



# Homomorphisms of higher categories

Richard Garner

*Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge CB3 0WB, UK*

Received 10 January 2009; accepted 18 January 2010

Available online 3 March 2010

Communicated by Ross Street

---

## Abstract

We describe a construction that to each algebraically specified notion of higher-dimensional category associates a notion of *homomorphism* which preserves the categorical structure only up to weakly invertible higher cells. The construction is such that these homomorphisms admit a strictly associative and unital composition. We give two applications of this construction. The first is to tricategories; and here we do not obtain the trihomomorphisms defined by Gordon, Power and Street, but rather something which is equivalent in a suitable sense. The second application is to Batanin's weak  $\omega$ -categories.

© 2010 Elsevier Inc. All rights reserved.

MSC: 18D05; 55U35

Keywords: Higher-dimensional categories; Weak morphisms; Abstract homotopy theory

---

## 1. Introduction

The purpose of this paper is to describe a notion of homomorphism for weak higher-dimensional categories. Let us at once say that we concern ourselves exclusively with those notions of higher-dimensional category which are essentially-algebraic in the sense described by Freyd [9]; for which composition and its associated coherence are realised by specified operations subject to equational laws. Of course any species of essentially-algebraic structure has a concomitant notion of homomorphism, given by functions on the underlying data commuting to the specified operations: but it is a commonplace that for higher-dimensional categories, such homomorphisms are too strict to be of practical use (though they retain significant *theoretical* importance), because they must preserve the categorical structure “on-the-nose” rather than up

---

*E-mail address:* [rhgg2@cam.ac.uk](mailto:rhgg2@cam.ac.uk).

to suitably coherent higher cells. It is this latter, looser notion of homomorphism that we shall concern ourselves with here.

In low dimensions, the homomorphisms we seek already have satisfactory descriptions: in the case of bicategories, they are Bénabou’s homomorphisms [5, §4], whilst for tricategories we have the *trihomomorphisms* of [12, §3]. This gives us little direct insight into how the general case should look; yet there is a particular aspect of the low-dimensional examples which can usefully be incorporated into a general theory, namely the idea that, as important as the homomorphisms are, of greater importance still is their relationship with the *strict* homomorphisms—the maps we described earlier as preserving the categorical structure “on-the-nose”. In the case of bicategories, this relationship is described by the two-dimensional monad theory of [6]. We write **CatGph** for the 2-category of **Cat**-enriched graphs—whose objects are given by a set  $X$  together with a functor  $X \times X \rightarrow \mathbf{Cat}$ —and  $T$  for the 2-monad thereupon whose algebras are small bicategories. There now arise both the category  $T\text{-Alg}_s$  of  $T$ -algebras and strict  $T$ -algebra morphisms—which is equally well the category **Bicat**<sub>s</sub> of bicategories and strict homomorphisms—and also the category  $T\text{-Alg}$  of  $T$ -algebras and  $T$ -algebra pseudomorphisms—which is equally well (after some work) the category **Bicat** of bicategories and homomorphisms (of course, each of these categories has additional 2-dimensional structure; but we will not concern ourselves with that here). Theorem 3.13 of [6] now describes the fundamental relationship between these two categories in terms of an adjunction:

$$\mathbf{Bicat}_s \begin{array}{c} \xleftarrow{(-)'} \\ \xrightarrow[\perp]{J} \\ \xrightarrow{J} \end{array} \mathbf{Bicat} \tag{1}$$

where  $J$  is the identity-on-objects inclusion functor. The force of this is that homomorphisms  $\mathcal{A} \rightarrow \mathcal{B}$  are classified by strict homomorphisms  $\mathcal{A}' \rightarrow \mathcal{B}$ , so that the seemingly inflexible strict homomorphisms are in fact the more general notion. The adjunction in (1) is of fundamental importance to the theory developed in [6], and a suitable generalisation of it seems a natural desideratum for a theory of higher-dimensional homomorphisms.

Let us examine the ramifications of incorporating such a generalisation into our theory. Suppose we are presented with some notion of higher-dimensional category: in accordance with our assumptions, it admits an essentially-algebraic presentation, and as such we have a notion of strict homomorphism, giving us the morphisms of a category **HCat**<sub>s</sub>. We wish to find the remaining elements of (1): thus a category **HCat** whose maps are the homomorphisms and an adjunction  $(-)' \dashv J : \mathbf{HCat}_s \rightarrow \mathbf{HCat}$  in which  $J$  is the identity on objects. Now to give these data is equally well to give a comonad on **HCat**<sub>s</sub>, since on the one hand, any adjunction  $(-)' \dashv J$  of the required form determines a comonad  $(-)' \circ J$  on **HCat**<sub>s</sub>; and on the other, any comonad  $Q$  on **HCat**<sub>s</sub> determines an adjunction of the required form upon taking **HCat** to be the *co-Kleisli category* of  $Q$  (whose definition we recall in Section 2 below). Consequently, we can restate the problem of defining a notion of homomorphism in terms of that of defining a suitable comonad on the category of strict homomorphisms.

One technique for constructing such a comonad is suggested in [15]. For this we must suppose the category **HCat**<sub>s</sub> to be presentable as the category of algebras for a symmetric operad  $\mathcal{O}$  on a suitable base category  $\mathcal{V}$ ; and may then consider *co-rings* over the operad  $\mathcal{O}$ —these being  $(\mathcal{O}\text{-}\mathcal{O})$ -bimodules equipped with comonoid structure in the monoidal category of  $(\mathcal{O}\text{-}\mathcal{O})$ -bimodules. Each such co-ring  $\mathcal{M}$  induces a comonad  $\mathcal{M} \otimes_{\mathcal{O}}$ —on **HCat**<sub>s</sub>, and hence a notion of homomorphism. The problem with this approach lies in the initial supposition of operadicity;

which though it may be appropriate for homological algebra is rather infrequently satisfied in the case of higher categories. We may try and rectify this by moving from symmetric operads to the higher operads of Batanin [2]; but here a different problem arises, namely that the tensor product of bimodules over a globular operad is ill-defined, for the reason that, in the category whose monoids are globular operads, the tensor product does not preserve reflexive coequalisers in both variables. Thus, though one can speak of bimodules—as Batanin himself does in [2, Definition 8.8]—one cannot speak of co-rings: and so the homomorphisms we obtain need not admit a composition.

In this paper, we adopt a quite different means of constructing a comonad on the category of strict homomorphisms, one informed by categorical homotopy theory. Lack, in [17], establishes that the comonad on  $\mathbf{Bicat}_s$  generated by the adjunction in (1) gives a notion of *cofibrant replacement* for a certain Quillen model structure on  $\mathbf{Bicat}_s$ ; whose generating cofibrations are the inclusions of the basic  $n$ -dimensional boundaries into the basic  $n$ -dimensional cells. For the general case, we can run this argument backwards: given a Quillen model structure on  $\mathbf{HCat}_s$ , we can—by the machinery of [10]—use it to generate a “cofibrant replacement comonad”, and so obtain a notion of homomorphism. In fact, to generate a cofibrant replacement comonad we do not need a full model structure on  $\mathbf{HCat}_s$ , but only a single weak factorisation system; and for this it suffices to give a set of generating cofibrations, which as in the bicategorical case will be given by the inclusions of  $n$ -dimensional boundaries into  $n$ -dimensional cells.

The plan of the paper is as follows. We begin in Section 2 by giving a detailed explanation of the general approach outlined above. We then give two applications. The first, in Section 3, is to the tricategories of [12]. In this case it may seem redundant to define a notion of homomorphism, since as noted above there is already one in the literature. However, the homomorphisms that we define are better-behaved: they form a category whereas the trihomomorphisms of [12] form, at best, a bicategory (see [11] for the details). Now this may lead us to question whether our homomorphisms are in fact sufficiently weak. In order to show that they are, we devote Section 4 to a demonstration that the two different notions of homomorphism, though not strictly the same, are at least equivalent in a bicategorical sense. With this as justification, we then give in Section 5 the main application of our theory, to the definition of homomorphisms between the weak  $\omega$ -categories of Michael Batanin [2].

## 2. Homotopy-theoretic framework

We saw in the Introduction that in order to obtain a notion of homomorphism for some essentially-algebraic notion of higher-dimensional category, it suffices to generate a suitable comonad  $\mathbf{Q} = (Q, \Delta, \epsilon)$  on the category  $\mathbf{HCat}_s$  of strict homomorphisms: for then we may then *define* a homomorphism from  $A$  to  $B$  to be a strict homomorphism  $QA \rightarrow B$ . Moreover, we may compose two such homomorphisms  $f : QA \rightarrow B$  and  $g : QB \rightarrow C$  according to the formula

$$QA \xrightarrow{\Delta_A} QQA \xrightarrow{Qf} QB \xrightarrow{g} C,$$

and, from the comonad laws, see that this composition is associative and has identities given by the counit maps  $\epsilon_A : QA \rightarrow A$ . Thus we obtain a category  $\mathbf{HCat}$  of homomorphisms: it is the *co-Kleisli category* of the comonad  $\mathbf{Q}$ .

