal{V}'}) \wedge G, \end{aligned} \tag{3.23}$$

where in the last line, we consider $\mathcal{V}'$ as a connection on X, but with the minus sign in the integral due to orientation reversal. Note that $\mathcal{V}$ and $\mathcal{V}'$ should agree on the boundary Y, as they are from a common $\bar{\mathcal{V}}$ on $\bar{X}$, hence we can rewrite the path integral over $\bar{\mathcal{V}}$ into

$$\int \mathcal{D}\bar{\mathcal{V}} = \int_{(\mathcal{V}-\mathcal{V}') \to 0} \mathcal{D}\mathcal{V}\, \mathcal{D}\mathcal{V}'. \tag{3.24}$$

From the above two observations, we have

$$\begin{aligned} &\int \mathcal{D}\bar{\mathcal{V}} \exp\left(\frac{i}{2\pi} \int_{\bar{X}} \mathcal{F}_{\bar{\mathcal{V}}} \wedge \bar{G}\right) \\ &\quad = \int_{(\mathcal{V}-\mathcal{V}') \to 0} \mathcal{D}\mathcal{V}\, \mathcal{D}\mathcal{V}' \exp\left(\frac{i}{2\pi} \int_X (\mathcal{F}_{\mathcal{V}} - \mathcal{F}_{\mathcal{V}'}) \wedge G\right) \\ &\quad = \left[\int \mathcal{D}\mathcal{V}'\right] \cdot \int_{\mathcal{V} \to 0} \mathcal{D}\mathcal{V} \exp\left(\frac{i}{2\pi} \int_X \mathcal{F}_{\mathcal{V}} \wedge G\right), \end{aligned} \tag{3.25}$$

where we have changed the variable $(\mathcal{V} - \mathcal{V}') \to \mathcal{V}$ in the last line. Thus, (3.21) indeed gives a desired delta function (up to a constant factor).

Now, we are ready to perform the duality procedure in a space–time with boundary. Introduce a 2-form field G and replace every $\mathcal{F}_\mathcal{A}$ in the action with $\mathcal{F}_\mathcal{A}+G$. Also introduce a connection $\mathcal{V}$ with the boundary condition that $\mathcal{V}$ vanishes on Y, and add the coupling

$$\frac{i}{2\pi}\int_X \mathcal{F}_\mathcal{V}\wedge G. \tag{3.26}$$

The resulting action is invariant under the extended gauge transform,

$$\mathcal{A}\to\mathcal{A}+\mathcal{C},\qquad G\to G-\mathcal{F}_\mathcal{C}, \tag{3.27}$$

where $\mathcal{C}$ is an arbitrary connection in X. Precisely because $\mathcal{V}$ vanishes on Y, (3.26) is invariant under (3.27) modulo $2\pi i$. Let us explain this fact in some detail. The vanishing connection on Y can be extended to a trivial (globally defined one form) connection on X, say $\mathcal{V}'$. We also know that

$$\frac{i}{2\pi}\int_X \mathcal{F}_{\mathcal{V}'}\wedge\mathcal{F}_\mathcal{C}=\frac{i}{2\pi}\int_X \mathcal{F}_\mathcal{V}\wedge\mathcal{F}_\mathcal{C}, \tag{3.28}$$

modulo $2\pi i$ because $\mathcal{V}'$ and $\mathcal{V}$ agree on Y. Being trivial, $\mathcal{F}_{\mathcal{V}'}$ can be written as $\mathcal{F}_{\mathcal{V}'}=d\mathcal{V}'$ globally on X, and performing partial integration, we have

$$\frac{1}{2\pi}\int_X \mathcal{F}_{\mathcal{V}'}\wedge\mathcal{F}_\mathcal{C}=\frac{1}{2\pi}\int_Y \mathcal{V}'\wedge\mathcal{F}_\mathcal{C}=0, \tag{3.29}$$

because $\mathcal{V}'$ vanishes on Y.