The purpose of this section is to describe how we may obtain suitable comonads by taking cofibrant replacements for a weak factorisation system on the category  $\mathbf{HCat}_s$ . As motivation, we first show how any weak factorisation system on a category gives rise to the data (though not

necessarily the axioms) for a comonad. We recall from [7] that a *weak factorisation system*, or *w.f.s.*,  $(\mathcal{L}, \mathcal{R})$  on a category  $\mathcal{C}$  is given by two classes  $\mathcal{L}$  and  $\mathcal{R}$  of morphisms in  $\mathcal{C}$  which are each closed under retracts when viewed as full subcategories of the arrow category  $\mathcal{C}^2$ , and which satisfy the two axioms of *factorisation*—that each  $f \in \mathcal{C}$  may be written as  $f = pi$  where  $i \in \mathcal{L}$  and  $p \in \mathcal{R}$ —and *lifting*—that for each  $i \in \mathcal{L}$  and  $p \in \mathcal{R}$ , we have  $i \pitchfork p$ , where to say that  $i \pitchfork p$  holds is to say that for each commutative square

$$\begin{array}{ccc} U & \xrightarrow{f} & W \\ i \downarrow & & \downarrow p \\ V & \xrightarrow{g} & X \end{array}$$

we may find a filler  $j : V \rightarrow W$  satisfying  $ji = f$  and  $pj = g$ . If  $(\mathcal{L}, \mathcal{R})$  is a w.f.s., then its two classes determine each other via the formulae

$$\begin{aligned} \mathcal{R} &= \mathcal{L}^{\pitchfork} := \{g \in \text{mor } \mathcal{C} \mid f \pitchfork g \text{ for all } f \in \mathcal{L}\} \quad \text{and} \\ \mathcal{L} &= \pitchfork \mathcal{R} := \{f \in \text{mor } \mathcal{C} \mid f \pitchfork g \text{ for all } g \in \mathcal{R}\}. \end{aligned}$$

For those weak factorisation systems that we will be considering, the following terminology will be appropriate: the maps in  $\mathcal{L}$  we call *cofibrations*, and the maps in  $\mathcal{R}$ , *acyclic fibrations*. Supposing  $\mathcal{C}$  to have an initial object  $0$ , we say that  $U \in \mathcal{C}$  is *cofibrant* just when the unique map  $0 \rightarrow U$  is a cofibration; and define a *cofibrant replacement* for  $X \in \mathcal{C}$  to be a cofibrant object  $Y$  together with an acyclic fibration  $p : Y \rightarrow X$ . The factorisation axiom implies that every  $X \in \mathcal{C}$  has a cofibrant replacement, obtained by factorising the unique map  $0 \rightarrow X$ . Suppose now that for every  $X$  we have made a choice of such, which we denote by  $\epsilon_X : QX \rightarrow X$ ; then by the lifting axiom, for every  $f : X \rightarrow Y$  in  $\mathcal{C}$  there exists a filler for the square on the left, and for every  $X \in \mathcal{C}$  a filler for the square on the right of the following diagram:

$$\begin{array}{ccc} 0 & \xrightarrow{!} & QY \\ ! \downarrow & \nearrow & \downarrow \epsilon_Y \\ QX & \xrightarrow{f \cdot \epsilon_X} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} 0 & \xrightarrow{!} & QQX \\ ! \downarrow & \nearrow & \downarrow \epsilon_{QX} \\ QX & \xrightarrow{1_{QX}} & QX. \end{array} \tag{2}$$

If we now suppose choices of such fillers to have been made—which we denote by  $Qf : QX \rightarrow QY$  and  $\Delta_X : QX \rightarrow QQX$  respectively—then we see that we have obtained all of the data required for a comonad  $(Q, \epsilon, \Delta)$ . However, because these data have been chosen arbitrarily, there is no reason to expect that the coassociativity and counit axioms should hold, that  $\Delta$  should be natural in  $X$ , or even that the assignation  $f \mapsto Qf$  should be functorial. Whilst in general we cannot resolve these issues, we may do so for a large class of w.f.s.’s, including those which in the sequel will interest us.

Recall that a w.f.s. is called *cofibrantly generated* by a set  $J \subseteq \mathcal{L}$  if  $\mathcal{R} = J^{\pitchfork}$ . The principal technique by which we build cofibrantly generated w.f.s.’s is the *small object argument* of Quillen [22, §II.3] and Bousfield [7], which tells us that if  $\mathcal{C}$  is a cocomplete category, and  $J$  a set of maps

in it satisfying a suitable smallness property, then there is a w.f.s.  $(\mathcal{L}, \mathcal{R})$  on  $\mathcal{C}$  given by  $\mathcal{R} = J^{\pitchfork}$  and  $\mathcal{L} = \pitchfork \mathcal{R}$ . These hypotheses are most easily satisfied if  $\mathcal{C}$  is a *locally finitely presentable (l.f.p.)* category—which is to say that it may be presented as the category of models for an essentially-algebraic theory, or equally well, the category of finite-limit preserving functors  $\mathcal{M} \rightarrow \mathbf{Set}$  for some finitely complete small category  $\mathcal{M}$ . In this case,  $\mathcal{C}$  is certainly cocomplete, and moreover any set of maps  $J$  in it will satisfy the required smallness property, and so generate a w.f.s. on  $\mathcal{C}$ .

Let us now define a *cofibrant replacement comonad* for a w.f.s.  $(\mathcal{L}, \mathcal{R})$  to be a comonad  $Q = (Q, \epsilon, \Delta)$  such that for each  $X \in \mathcal{C}$ , the map  $\epsilon_X : QX \rightarrow X$  provides a cofibrant replacement for  $X$ .

**Proposition 2.1.** *If  $\mathcal{C}$  is a l.f.p. category, and  $J$  a set of maps in it, then the w.f.s.  $(\mathcal{L}, \mathcal{R})$  cofibrantly generated by  $J$  may be equipped with a cofibrant replacement comonad.*

**Proof.** By examination of the construction used in the small object argument, we see that it provides a choice of  $(\mathcal{L}, \mathcal{R})$ -factorisation

$$X \xrightarrow{f} Y \quad \mapsto \quad X \xrightarrow{\lambda_f} Pf \xrightarrow{\rho_f} Y \quad (\text{for all } f \in \mathcal{C}) \tag{3}$$

that is *functorial*, in the sense that it provides the assignation on objects of a functor  $\mathcal{C}^2 \rightarrow \mathcal{C}^3$  which is a section of the “composition” functor  $\mathcal{C}^3 \rightarrow \mathcal{C}^2$ . In particular, by fixing  $X$  to be 0, we obtain a choice of cofibrant replacements  $\epsilon_Y : QY \rightarrow Y$  and of fillers  $Qf : QY \rightarrow QZ$  such that  $f \mapsto Qf$  is a functorial assignment and  $\epsilon$  a natural transformation. It remains only to construct natural maps  $\Delta_Y : QY \rightarrow QQY$  for which the comonad laws are satisfied, and this is done by Radulescu-Banu in [23, §1.1]; we omit the details.  $\square$

In principle, we could end this section here, since we have now shown how to associate a cofibrant replacement comonad to any (well-behaved) category equipped with a (well-behaved) w.f.s. However, there is something unsatisfactory about the previous proposition. An examination of its proof shows that a cofibrantly generated w.f.s.  $(\mathcal{L}, \mathcal{R})$  may well admit many different cofibrant replacement comonads, since the given construction relies on arbitrary choices of data which, in general, will induce non-isomorphic choices of  $Q$ . Firstly, we must choose a generating set  $J$  for  $(\mathcal{L}, \mathcal{R})$ ; and secondly, we must choose a (sufficiently large) regular cardinal  $\kappa$  that governs the length of the transfinite induction used in the application of the small object argument. The first of these choices should not worry us unduly, since in practice, it is the set  $J$  that one starts from, rather than the w.f.s. it generates. However, the second is a more substantial concern, since the piece of data on which it is predicated is one that ought to remain entirely internal to the workings of the small object argument. This raises the question as to whether there is a canonical—or better yet, *universal*—choice of cofibrant replacement comonad associated to a w.f.s.  $(\mathcal{L}, \mathcal{R})$ . We will now show that there is, at least once we have fixed a generating set  $J$ . To do so we will need to recall some definitions from [10].

**Definition 2.2.** Let  $(\mathcal{L}, \mathcal{R})$  be a w.f.s. on a category  $\mathcal{C}$ . An *algebraic realisation* of  $(\mathcal{L}, \mathcal{R})$  is given by the following pieces of data:

- For each  $f : X \rightarrow Y$  in  $\mathcal{C}$ , a choice of  $(\mathcal{L}, \mathcal{R})$  factorisation as in (3);

- For each commutative square as on the left of the following diagram, a choice of filler as on the right:

$$\begin{array}{ccc}
 U & \xrightarrow{h} & W \\
 f \downarrow & & \downarrow g \\
 V & \xrightarrow{k} & X
 \end{array}
 \quad \dashrightarrow \quad
 \begin{array}{ccc}
 U & \xrightarrow{\lambda_g \cdot h} & P g \\
 \lambda_f \downarrow & \nearrow P(h,k) & \downarrow \rho_g \\
 P f & \xrightarrow{k \cdot \rho_f} & X;
 \end{array}$$

- For each  $f : X \rightarrow Y$  in  $\mathcal{C}$ , choices of fillers for the following squares:

$$\begin{array}{ccc}
 X & \xrightarrow{\lambda_{\lambda_f}} & P \lambda_f \\
 \lambda_f \downarrow & \nearrow \sigma_f & \downarrow \rho_{\lambda_f} \\
 P f & \xrightarrow{1_{P f}} & P f
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 P f & \xrightarrow{1_{P f}} & P f \\
 \lambda_{\rho_f} \downarrow & \nearrow \pi_f & \downarrow \rho_f \\
 P \rho_f & \xrightarrow{\rho_{\rho_f}} & Y;
 \end{array}$$

subject to the following axioms:

- The assignment  $f \mapsto \lambda_f$  is the functor part of a comonad  $L$  on  $\mathcal{C}^2$  whose counit map at  $f$  is  $(1, \rho_f) : \lambda_f \rightarrow f$  and whose comultiplication is  $(1, \sigma_f) : \lambda_f \rightarrow \lambda_{\lambda_f}$ ;
- The assignment  $f \mapsto \rho_f$  is the functor part of a monad  $R$  on  $\mathcal{C}^2$  whose unit map at  $f$  is  $(\lambda_f, 1) : f \rightarrow \rho_f$  and whose multiplication is  $(\pi_f, 1) : \lambda_f \rightarrow \lambda_{\lambda_f}$ ;
- The natural transformation  $LR \Rightarrow RL : \mathcal{C}^2 \rightarrow \mathcal{C}^2$  whose component at  $f$  is  $(\sigma_f, \pi_f) : \lambda_{\rho_f} \rightarrow \rho_{\lambda_f}$  describes a distributive law in the sense of [4] between  $L$  and  $R$ .

The data for an algebraic realisation is sufficient to reconstruct the underlying w.f.s.  $(\mathcal{L}, \mathcal{R})$ : indeed, the classes  $\mathcal{L}$  and  $\mathcal{R}$  are the closure under retracts of the classes of maps admitting an  $L$ -coalgebra structure, respectively an  $R$ -algebra, structure. Hence the pairs  $(L, R)$  arising from algebraic realisations are objects worthy of study on their own: they are the *natural weak factorisation systems* of [13]. Note that the *data* for an algebraic realisation will exist for any weak factorisation system; the issue is whether or not we may choose it such that the *axioms* are satisfied. The main result of [10] is to show that for a cofibrantly generated w.f.s., we can, and moreover, that there is a best possible way of doing so.

**Proposition 2.3.** *Let  $\mathcal{C}$  be a l.f.p. category, and let  $J$  be a set of maps in it. Then the w.f.s.  $(\mathcal{L}, \mathcal{R})$  cofibrantly generated by  $J$  has a universally determined algebraic realisation.*

The sense of this universality is discussed in detail in [10, §3]; in brief, it says that the universal algebraic realisation  $(L, R)$  is freely generated by the requirement that each map  $j \in J$  should come equipped with a distinguished structure of  $L$ -coalgebra. In other words, given any other natural w.f.s.  $(L', R')$  on  $\mathcal{C}$  and a distinguished  $L'$ -coalgebra structure on each  $j \in J$ , we can find a unique morphism of natural w.f.s.'s (see [10, §3.3] for the definition)  $(L, R) \rightarrow (L', R')$  preserving the distinguished coalgebras. Note in particular that this universality is determined by

the set  $J$ , and not merely by the w.f.s. it generates; but as we have remarked before, this should not worry us unduly, since in practice it is the set  $J$ , rather than the w.f.s., from which one starts.

**Proof of Proposition 2.3.** For a full proof see [10, Theorem 4.4]: we recall only the salient details here. To construct the factorisation of a map  $f : X \rightarrow Y$  of  $\mathcal{C}$ , we begin exactly as in the small object argument. We form the set  $S$  whose elements are squares

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ j \downarrow & & \downarrow f \\ B & \xrightarrow{k} & Y \end{array}$$

such that  $j \in J$ . We then form the coproduct

$$\begin{array}{ccc} \sum_{x \in S} A_x & \xrightarrow{\langle h_x \rangle_{x \in S}} & X \\ \sum_{x \in S} j_x \downarrow & & \downarrow f \\ \sum_{x \in S} B_x & \xrightarrow{\langle h_x \rangle_{x \in S}} & Y \end{array}$$

and define an object  $P'g$  and morphisms  $\lambda'_g$  and  $\rho'_g$  by factorising this square as

$$\begin{array}{ccccc} \sum_{x \in S} A_x & \xrightarrow{\langle h_x \rangle_{x \in S}} & X & \xrightarrow{\text{id}_X} & X \\ \sum_{x \in S} j_x \downarrow & & \downarrow \lambda'_f & & \downarrow f \\ \sum_{x \in S} B_x & \longrightarrow & P'f & \xrightarrow{\rho'_f} & Y \end{array}$$

where the left-hand square is a pushout. The assignment  $f \mapsto \rho'_f$  may now be extended to a functor  $R' : \mathcal{C}^2 \rightarrow \mathcal{C}^2$ ; whereupon the map  $(\lambda'_f, \text{id}_Y) : f \rightarrow R'f$  provides the component at  $f$  of a natural transformation  $\Lambda' : \text{id}_{\mathcal{C}^2} \Rightarrow R'$ . We now obtain the monad part  $\mathbb{R}$  of the desired algebraic realisation as the free monad on the pointed endofunctor  $(R', \Lambda')$ . We may construct this using the techniques of [16]. To obtain the comonad part  $\mathbb{L}$  we proceed as follows. The assignment  $f \mapsto \lambda'_f$  underlies a functor  $L' : \mathcal{C}^2 \rightarrow \mathcal{C}^2$ ; and a little manipulation shows that this functor in turn underlies a comonad  $L'$  on  $\mathcal{C}^2$ . We may now adapt the free monad construction so that at the same time as it produces  $\mathbb{R}$  from  $(R', \Lambda')$ , it also produces  $\mathbb{L}$  from  $L'$ .  $\square$

**Corollary 2.4.** *Let  $\mathcal{C}$  be a l.f.p. category, and let  $J$  be a set of maps in it. Then the w.f.s.  $(\mathcal{L}, \mathcal{R})$  generated by  $J$  may be equipped with a universally determined cofibrant replacement comonad.*

**Proof.** Form the universal algebraic realisation  $(\mathbb{L}, \mathbb{R})$  of  $J$ ; now define the universal cofibrant replacement comonad to be the restriction of the comonad  $\mathbb{L}$  to the coslice category  $0/\mathcal{C} \cong \mathcal{C}$ .  $\square$

The preceding proofs provide us with a very general machinery for building the universal cofibrant replacement comonad of a w.f.s. In practice, however, it is often easier to describe directly what we think this comonad should be; and so we now give a recognition principle that will allow us to prove such a description to be correct.

**Definition 2.5.** Let  $J$  be a fixed set of maps in a category  $\mathcal{C}$ . Now for any  $f: Y \rightarrow X$  in  $\mathcal{C}$ , a *choice of liftings* for  $f$  (with respect to  $J$ ) is a function  $\phi$  which to every  $j \in J$  and commutative square

$$\begin{array}{ccc}
 A & \xrightarrow{h} & Y \\
 j \downarrow & \nearrow & \downarrow f \\
 B & \xrightarrow{k} & X
 \end{array}
 \tag{4}$$

in  $\mathcal{C}$  assigns a diagonal filler  $\phi(j, h, k): B \rightarrow Y$  making both triangles commute as indicated. We call the pair  $(f, \phi)$  an *algebraic acyclic fibration*. Given an object  $X \in \mathcal{C}$ , we define the category  $\mathbf{AAF}/X$  to have as objects, algebraic acyclic fibrations into  $X$ , and as morphisms  $(f, \phi) \rightarrow (g, \psi)$ , commutative triangles

$$\begin{array}{ccc}
 Y & \xrightarrow{u} & Z \\
 f \searrow & & \swarrow g \\
 & X &
 \end{array}$$

such that for any square of the form (4) we have  $u \cdot \phi(j, h, k) = \psi(j, uh, k)$ .

Our recognition principle is now the following:

**Proposition 2.6.** *Let  $J$  be a set of maps in a l.f.p. category  $\mathcal{C}$ . Then for each  $X \in \mathcal{C}$ , the universal cofibrant replacement  $\epsilon_X: QX \rightarrow X$  with respect to  $J$  may be equipped with a choice of liftings  $\phi_X$  such that  $(\epsilon_X, \phi_X)$  becomes an initial object of  $\mathbf{AAF}/X$ .*

**Proof.** Let  $(L, R)$  be the universal algebraic realisation of  $J$ . It follows from [10, Proposition 5.4] that  $\mathbf{AAF}/X$  is isomorphic to the category of algebras for the monad  $\mathbf{R}_X$  obtained by restricting and corestricting the monad  $\mathbf{R}: \mathcal{C}^2 \rightarrow \mathcal{C}^2$  to the slice category  $\mathcal{C}/X$ . As such, it has an initial object obtained by applying the free functor  $\mathcal{C}/X \rightarrow \mathbf{AAF}/X$  to the initial object  $0 \rightarrow X$  of  $\mathcal{C}/X$ . Moreover, the underlying map of this initial object is obtained by applying  $R$  to  $0 \rightarrow X$ , and hence is the universal cofibrant replacement  $\epsilon_X: QX \rightarrow X$ .  $\square$