We then consider G and $\mathcal{V}$ as dynamical, and mod out the theory with gauge equivalence. If we integrate over $\mathcal{V}$ first, it gives a constraint that G is a field strength of some connection C by the discussion in the previous paragraphs. Then, by gauge fixing, we can set $G=0$ and recover the original theory of $\mathcal{A}$. The equivalent dual theory in terms of $\mathcal{V}$ is obtained by first gauge fixing $\mathcal{A}=0$, and integrating over G. Applying this to (3.20), we get

$$\begin{aligned}
&\int\mathcal{D}\mathcal{A}\exp\left(iS(\mathcal{F}_\mathcal{A})+\frac{i}{2\pi}\int_X\mathcal{F}_\mathcal{B}\wedge\mathcal{F}_\mathcal{A}\right)\\
&\quad=\int_{\mathcal{V}\to 0}\mathcal{D}\mathcal{V}\int\mathcal{D}G\\
&\qquad\times\exp\left(iS(G)+\frac{i}{2\pi}\int_X(\mathcal{F}_\mathcal{B}+\mathcal{F}_\mathcal{V})\wedge G\right)\\
&\quad=\int_{\mathcal{V}\to B_i}\mathcal{D}\mathcal{V}\int\mathcal{D}G\exp\left(iS(G)+\frac{i}{2\pi}\int_X\mathcal{F}_\mathcal{V}\wedge G\right)\\
&\quad=\int_{\mathcal{V}\to B_i}\mathcal{D}\mathcal{V}\exp\big(iS_D(\mathcal{F}_\mathcal{V})\big),
\end{aligned}\tag{3.30}$$

where in the fourth line, we changed the variable $\mathcal{B}+\mathcal{V}\to\mathcal{V}$ with the new boundary condition that $\mathcal{V}$ goes to the specified B_i on Y. In the last line, integrating over G gives the dual bulk action $S_D(\mathcal{F}_\mathcal{V})$ in terms of the dual gauge field $\mathcal{V}$ with the coupling constant $-1/\tau$, and we have the desired boundary condition for $\mathcal{V}$ on Y.

At this point, it would be clarifying to see explicitly the relation between the boundary condition for the dual field $\mathcal{V}$ that we derived above, and the boundary condition in terms of the original field $\mathcal{A}$ [1]. In the bulk AdS, the dual field $\mathcal{V}$ is nothing but a non-local change of variable from the original variable $\mathcal{A}$. In the case of vanishing θ angle,[1] they are related by

$$(\mathcal{F}_\mathcal{A})_{\mu\nu}=\frac{e^2}{8\pi}\epsilon_{\mu\nu\alpha\beta}(\mathcal{F}_\mathcal{V})^{\alpha\beta}. \tag{3.31}$$

This is easily seen in a naive dualization procedure of making the field strength of $\mathcal{A}$ as a fundamental integration variable by imposing the Bianchi identity. The dual field $\mathcal{V}$ is introduced as a Lagrange multiplier

$$\begin{aligned}
\int\mathcal{D}\mathcal{V}\mathcal{D}\mathcal{F}_\mathcal{A}\exp\bigg(i\int d^4x\,\sqrt{g}\bigg(&\frac{1}{2e^2}(\mathcal{F}_\mathcal{A})_{\mu\nu}(\mathcal{F}_\mathcal{A})^{\mu\nu}\\
&+\frac{1}{8\pi}\epsilon_{\mu\nu\alpha\beta}(\mathcal{F}_\mathcal{V})^{\mu\nu}(\mathcal{F}_\mathcal{A})^{\alpha\beta}\bigg)\bigg).
\end{aligned}\tag{3.32}$$

Integrating out $\mathcal{F}_\mathcal{A}$ gives the dual description. The equation of motion of $\mathcal{F}_\mathcal{A}$ is (3.31).

Now, in Poincaré coordinate $(x_0,\vec{x})$, (with the boundary at $x_0=0$)

$$ds^2=\frac{dx_0^2+d\vec{x}^2}{x_0^2}, \tag{3.33}$$

[1] This is just for simplicity. The case with non-vanishing θ angle is similar [2].