**Example 2.7.** Let  $S$  be a commutative ring, and consider the category  $\mathbf{Ch}(S)$  of positively graded chain complexes of  $S$ -modules, equipped with the set of generating cofibrations  $J := \{2_i \hookrightarrow \partial_i \mid i \in \mathbb{N}\}$ . Here  $2_i$  is the representable chain complex at  $i$ , with components given by

$$(2_i)_n = \begin{cases} S & \text{if } n = i \text{ or } n = i - 1; \\ 0 & \text{otherwise,} \end{cases}$$



and differential being the identity map at stage  $i$  and the zero map elsewhere. The chain complex  $\partial_i$  is its boundary, whose components are

$$(\partial_i)_n = \begin{cases} S & \text{if } n = i - 1; \\ 0 & \text{otherwise,} \end{cases}$$

and whose differential is everywhere zero.  $\mathbf{Ch}(S)$  is a l.f.p. category, and so by Corollary 2.4 we may take universal cofibrant replacements with respect to  $J$ . We now give an explicit description of these cofibrant replacements. Given a chain complex  $X$ , the chain complex  $QX$  will be free in every dimension; and so it suffices to specify a set of free generators for each  $(QX)_i$  and to specify where each generator should be sent by the differential  $d_i : (QX)_i \rightarrow (QX)_{i-1}$  and the counit  $\epsilon_i : (QX)_i \rightarrow X_i$ . We do this by induction over  $i$ :

- For the base step,  $(QX)_0$  is generated by the set  $\{[x] \mid x \in X_0\}$ , and  $\epsilon_0 : (QX)_0 \rightarrow X_0$  is specified by  $\epsilon_0([x]) = x$ ;
- For the inductive step,  $(QX)_{i+1}$  (for  $i \geq 0$ ) is generated by the set

$$\{[x, z] \mid x \in X_{i+1}, z \in Z(QX)_i, \epsilon_i(z) = d_{i+1}(x)\},$$

whilst  $\epsilon_{i+1} : (QX)_{i+1} \rightarrow X_{i+1}$  and  $d'_{i+1} : (QX)_{i+1} \rightarrow (QX)_i$  are specified by  $\epsilon_{i+1}([x, z]) = x$  and  $d'_{i+1}([x, z]) = z$ ,

where given a chain complex  $A$ , we are writing  $ZA_i$  for the kernel of the map  $d_i : A_i \rightarrow A_{i-1}$ . To prove that  $\epsilon_X$  is the universal cofibrant replacement for  $X$ , it suffices, by Proposition 2.6, to equip it with a choice of liftings such that it becomes an initial object of  $\mathbf{AAF}/X$ . By inspection, to equip a chain map  $f : Y \rightarrow X$  with a choice of liftings is to give:

- A set function  $k_0 : X_0 \rightarrow Y_0$  which is a section of  $f_0 : Y_0 \rightarrow X_0$ ;
- For every  $i \geq 0$ , a set function  $k_{i+1} : X_{i+1} \times ZY_i \rightarrow Y_{i+1}$  which is a section of  $(f_{i+1}, d_{i+1}) : Y_{i+1} \rightarrow X_{i+1} \times ZY_i$ .

The map  $\epsilon_X : QX \rightarrow X$  has an obvious choice of liftings given by the inclusion of generators. We claim that this makes it initial in  $\mathbf{AAF}/X$ . Indeed, given  $f : Y \rightarrow X$  equipped with a choice of liftings  $\{k_i\}$ , there is a chain map  $h : QX \rightarrow Y$  given by the following recursion:

- For the base step,  $h_0$  is specified by  $h_0([x]) = k_0(x)$ ;
- For the inductive step,  $h_{i+1}$  is specified by  $h_{i+1}([x, z]) = k_{i+1}(x, h_i(z))$ .

It's easy to see that this  $h$  commutes with the projections to  $X$ , and with the given choices of liftings; and moreover, that it is the unique chain map  $QX \rightarrow Y$  with these properties. Hence, by Proposition 2.6,  $\epsilon_X : QX \rightarrow X$  is the universal cofibrant replacement of  $X$ .

Now, although Proposition 2.6 allows us to recognise the functor and the counit part of the universal cofibrant replacement comonad, it says nothing about its comultiplication. In fact, we may recover this using the initiality exhibited in Proposition 2.6. We first observe that if  $f : C \rightarrow D$  and  $g : D \rightarrow E$  are equipped with choices of liftings  $\phi$  and  $\psi$ , then their composite  $gf : C \rightarrow E$  may also be so equipped, via the assignment  $(\phi \bullet \psi)(j, h, k) := \phi(j, h, \psi(j, fh, k))$ .

**Proposition 2.8.** *Let  $J$  be a set of maps in a l.f.p. category  $\mathcal{C}$ . Then for each  $X \in \mathcal{C}$ , the unique map*

$$\begin{array}{ccc}
 QX & \xrightarrow{\quad\quad\quad} & QQX \\
 \searrow^{(\epsilon_X, \phi_X)} & & \swarrow_{(\epsilon_X \cdot \epsilon_{QX}, \phi_X \bullet \phi_{QX})} \\
 & X &
 \end{array} \tag{5}$$

of  $\mathbf{AAF}/X$  is the comultiplication  $\Delta_X : QX \rightarrow QQX$  of the universal cofibrant replacement comonad generated by  $J$ .

**Proof.** It suffices to check that  $\Delta_X : QX \rightarrow QQX$  renders (5) commutative, and that it respects the chosen liftings. The first of these conditions follows from the comonad axioms. For the second, we again make use of the isomorphism between  $\mathbf{AAF}/X$  and the category of algebras for the monad  $R_X : \mathcal{C}/X \rightarrow \mathcal{C}/X$  obtained from the universal algebraic realisation of  $J$ . To show that  $\Delta_X$  respects the chosen liftings in (5) is equally well to show that it respects the corresponding  $R_X$ -algebra structures on  $\epsilon_X$  and  $\epsilon_X \cdot \epsilon_{QX}$ , and we now do so by explicit calculation.

First let us introduce some notation: we write  $f$  to denote the unique map  $0 \rightarrow X$  in  $\mathcal{C}$ . Now the map  $\epsilon_X : QX \rightarrow X$  is equally well the map  $\rho_f : Pf \rightarrow X$ , and in these terms its  $R_X$ -algebra structure is the morphism

$$\begin{array}{ccc}
 P\rho_f & \xrightarrow{\pi_f} & Pf \\
 \searrow_{\rho_f} & & \swarrow_{\rho_f} \\
 & X &
 \end{array}$$

of  $\mathcal{C}/X$ . Likewise, the map  $\epsilon_X \cdot \epsilon_{QX} : QQX \rightarrow X$  is equally well the map  $\rho_f \cdot \rho_{\lambda_f} : P\lambda_f \rightarrow X$ , in which terms its  $R_X$ -algebra structure will be given by a morphism  $\theta_f : P(\rho_f \cdot \rho_{\lambda_f}) \rightarrow P\lambda_f$  over  $X$ . To describe this map we appeal to Theorem A.1 of [10], which shows that it is given by the following composite

$$P(\rho_f \cdot \rho_{\lambda_f}) \xrightarrow{\sigma_{\rho_f \cdot \rho_{\lambda_f}}} P\lambda_{\rho_f \cdot \rho_{\lambda_f}} \xrightarrow{P(1, P(\rho_{\lambda_f}, 1))} P(\lambda_{\rho_f} \cdot \rho_{\lambda_f}) \xrightarrow{P(1, \pi_f)} P\rho_{\lambda_f} \xrightarrow{\pi_{\lambda_f}} P\lambda_f.$$

Now, the map  $\Delta_X : QX \rightarrow QQX$  is equally well the map  $\sigma_f : Pf \rightarrow P\lambda_f$ , and so to check that it is an  $R_X$ -algebra map, and thereby complete the proof, it suffices to show that the square

$$\begin{array}{ccc}
 P\rho_f & \xrightarrow{P(\sigma_f, 1)} & P(\rho_f \cdot \rho_{\lambda_f}) \\
 \pi_f \downarrow & & \downarrow \theta_f \\
 Pf & \xrightarrow{\sigma_f} & P\lambda_f
 \end{array}$$

commutes; and this follows by a short calculation with the axioms for a natural w.f.s.  $\square$

**Example 2.9.** We consider again the situation of Example 2.7. Given a chain complex  $X$ , the canonical choice of liftings for the map  $\epsilon_X : \epsilon_{\mathcal{Q}X} : \mathcal{Q}\mathcal{Q}X \rightarrow X$  is given as follows:

- For the base step,  $k_0 : X_0 \rightarrow (\mathcal{Q}\mathcal{Q}X)_0$  is given by  $k_0(x) = [[x]]$ ;
- For the inductive step,  $k_{i+1} : X_{i+1} \times Z(\mathcal{Q}\mathcal{Q}X)_i \rightarrow (\mathcal{Q}\mathcal{Q}X)_{i+1}$  is given by  $k_{i+1}(x, z) = [[x, \epsilon_{\mathcal{Q}X}(z)], z]$ .

It follows from this, the description of the initiality of  $\epsilon_X$  given in Example 2.7, and Proposition 2.8, that the comultiplication map  $\Delta_X : \mathcal{Q}X \rightarrow \mathcal{Q}\mathcal{Q}X$  has components given by the following recursion:

- For the base step,  $\Delta_0 : (\mathcal{Q}X)_0 \rightarrow (\mathcal{Q}\mathcal{Q}X)_0$  is specified by  $\Delta_0([x]) = [[x]]$ ;
- For the inductive step,  $(\Delta_X)_{i+1} : (\mathcal{Q}X)_{i+1} \rightarrow (\mathcal{Q}\mathcal{Q}X)_{i+1}$  is specified by  $\Delta_{i+1}([x, z]) = [[x, z], \Delta_i(z)]$ .

### 3. Homomorphisms of tricategories

In the following sections we give two applications of the general theory described above. In the present section, we shall use it to develop a notion of homomorphism between the *tricategories* of [12]. We begin in Section 3.1 by defining a category **Tricat**<sub>s</sub> of tricategories and strict homomorphisms, and distinguishing in it a suitable set of generating cofibrations. Then in Section 3.2 we characterise the universal cofibrant replacement comonad this generates; and finally in Section 3.3, we extract a concrete description of the co-Kleisli category of this comonad, which will be the desired category of trihomomorphisms.

Since there is already in the literature a notion of trihomomorphism (see [12, §3], for instance), it is reasonable to ask why we should go to the effort of defining another one. There are two main reasons to do so. The first is that it illustrates the operation of our machinery in a relatively elementary case, which will prove useful in understanding the  $\omega$ -categorical application of Section 5 below. The second is that the trihomomorphisms we describe are better-behaved than the existing ones: in particular, ours admit a *strictly* associative and unital composition.

Now, the fact that our trihomomorphisms are better-behaved could suggest that they are insufficiently weak, and hence that our general machinery is not fit for the task. In order to show that this is not the case, we give in Section 4 a careful comparison between our trihomomorphisms and those of [12], and show that the two are the same in a suitable sense, by proving a biequivalence between two bicategories whose 0-cells are tricategories, and whose 1-cells are trihomomorphisms of the two different kinds.

#### 3.1. Generating cofibrations

The notion of tricategory was introduced in [12], yet the formulation given there is unsuitable for our purposes since it is not wholly algebraic: it asserts certain morphisms of a hom-bicategory to be equivalences without requiring choices of pseudo-inverse to be provided. Instead we shall adopt<sup>1</sup> the definition of [14], for which such choices are part of the data.

---

<sup>1</sup> With one minor alteration: we ask that the homomorphisms of bicategories  $1 \rightarrow \mathcal{T}(x, x)$  picking out units should be normalised. This change is not substantive, since any homomorphism of bicategories can be replaced with a normal one; but it does reduce slightly the amount of coherence data we have to deal with.

**Definition 3.1.** The category  $\mathbf{Tricat}_s$  has as objects, tricategories in the sense of [14, Chapter 4]; and as morphisms  $\mathcal{T} \rightarrow \mathcal{U}$ , assignments on 0-, 1-, 2- and 3-cells which commute with the tricategorical operations on the nose.

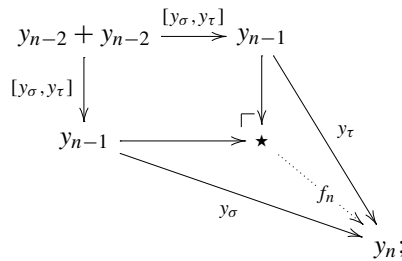
We observe that  $\mathbf{Tricat}_s$  is the category of models of an essentially-algebraic theory, and as such is locally finitely presentable. Therefore we may use Corollary 2.4 to describe a cofibrant replacement comonad on it, as soon as we have distinguished in it a suitable set of generating cofibrations. Before doing so, we observe that underlying any tricategory is a three-dimensional globular set; that is, a presheaf over the category  $\mathbf{G}_3$  generated by the graph

$$0 \begin{array}{c} \xrightarrow{\sigma} \\ \xrightarrow{\tau} \end{array} 1 \begin{array}{c} \xrightarrow{\sigma} \\ \xrightarrow{\tau} \end{array} 2 \begin{array}{c} \xrightarrow{\sigma} \\ \xrightarrow{\tau} \end{array} 3,$$

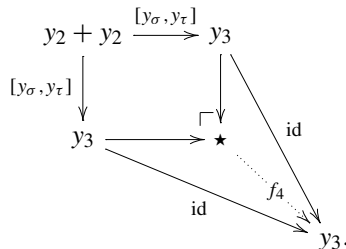
subject to the equations  $\sigma\sigma = \tau\sigma$  and  $\sigma\tau = \tau\tau$ . Thus there is a functor  $V : \mathbf{Tricat}_s \rightarrow [\mathbf{G}_3^{\text{op}}, \mathbf{Set}]$  which, because it is given by forgetting essentially-algebraic structure, has a left adjoint  $K : [\mathbf{G}_3^{\text{op}}, \mathbf{Set}] \rightarrow \mathbf{Tricat}_s$ .

**Definition 3.2.** The *generating cofibrations* of  $\mathbf{Tricat}_s$  are the maps  $\{t_n : \partial_n \rightarrow 2_n \mid 0 \leq n \leq 4\}$  obtained by applying the functor  $K$  to the morphisms  $f_0, \dots, f_4$  of  $[\mathbf{G}_3^{\text{op}}, \mathbf{Set}]$  given as follows (where we write  $y$  for the Yoneda embedding  $\mathbf{G}_3 \rightarrow [\mathbf{G}_3^{\text{op}}, \mathbf{Set}]$ ):

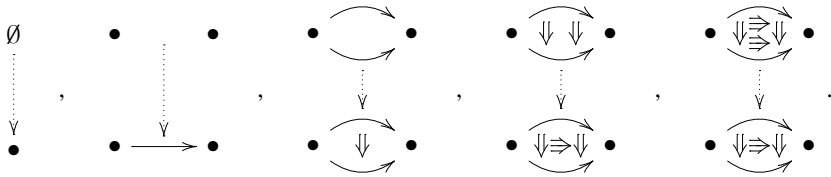
- $f_0$  is the unique map  $0 \rightarrow y_0$ ;
- $f_1$  is the map  $[y_\sigma, y_\tau] : y_0 + y_0 \rightarrow y_1$ ;
- $f_2$  and  $f_3$  are the maps induced by the universal property of pushout in the following diagram (for  $n = 2, 3$ ):



- $f_4$  is the map induced by the universal property of pushout in the following diagram:



In diagrammatic terms,  $f_0, \dots, f_4$  are the following maps:



**Definition 3.3.** We define  $\mathbf{Q} : \mathbf{Tricat}_s \rightarrow \mathbf{Tricat}_s$  to be the universal cofibrant replacement comonad for the generating cofibrations of Definition 3.2, and define the category **Tricat** of tricategories and trihomomorphisms to be the co-Kleisli category of this comonad.

3.2. Universal cofibrant replacement

The aim of this section is to obtain a concrete description of the comonad  $\mathbf{Q}$ . As in Example 2.7, the easiest way of doing this will not be to work through the construction given in Proposition 2.3; rather, it will be to describe directly the universal cofibrant replacements and then prove our description correct by appealing to Proposition 2.6. In order to give this description, we will need to develop some constructions on tricategories. First we observe that any tricategory  $\mathcal{T}$  has an underlying one-dimensional globular set, comprised of the 0- and 1-cells of  $\mathcal{T}$ ; and so we have an adjunction

$$L \dashv W : \mathbf{Tricat}_s \rightarrow [\mathbf{G}_1^{\text{op}}, \mathbf{Set}]$$

(where  $\mathbf{G}_1$  is the category  $1 \rightrightarrows 0$ ). Given some  $X \in [\mathbf{G}_1^{\text{op}}, \mathbf{Set}]$ , we may take  $LX$  to have the same 0-cells as  $X$ , and write  $[f] : x \rightarrow y$  for the image in  $LX$  of a 1-cell  $f : x \rightarrow y$  of  $X$ . We next describe what it means to *adjoin a 2-cell* to a tricategory  $\mathcal{T}$ . Given a pair of parallel 1-cells  $f, g : X \rightarrow Y$  in  $\mathcal{T}$ , there is a unique strict homomorphism  $(f, g) : \partial_2 \rightarrow \mathcal{T}$  sending the generating 1-cells of  $\partial_2$  to  $f$  and  $g$  respectively. Since  $\mathbf{Tricat}_s$  is locally finitely presentable, it is in particular cocomplete, and so we may define a new tricategory  $\mathcal{T}[\alpha]$  by means of the following pushout:

$$\begin{array}{ccc}
 \partial_2 & \xrightarrow{(f,g)} & \mathcal{T} \\
 \iota_2 \downarrow & & \downarrow \eta \\
 2_2 & \xrightarrow{\bar{\alpha}} & \mathcal{T}[\alpha].
 \end{array} \tag{6}$$

We say that  $\mathcal{T}[\alpha]$  is obtained from  $\mathcal{T}$  by adjoining a 2-cell  $\alpha : f \rightrightarrows g$ . Indeed, to give a strict homomorphism  $F : \mathcal{T}[\alpha] \rightarrow \mathcal{U}$  is equally well to give its restriction  $F\eta : \mathcal{T} \rightarrow \mathcal{U}$  together with the 2-cell  $Ff \rightrightarrows Fg$  named by  $F\bar{\alpha} : D_2 \rightarrow \mathcal{V}$ . By replacing the morphism  $\iota_2$  in (6) with a suitable coproduct of  $\iota_2$ 's, we may extend this definition to deal with the simultaneous adjunction to  $\mathcal{T}$  of any set-sized collection of 2-cells.