the usual boundary condition specifying the value of gauge field on the boundary corresponds to specifying the (gauge invariant) 'magnetic' component $M_i = \frac{1}{2}\epsilon_{ijk}F^{jk}$, $i, j, k = 1, 2, 3$. Because (3.31) interchanges the 'magnetic' component of $\mathcal{V}$ with the 'electric' component of $\mathcal{A}$, $E_i = \partial_0\mathcal{A}_i$ (in the gauge $A_0 = 0$), we see that in terms of the original field $\mathcal{A}$, the S-transformed CFT is mapped to the bulk AdS theory with 'electric' boundary condition. Note that 'magnetic' and 'electric' boundary conditions are natural counterparts of Dirichlet and Neumann boundary conditions for scalar field, and they are naturally expected to be conjugate with each other in AdS/CFT. We will come to this point in the next section. In fact, we need to look at the T-transformation more carefully in this respect. The AdS dual of the T-transformation of 3D CFT was identified as a 2π shift of the bulk θ-angle, while the 'magnetic' boundary condition is unchanged. In the presence of θ-angle, the 'electric' component naturally conjugate to the 'magnetic' component (or more precisely, the value A_i on the boundary) has a term proportional to θ-angle. This is most easily seen from the fact that the natural conjugate variable to A_i is obtained by varying the action w.r.t. $\partial_0\mathcal{A}_i$.[2] Denoting this as D_i, we have

$$D_i = \frac{1}{e^2}\partial_0\mathcal{A}_i + \frac{\theta}{8\pi^2}M_i, \tag{3.34}$$

and shifting θ results in shifting of D_i by a unit of 'magnetic' component M_i.

4. In view of boundary deformations

In the last section, we observed that the AdS dual of S-operation on 3D CFT interchanges the 'magnetic' and 'electric' boundary conditions, while T-operation corresponds to shifting the 'electric' component D_i by a unit of 'magnetic' component. Though T-operation does not really change the boundary condition by itself, it has a non-trivial effect when combined with S. With appropriate normalization, we can represent the S and T action on boundary conditions as

$$S:\quad \begin{pmatrix} D_i \\ M_i \end{pmatrix} \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} D_i \\ M_i \end{pmatrix},$$
$$T:\quad \begin{pmatrix} D_i \\ M_i \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} D_i \\ M_i \end{pmatrix}, \tag{4.35}$$

where we take the usual 'magnetic' boundary condition in terms of transformed variable. This gives a natural correspondence between the $SL(2, Z)$ action on 3D CFTs with the $SL(2, Z)$ action on boundary condition (or bulk gauge field).

In this section, we give another concrete evidence of this picture in the context of the recently proposed prescription [17] on boundary conditions when we deform the boundary CFT in a non-standard manner. The proposed prescription in Ref. [17] is for scalar fields in the bulk, and it goes as follows. Suppose we have a scalar field ϕ in the bulk, whose asymptotic behavior near the boundary $x_0 = 0$ is

$$\phi(x_0, \vec{x}) \sim A(\vec{x})x_0^{\Delta+} + B(\vec{x})x_0^{\Delta-}. \tag{4.36}$$

We consider the CFT on $x_0 = 0$ defined by the boundary condition $A(\vec{x}) = 0$. By standard AdS/CFT dictionary, $A(\vec{x})$ couples to a scalar operator $\mathcal{O}$ of dimension Δ_+ on the boundary, that is, a boundary condition with non-vanishing $A(\vec{x})$ maps to a deformation of CFT side by

$$S_{\mathrm{CFT}} \to S_{\mathrm{CFT}} + \int d\vec{x}\, A(\vec{x})\mathcal{O}(\vec{x}). \tag{4.37}$$

The expectation value of $\mathcal{O}$ in this deformed CFT is given by $B(\vec{x})$,

$$\langle\mathcal{O}(\vec{x})\rangle_A \sim B(\vec{x}). \tag{4.38}$$

They are natural conjugate pair of source and expectation value. For specific range of ϕ mass, it is possible also to consider the boundary CFT defined by $B(\vec{x}) = 0$. In this CFT, the roles of $A(\vec{x})$ and $B(\vec{x})$ are reversed, and the partition function is just a Legendre transform of the previous CFT [18]. It has been argued that this CFT is the IR fixed point of the previous CFT deformed by a term which is quadratic in $\mathcal{O}(\vec{x})$. The question is what would be the boundary condition when we deform the boundary CFT (defined by $A(\vec{x}) = 0$) in a more general manner,

$$S_{\mathrm{CFT}} \to S_{\mathrm{CFT}} + W(\mathcal{O}), \tag{4.39}$$

[2] If we consider x_0 as a time variable, this is the Witten effect. This should be true even in Euclidean case when considering 'naturally' conjugate boundary variables on $x_0 = 0$.