Finally, we observe that there is an orthogonal (or strong) factorisation system on  $\mathbf{Tricat}_s$  whose left class comprises those strict homomorphisms which are bijective on 0-, 1- and 2-cells,

and whose right class consists of those strict homomorphisms which are locally locally fully faithful; that is, those  $F : \mathcal{T} \rightarrow \mathcal{U}$  for which the following diagram of sets is a pullback:

$$\begin{array}{ccc}
 (VT)_3 & \xrightarrow{(VF)_3} & (VU)_3 \\
 (s,t) \downarrow & & \downarrow (s,t) \\
 (VT)_2 \times_{(VT)_1} (VT)_2 & \xrightarrow{(VF)_2 \times_{(VF)_1} (VF)_2} & (VU)_2 \times_{(VU)_1} (VU)_2.
 \end{array}$$

We now give an explicit construction of the universal cofibrant replacement  $\epsilon_{\mathcal{T}} : Q\mathcal{T} \rightarrow \mathcal{T}$  of a tricategory  $\mathcal{T}$ . We begin by defining  $\mathcal{T}_1$  to be  $LWT$ , the free tricategory on the underlying graph of  $\mathcal{T}$ , and  $e_1 : \mathcal{T}_1 \rightarrow \mathcal{T}$  to be the counit morphism. We now let  $\mathcal{T}_2$  be the tricategory obtained by adjoining the set of 2-cells

$$\{[\alpha] : f \Rightarrow g \mid f, g : X \rightarrow Y \text{ in } \mathcal{T}_1 \text{ and } \alpha : e_1(f) \Rightarrow e_1(g) \text{ in } \mathcal{T}\}$$

to  $\mathcal{T}_1$ , and define  $e_2 : \mathcal{T}_2 \rightarrow \mathcal{T}$  to be the unique strict homomorphism whose restriction to  $\mathcal{T}_1$  is  $e_1$ , and whose value at an adjoined 2-cell  $[\alpha] : f \Rightarrow g$  is  $\alpha : e_1(f) \Rightarrow e_1(g)$ . Finally, we obtain  $Q\mathcal{T}$  and  $\epsilon_{\mathcal{T}}$  by factorising  $e_2$  as

$$e_2 = \mathcal{T}_2 \xrightarrow{\psi_{\mathcal{T}}} Q\mathcal{T} \xrightarrow{\epsilon_{\mathcal{T}}} \mathcal{T} \tag{7}$$

where  $\psi_{\mathcal{T}}$  is the identity on 0-, 1- and 2-cells, and  $\epsilon_{\mathcal{T}}$  is locally locally fully faithful.

**Proposition 3.4.** *The strict homomorphism  $\epsilon_{\mathcal{T}} : Q\mathcal{T} \rightarrow \mathcal{T}$  is the universal cofibrant replacement of  $\mathcal{T}$ .*

**Proof.** We appeal to our recognition principle Proposition 2.6. First observe that a strict homomorphism  $F : \mathcal{U} \rightarrow \mathcal{T}$  may be equipped with a choice of liftings with respect to the generating cofibrations only if it is locally locally fully faithful; and that in this case, to give such a choice is to give:

- For each 0-cell  $t \in \mathcal{T}$ , a 0-cell  $k(t) \in \mathcal{U}$  with  $Fk(t) = t$ ;
- For each pair of 0-cells  $u, u'$  of  $\mathcal{U}$  and each 1-cell  $f : Fu \rightarrow Fu'$  of  $\mathcal{T}$ , a 1-cell  $k(f, u, u') : u \rightarrow u'$  of  $\mathcal{U}$  with  $Fk(f, u, u') = f$ ;
- For each parallel pair of 1-cells  $f, g : u \rightarrow u'$  of  $\mathcal{U}$  and each 2-cell  $\alpha : Ff \Rightarrow Fg$  of  $\mathcal{T}$ , a 2-cell  $k(\alpha, f, g) : f \Rightarrow g$  of  $\mathcal{U}$  with  $Fk(\alpha, f, g) = \alpha$ .

Observe now that  $\epsilon_{\mathcal{T}} : Q\mathcal{T} \rightarrow \mathcal{T}$  is locally locally fully faithful, and can be equipped with the following choice of liftings:

- Since  $Q\mathcal{T}$  has the same 0-cells as  $\mathcal{T}$ , we may take  $k(t) := t$ ;
- Since  $Q\mathcal{T}$  has the same 1-cells as  $\mathcal{T}_1$ , we may take  $k(f, u, u') := [f]$ ;
- Since  $Q\mathcal{T}$  has the same 2-cells as  $\mathcal{T}_2$ , we may take  $k(\alpha, f, g) := [\alpha]$ .

By Proposition 2.6, if we can show that this data determines an initial object of  $\mathbf{AAF}/\mathcal{T}$ , then we will have shown  $\epsilon_{\mathcal{T}} : Q\mathcal{T} \rightarrow \mathcal{T}$  to be the universal cofibrant replacement of  $\mathcal{T}$ . So suppose  $F : \mathcal{V} \rightarrow \mathcal{T}$  is another locally locally fully faithful strict homomorphism equipped with a choice of liftings  $k'$ . From this we first construct a strict homomorphism  $H : \mathcal{T}_2 \rightarrow \mathcal{V}$ ; and to do so, it suffices to specify where  $H$  should send each 0-cell  $t$ , each generating 1-cell  $[f] : t \rightarrow t'$ , and each generating 2-cell  $[\alpha] : f \Rightarrow g$ . So we take:

- $H(t) = k'(t)$ ;
- $H([f] : t \rightarrow t') = k'(f, Ht, Ht')$ ;
- $H([\alpha] : f \Rightarrow g) = k'(\alpha, Hf, Hg)$ .

Now we observe that the outside of the following diagram commutes:

$$\begin{array}{ccc}
 \mathcal{T}_2 & \xrightarrow{H} & \mathcal{V} \\
 \psi_{\mathcal{T}} \downarrow & \nearrow & \downarrow F \\
 Q\mathcal{T} & \xrightarrow{\epsilon_{\mathcal{T}}} & \mathcal{T}
 \end{array}$$

and since  $\psi_{\mathcal{T}}$  is bijective on 0-, 1- and 2-cells, and  $F$  is locally locally fully faithful, it follows that there is a unique strict homomorphism  $K : Q\mathcal{T} \rightarrow \mathcal{V}$  (as indicated) rendering both induced triangles commutative. It's now straightforward to prove that  $K$  commutes with the specified choices of lifting, and that moreover it is the unique strict homomorphism that does so.  $\square$

Thus we have characterised the functor  $Q$  and its counit  $\epsilon : Q \Rightarrow \text{id}$ ; and it remains only to describe the comultiplication  $\Delta : Q \Rightarrow QQ$ .

**Proposition 3.5.** *The strict homomorphism  $\Delta_{\mathcal{T}} : Q\mathcal{T} \rightarrow QQ\mathcal{T}$  is uniquely determined by the following assignments:*

- On 0-cells,  $\Delta_{\mathcal{T}}(t) = t$ ;
- On 1-cells,  $\Delta_{\mathcal{T}}([f] : t \rightarrow t') = [[f]] : t \rightarrow t'$ ;
- On 2-cells,  $\Delta_{\mathcal{T}}([\alpha] : f \Rightarrow g) = [[\alpha]] : \Delta_{\mathcal{T}}(f) \Rightarrow \Delta_{\mathcal{T}}(g)$ ;
- On 3-cells,  $\Delta_{\mathcal{T}}(\Gamma : \alpha \Rrightarrow \beta) = \Gamma : \Delta_{\mathcal{T}}(\alpha) \Rrightarrow \Delta_{\mathcal{T}}(\beta)$ .

**Proof.** Observe first that the above data determine a unique homomorphism  $K : Q\mathcal{T} \rightarrow QQ\mathcal{T}$  commuting with the maps into  $\mathcal{T}$ . Therefore by Proposition 2.8 it suffices to check that  $K$  also commutes with the canonical choices of liftings for these maps. For  $\epsilon_{\mathcal{T}}$ , these liftings are given as in Proposition 3.4; whilst for  $\epsilon_{\mathcal{T}} \cdot \epsilon_{Q\mathcal{T}}$ , they are given as follows:

- On 0-cells,  $k(t) = t$ ;
- On 1-cells,  $k(f, u, u') = [[f]]$ ;
- On 2-cells,  $k(\alpha, f, g) = [[\alpha]]$ .

These liftings are manifestly preserved by  $K$ , so that  $K = \Delta_X$  as required.  $\square$

### 3.3. Trihomomorphisms

Recall that **Tricat**, the category of tricategories and trihomomorphisms, is defined to be the co-Kleisli category of the comonad  $\mathbf{Q}$ . Our goal in the remainder of this section is to give an elementary description of this category that does not require us to invoke the comonad  $\mathbf{Q}$ .