where W is an arbitrary (possibly non-local) function of $\mathcal{O}(\vec{x})$. The proposal in Ref. [17] is to take the following boundary condition on $A(\vec{x})$ and $B(\vec{x})$,

$$A(\vec{x}) = \frac{\delta W(\mathcal{O})}{\delta \mathcal{O}(\vec{x})}\bigg|_{\mathcal{O}(\vec{x})\to B(\vec{x})}. \tag{4.40}$$

The situation with bulk gauge field in AdS_4/CFT_3 correspondence might look similar to the case of bulk scalar field with its mass such that two CFTs are possible. We have two naturally conjugate variables (A_i, D_i) in the boundary, and two different boundary conditions are possible. However, contrary to the scalar field case, these two boundary CFTs cannot possibly be related by an RG flow, because acting S twice gives us the original theory and the degrees of freedom are not lost.

However, the proposal (4.40) can be naturally extended to include the case of gauge fields, and we will show that this extension indeed reproduces the results in the previous sections, providing a compelling check for the proposal applied to gauge fields. We start with the CFT defined by the usual 'magnetic' boundary condition specifying gauge field components tangential to the boundary as $x_0 \to 0$,

$$\mathcal{A}_i \to A_i, \quad i = 1, 2, 3. \tag{4.41}$$

As usual, this corresponds to deforming the CFT by adding the coupling,

$$\delta S_{\mathrm{CFT}} = \int d^3x\, A_i J^i, \tag{4.42}$$

where J^i is the 3D $U(1)$ current. Now, we ask the question of what boundary condition we take when we deform the CFT with an arbitrary function of $J^i(\vec{x})$,

$$\delta S_{\mathrm{CFT}} = W\big[J^i\big], \tag{4.43}$$

instead of a linear one (4.42). Recalling that

$$D^i(\vec{x}) = \frac{\delta S_{\mathrm{bulk}}}{\delta \partial_0 \mathcal{A}_i(\vec{x})} \tag{4.44}$$

is the "electric" field that is canonically conjugate to $\mathcal{A}_i(\vec{x})$, a direct analogy with the case of scalar fields (4.40) suggests the following prescription on the boundary condition as $x_0 \to 0$,

$$A_i(\vec{x}) = \frac{\delta W[J]}{\delta J^i(\vec{x})}\bigg|_{J^i(\vec{x})\to D^i(\vec{x})}, \tag{4.45}$$

where $\mathcal{A}_i \to A_i$.

Having this in mind, let us go back to our $SL(2, Z)$ actions on 3D CFT and consider the action given by ST^n, which corresponds to the matrix,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & n \end{pmatrix}. \tag{4.46}$$

By definition, the partition function of the transformed CFT is given by

$$Z_{\mathrm{after}} = \int \mathcal{D}B_i \bigg\langle \exp\bigg(i\int d^3x\, B_i J^i + n\cdot\frac{i}{4\pi}\int d^3x\, \epsilon^{ijk} B_i \partial_j B_k \bigg)\bigg\rangle, \tag{4.47}$$

where $\langle\cdots\rangle$ and J^i are expectation values and $U(1)$ current, respectively, of the original CFT, and B_i is the intermediate connection variable in defining the S operation. Because the exponent is quadratic in B_i, we can perform the path integral over B_i explicitly. We introduce the gauge fixing term $i\int d^3x\, \xi(\partial_i B_i)^2$, and the B_i propagator is

$$S_{ij}(p) \equiv \big\langle B_i(p) B_j(-p)\big\rangle = i\frac{p_i p_j}{2\xi(p^2)^2} + \frac{2\pi}{np^2}\epsilon_{ijk} p^k. \tag{4.48}$$