Now, morphisms  $\mathcal{T} \rightarrow \mathcal{U}$  in **Tricat** are given by strict homomorphisms  $Q\mathcal{T} \rightarrow \mathcal{U}$ , and so we wish to characterise these latter maps in a manner that does not refer to  $Q$ . First let us observe that precomposition with  $\psi_{\mathcal{T}}$  sends each such map to a strict homomorphism  $\mathcal{T}_2 \rightarrow \mathcal{U}$ , and these latter have an easy characterisation: they are given by a map  $F : \mathcal{T}_1 \rightarrow \mathcal{U}$ —which, since  $\mathcal{T}_1 = LWT$ , is equally well a map of underlying 1-globular sets  $WT \rightarrow W\mathcal{U}$ —together with, for every pair of arrows  $f, g : X \rightarrow Y$  in  $\mathcal{T}_1$  and 2-cell  $\alpha : e_1(f) \Rightarrow e_1(g)$  in  $\mathcal{T}$ , a 2-cell  $F\alpha : Ff \Rightarrow Fg$  of  $\mathcal{U}$ . Thus, in order to characterise the trihomomorphisms  $\mathcal{T} \rightarrow \mathcal{U}$ , it will be enough to determine what extra data is required in order to extend a strict homomorphism  $\mathcal{T}_2 \rightarrow \mathcal{U}$  to one  $Q\mathcal{T} \rightarrow \mathcal{U}$ . However, the construction we have of  $Q\mathcal{T}$  from  $\mathcal{T}_2$ , in terms of the factorisation (7), is not suitable for this purpose; and so we shall now give an alternative construction, one that builds  $Q\mathcal{T}$  from  $\mathcal{T}_2$  through the adjunction of 3-cells and of 3-cell equations.

Recall that to *adjoin a 2-cell* to a tricategory  $\mathcal{T}$  is to take a pushout of the form (6). By replacing the morphism  $\iota_2 : \partial_2 \rightarrow 2_2$  in this diagram with  $\iota_3$  or  $\iota_4$ , we can say what it means to *adjoin a 3-cell* or to *adjoin a 3-cell equation* to  $\mathcal{T}$ : and hence what it means to adjoin an *invertible* 3-cell to  $\mathcal{T}$ —namely, to adjoin 3-cells  $\alpha \Rightarrow \beta$  and  $\beta \Rightarrow \alpha$  together with equations asserting these 3-cells to be mutually inverse. We shall now give a construction of  $Q\mathcal{T}$  from  $\mathcal{T}_2$  through the adjunction first of a number of (invertible) 3-cells, and then of a number of 3-cell equations.

**Definition 3.6.** The tricategory  $\mathcal{T}_3$  is the result of adjoining the following 3-cells to  $\mathcal{T}_2$ :

- 3-cells  $[\Gamma] : [\alpha] \Rightarrow [\beta] : f \Rightarrow g$  for  $\Gamma : \alpha \Rightarrow \beta : e_2(f) \Rightarrow e_2(g)$  in  $\mathcal{T}$ ;
- Invertible 3-cells  $V_{\alpha, \beta} : [\beta] \circ [\alpha] \Rightarrow [\beta \circ \alpha] : f \Rightarrow h$  for  $[\alpha] : f \Rightarrow g$  and  $[\beta] : g \Rightarrow h$  in  $\mathcal{T}_2$ ;
- Invertible 3-cells  $H_{\alpha, \beta} : [\beta] \otimes [\alpha] \Rightarrow [\beta \otimes \alpha] : h \otimes f \Rightarrow k \otimes g$  for  $[\alpha] : f \Rightarrow g : x \rightarrow y$  and  $[\beta] : h \Rightarrow k : y \rightarrow z$  in  $\mathcal{T}_2$ ;
- Invertible 3-cells  $U_f : 1_f \Rightarrow [1_{e_2(f)}] : f \Rightarrow f$  for  $f : x \rightarrow y$  in  $\mathcal{T}_2$ ;
- Invertible 3-cells  $L_f : l_f \Rightarrow [l_{e_2(f)}] : I_y \otimes f \Rightarrow f$  for  $f : x \rightarrow y$  in  $\mathcal{T}_2$ ;
- Invertible 3-cells  $R_f : r_f \Rightarrow [r_{e_2(f)}] : f \otimes I_x \Rightarrow f$  for  $f : x \rightarrow y$  in  $\mathcal{T}_2$ ; and
- Invertible 3-cells  $A_{fgh} : a_{fgh} \Rightarrow [a_{e_2(f), e_2(g), e_2(h)}] : (h \otimes g) \otimes f \Rightarrow h \otimes (g \otimes f)$  for  $f : x \rightarrow y, g : y \rightarrow z$  and  $h : z \rightarrow w$  in  $\mathcal{T}_2$ .

The next step will be to adjoin a number of 3-cell equations to  $\mathcal{T}_3$  to obtain a tricategory  $\mathcal{T}_4$ , which in Proposition 3.9 below we will be able to prove isomorphic to  $Q\mathcal{T}$ . Before constructing  $\mathcal{T}_4$ , we give an auxiliary definition which will make the task appreciably simpler.

**Definition 3.7.** For every 2-cell  $\gamma$  of  $\mathcal{T}_2$ , we define a 3-cell  $\rho_\gamma : \gamma \Rightarrow [e_2(\gamma)]$  of  $\mathcal{T}_3$  by structural induction over  $\gamma$ , exploiting the fact that the 2-cells under consideration are *freely* generated by those of the form  $[\alpha] : f \Rightarrow g$ .

- If  $\gamma = [\alpha] : f \Rightarrow g$  for some  $\alpha : e_2(f) \Rightarrow e_2(g)$  in  $\mathcal{T}$ , we take  $\rho_\gamma = \text{id}_{[\alpha]}$ ;
- If  $\gamma = \beta \circ \alpha : f \Rightarrow h$  for some  $\alpha : f \Rightarrow g$  and  $\beta : g \Rightarrow h$ , then we take  $\rho_\gamma$  to be the composite

$$\beta \circ \alpha \xrightarrow{\rho_\beta \circ \rho_\alpha} [\beta] \circ [\alpha] \xrightarrow{V_{e_2(\alpha), e_2(\beta)}} [e_2(\beta) \circ e_2(\alpha)] = [e_2(\beta \circ \alpha)];$$



- If  $\gamma = \beta \otimes \alpha : h \otimes f \Rightarrow k \otimes g$  for some  $\alpha : f \Rightarrow g : x \rightarrow y$  and some  $\beta : h \Rightarrow k : y \rightarrow z$ , then we take  $\rho_\gamma$  to be the composite

$$\beta \otimes \alpha \xrightarrow{\rho_\beta \otimes \rho_\alpha} [e_2(\beta)] \otimes [e_2(\alpha)] \xrightarrow{H_{e_2(\alpha), e_2(\beta)}} [e_2(\beta) \otimes e_2(\alpha)] = [e_2(\beta \otimes \alpha)];$$

- If  $\gamma = 1_f : f \Rightarrow f$  for some  $f : x \rightarrow y$ , then we take  $\rho_\gamma = U_f$ ;
- If  $\gamma = l_f, r_f$  or  $a_{fgh}$ , then we take  $\rho_\gamma = L_f, R_f$  or  $A_{fgh}$  respectively;
- If  $\gamma = l'_f : f \Rightarrow I_y \otimes f$ —where we recall from [14] that such a 2-cell participates in a specified adjoint equivalence  $(\eta_f, \epsilon_f)$  with  $l_f$ —then we obtain  $\rho_\gamma$  as follows. First we define a 3-cell  $\tilde{\eta}_f : 1_f \Rightarrow [l_{e_2(f)}] \circ [l'_{e_2(f)}]$  as the composite

$$1_f \xrightarrow{U_f} [1_{e_2(f)}] \xrightarrow{[\eta_{e_2(f)}]} [l_{e_2(f)} \circ l'_{e_2(f)}] \xrightarrow{V^{-1}} [l_{e_2(f)}] \circ [l'_{e_2(f)}]; \tag{8}$$

and now we take  $\rho_\gamma$  to be the pasting composite

$$\begin{array}{ccccc}
 & & f & \xlongequal{\quad} & f & \xrightarrow{l'_f} & I_y \otimes f. \\
 & & \uparrow & & \uparrow & \Rightarrow & \uparrow \\
 & & \tilde{\eta}_f & \Rightarrow & [l_{e_2(f)}] & \xrightarrow{L_f^{-1}} & [l'_{e_2(f)}] \\
 & & \uparrow & & \uparrow & & \uparrow \\
 f & \xrightarrow{\quad} & I_y \otimes f & \xlongequal{\quad} & I_y \otimes f & & f \\
 & & [l'_{e_2(f)}] & & & & 
 \end{array} \tag{9}$$

The cases  $\gamma = r'_f$  and  $\gamma = a'_{fgh}$  proceed analogously.

**Definition 3.8.** The tricategory  $\mathcal{T}_4$  is obtained by adjoining the following 3-cell equalities to  $\mathcal{T}_3$ . First we force compatibility with composition in every dimension.