Using this, (4.47) is given by

$$\begin{aligned} Z_{\mathrm{after}} &= \bigg\langle \exp\bigg(-\frac{1}{2}\int \frac{d^3p}{(2\pi)^3} J^i(-p) S_{ij}(p) J^j(p)\bigg)\bigg\rangle \\ &= \bigg\langle \exp\bigg(-\frac{\pi}{n}\int \frac{d^3p}{(2\pi)^3} J^i(-p)\frac{\epsilon_{ijk}p^k}{p^2} J^j(p)\bigg)\bigg\rangle, \end{aligned} \tag{4.49}$$

where $J^i(p) \equiv \int d^3x\, e^{-ipx} J^i(x)$, and we have used the Ward identity $\langle p_i J^i(p)\ldots\rangle = 0$. Now, looking at the last expression, it is clear that the transformed theory is nothing but the original CFT with the deformation $W[J]$ given by

$$\begin{aligned} \delta S_{\mathrm{CFT}} = W[J] &= \frac{i\pi}{n}\int \frac{d^3p}{(2\pi)^3} J^i(-p)\frac{\epsilon_{ijk}p^k}{p^2} J^j(p) \\ &= \frac{i\pi}{n}\int \frac{d^3p}{(2\pi)^3}\int d^3x \\ &\quad\times \int d^3y\, e^{ipx} e^{-ipy}\frac{\epsilon_{ijk}p^k}{p^2} J^i(x) J^j(y). \end{aligned} \tag{4.50}$$

Hence, according to our proposal (4.45), the bulk AdS gauge theory of the transformed CFT has a modified boundary condition,

$$A_i(x) = \frac{2\pi i}{n} \int \frac{d^3 p}{(2\pi)^3} \times \int d^3 y\, e^{ipx} e^{-ipy} \frac{\epsilon_{ijk} p^k}{p^2} J^j(y)\bigg|_{J^j(y)\to D^j(y)}. \tag{4.51}$$

To see clearly what this means, take the x-derivative, $\frac{1}{2}\epsilon^{mni}\partial_n$, on both sides. The left-hand side gives the 'magnetic' component $M^m(x) = \frac{1}{2}\epsilon^{mni}\partial_n A_i(x)$, while the right-hand side becomes

$$\begin{aligned}
&-\frac{\pi}{n}\int \frac{d^3p}{(2\pi)^3}\int d^3y\, e^{ipx}e^{-ipy}\epsilon^{mni}\epsilon_{ijk} \times \frac{p_n p^k}{p^2} J^j(y)\bigg|_{J^j(y)\to D^j(y)}\\
&\quad= -\frac{\pi}{n}\int \frac{d^3p}{(2\pi)^3}\int d^3y\, e^{ipx}e^{-ipy}\left(\delta^m_j\delta^n_k - \delta^m_k\delta^n_j\right) \times \frac{p_n p^k}{p^2} J^j(y)\bigg|_{J^j(y)\to D^j(y)}\\
&\quad= -\frac{\pi}{n}\int \frac{d^3p}{(2\pi)^3}\int d^3y\, e^{ipx}e^{-ipy}\left(\delta^m_j - \frac{p^m p_j}{p^2}\right) \times J^j(y)\bigg|_{J^j(y)\to D^j(y)}\\
&\quad= -\frac{\pi}{n}\int \frac{d^3p}{(2\pi)^3}\int d^3y\, e^{ipx}e^{-ipy} \times J^m(y)\bigg|_{J^j(y)\to D^j(y)}\\
&\quad= -\frac{\pi}{n} J^m(x)\bigg|_{J^j(x)\to D^j(x)} = -\frac{\pi}{n} D^m(x),
\end{aligned} \tag{4.52}$$

where in going from the fifth line to the seventh, we again used the Ward identity for J^i.

In summary, the AdS bulk gauge field for the transformed CFT has the boundary condition; $n \cdot M_i(\vec{x}) + D_i(\vec{x}) = 0$ (with appropriate normalization absorbing π). Observe that this matches precisely with the result of the previous sections, because ST^n corresponds to performing first the change

$$\begin{pmatrix} D_i \\ M_i \end{pmatrix} \to \begin{pmatrix} 0 & -1 \\ 1 & n \end{pmatrix}\begin{pmatrix} D_i \\ M_i \end{pmatrix} = \begin{pmatrix} -M_i \\ n\cdot M_i + D_i \end{pmatrix}, \tag{4.53}$$

before taking the usual 'magnetic' boundary condition $M_i = 0$.

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