- For each  $[\Gamma] : [\alpha] \Rightarrow [\beta]$  and  $[\Delta] : [\beta] \Rightarrow [\gamma]$ , we require that

$$[\Delta] \circ [\Gamma] = [\Delta \circ \Gamma] : [\alpha] \Rightarrow [\gamma];$$

- For each  $[\alpha] : f \Rightarrow g$  we require that

$$\text{id}_{[\alpha]} = [\text{id}_\alpha] : [\alpha] \Rightarrow [\alpha];$$

- For each  $[\Gamma] : [\alpha] \Rightarrow [\beta] : f \Rightarrow g$  and  $[\Delta] : [\gamma] \Rightarrow [\delta] : g \Rightarrow h$  we require that the following diagram should commute:

$$\begin{array}{ccc}
 [\gamma] \circ [\alpha] & \xrightarrow{V} & [\gamma \circ \alpha] \\
 [\Delta] \circ [\Gamma] \downarrow & & \downarrow [\Delta \circ \Gamma] \\
 [\delta] \circ [\beta] & \xrightarrow{V} & [\delta \circ \beta];
 \end{array}$$

- For each  $[\Gamma]: [\alpha] \Rightarrow [\beta]: x \rightarrow y$  and  $[\Delta]: [\gamma] \Rightarrow [\delta]: y \rightarrow z$  we require that the following diagram should commute:

$$\begin{array}{ccc}
 [\gamma] \otimes [\alpha] & \xrightarrow{H} & [\gamma \otimes \alpha] \\
 [\Delta] \otimes [\Gamma] \downarrow & & \downarrow [\Delta \otimes \Gamma] \\
 [\delta] \otimes [\beta] & \xrightarrow{H} & [\delta \otimes \beta].
 \end{array}$$

The remaining equations we adjoin ensure compatibility between the structural 3-cells of  $\mathcal{T}$  and those of the tricategory we are defining. We begin by considering associativity and unitality constraints in the hom-bicategories.

- For each  $[\alpha]: f \rightarrow g$ , we require that the following diagrams should commute:

$$\begin{array}{ccc}
 1_g \circ [\alpha] \xrightarrow{\rho} [1_{e_2(g)} \circ \alpha] & & [\alpha] \circ 1_f \xrightarrow{\rho} [\alpha \circ 1_{e_2(f)}] \\
 \cong \downarrow & \text{and} & \cong \downarrow \\
 [\alpha] \xlongequal{\quad} [\alpha] & & [\alpha] \xlongequal{\quad} [\alpha]; \\
 \downarrow [\cong] & & \downarrow [\cong]
 \end{array}$$

- For each  $[\alpha]: f \rightarrow g$ ,  $[\beta]: g \rightarrow h$  and  $[\gamma]: h \rightarrow k$ , we require that the following diagram should commute:

$$\begin{array}{ccc}
 ([\gamma] \circ [\beta]) \circ [\alpha] \xrightarrow{\rho} [(\gamma \circ \beta) \circ \alpha] \\
 \cong \downarrow & & \downarrow [\cong] \\
 [\gamma] \circ ([\beta] \circ [\alpha]) \xrightarrow{\rho} [\gamma \circ (\beta \circ \alpha)].
 \end{array}$$

Next we require compatibility with the 3-cells which mediate middle-four interchange and its nullary analogue.

- For each suitable  $[\alpha]$ ,  $[\beta]$ ,  $[\gamma]$  and  $[\delta]$  we require the following diagram to commute:

$$\begin{array}{ccc}
 ([\delta] \otimes [\beta]) \circ ([\gamma] \otimes [\alpha]) \xrightarrow{\rho} [(\delta \otimes \beta) \circ (\gamma \otimes \alpha)] \\
 \cong \downarrow & & \downarrow [\cong] \\
 ([\delta] \circ [\gamma]) \otimes ([\beta] \circ [\alpha]) \xrightarrow{\rho} [(\delta \circ \gamma) \otimes (\beta \circ \alpha)];
 \end{array}$$

- For each  $f : x \rightarrow y$  and  $g : y \rightarrow z$ , we require the following diagram to commute:

$$\begin{array}{ccc} 1_g \otimes f & \xrightarrow{\rho} & [1_{e_2(g)} \otimes e_2(f)] \\ \cong \downarrow & & \downarrow [\cong] \\ 1_g \otimes 1_f & \xrightarrow{\rho} & [1_{e_2(g)} \otimes 1_{e_2(f)}]. \end{array}$$

Next we ensure compatibility with the pseudonaturality cells for the associativity and unitality constraints  $a$ ,  $l$  and  $r$ .

- For all suitable 2-cells  $[\alpha] : f \Rightarrow m$ ,  $[\beta] : g \Rightarrow n$  and  $[\gamma] : h \Rightarrow p$ , we require that the following diagram should commute:

$$\begin{array}{ccc} a_{m,n,p} \circ (([\gamma] \otimes [\beta]) \otimes [\alpha]) & \xrightarrow{\rho} & [a_{e_2(m),e_2(n),e_2(p)} \circ ((\gamma \otimes \beta) \otimes \alpha)] \\ \cong \downarrow & & \downarrow [\cong] \\ ([\gamma] \otimes ([\beta] \otimes [\alpha])) \circ a_{f,g,h} & \xrightarrow{\rho} & [(\gamma \otimes (\beta \otimes \alpha)) \circ a_{e_2(f),e_2(g),e_2(h)}]; \end{array}$$

- For each  $[\alpha] : f \Rightarrow g : x \rightarrow y$ , we require that the following diagrams should commute:

$$\begin{array}{ccc} l_g \circ (I_y \otimes [\alpha]) & \xrightarrow{\rho} & [l_{e_2(g)} \circ (I_y \otimes \alpha)] \\ \cong \downarrow & & \downarrow [\cong] \\ [\alpha] \circ l_f & \xrightarrow{\rho} & [\alpha \circ l_{e_2(f)}] \end{array} \quad \text{and} \quad \begin{array}{ccc} r_g \circ ([\alpha] \otimes I_x) & \xrightarrow{\rho} & [r_{e_2(g)} \circ (\alpha \otimes I_x)] \\ \cong \downarrow & & \downarrow [\cong] \\ [\alpha] \circ r_f & \xrightarrow{\rho} & [\alpha \circ r_{e_2(f)}]. \end{array}$$

Finally, we ensure compatibility with the coherence 3-cells  $\pi$  and  $\mu$ .

- For all composable 1-cells  $f$ ,  $g$ ,  $h$  and  $k$ , we require that the following diagram should commute:

$$\begin{array}{ccc} (k \otimes a_{fgh}) \circ (a_{f,h \otimes g,k} \circ (a_{ghk} \otimes f)) & \xrightarrow{\pi} & a_{g \otimes f,h,k} \circ a_{f,g,k \otimes h} \\ \rho \downarrow & & \downarrow \rho \\ [(\dot{k} \otimes a_{\dot{f},\dot{g},\dot{h}}) \circ (a_{\dot{f},\dot{h} \otimes \dot{g},\dot{k}} \circ (a_{\dot{g},\dot{h},\dot{k}} \otimes \dot{f})))] & \xrightarrow{[\pi]} & [a_{\dot{g} \otimes \dot{f},\dot{h},\dot{k}} \circ a_{\dot{f},\dot{g},\dot{k} \otimes \dot{h}}] \end{array}$$

where we write  $\dot{f}$  as an abbreviation for  $e_2(f)$ , and so on;

- For all 1-cells  $f : x \rightarrow y$  and  $g : y \rightarrow z$  we require that the following diagram should commute:

$$\begin{array}{ccc}
 (g \otimes l_f) \circ (a_{f, l_y, g} \circ (r_g^* \otimes f)) & \xrightarrow{\mu} & 1_{g \otimes f} \\
 \rho \downarrow & & \downarrow \rho \\
 [(e_2(g) \otimes l_{e_2(f)}) \circ (a_{e_2(f), l_y, e_2(g)} \circ (r_{e_2(g)}^* \otimes e_2(f)))] & \xrightarrow{[\mu]} & [1_{e_2(g) \otimes e_2(f)}].
 \end{array}$$

**Proposition 3.9.** *The tricategory  $\mathcal{T}_4$  is isomorphic to  $QT$  in  $\mathbf{Tricat}_s$ .*

**Proof.** Let us write  $\phi$  for the canonical map  $\mathcal{T}_2 \rightarrow \mathcal{T}_4$ . We begin by factorising  $e_2 : \mathcal{T}_2 \rightarrow \mathcal{T}$  as

$$\mathcal{T}_2 \xrightarrow{\phi} \mathcal{T}_4 \xrightarrow{e_4} \mathcal{T}. \tag{10}$$

To do so we must first specify where  $e_4$  will take each of the adjoined 3-cells in  $\mathcal{T}_4$ ; and then check that the images under  $e_4$  of the adjoined 3-cell equations are satisfied. We do this by sending each 3-cell  $[\Gamma] : [\alpha] \Rightarrow [\beta]$  to  $\Gamma : \alpha \Rightarrow \beta$ ; and each of the other 3-cells  $U, V, H, L, R$  and  $A$  to the appropriate identity morphism. It’s easy to see that the requisite 3-cell equations are then satisfied, and so we obtain the desired factorisation (10). We observe that  $\phi$  is bijective on 0-, 1- and 2-cells, and so if we are able to show  $e_4$  to be locally locally fully faithful, then—by the essential uniqueness of such factorisations—we can deduce the existence of an isomorphism  $\theta : QT \cong \mathcal{T}_4$  as desired.

Thus, given 2-cells  $\gamma$  and  $\delta$  of  $\mathcal{T}_4$  we aim to show that every 3-cell  $\Gamma : e_4(\gamma) \Rightarrow e_4(\delta)$  of  $\mathcal{T}$  has the form  $e_4(\tilde{\Gamma})$  for a unique 3-cell  $\tilde{\Gamma} : \gamma \Rightarrow \delta$  of  $\mathcal{T}_4$ . Now, by Definition 3.7, we have invertible 3-cells  $\rho_\gamma : \gamma \Rightarrow [e_4(\gamma)]$  and  $\rho_\delta : \delta \Rightarrow [e_4(\delta)]$ , and by structural induction can show that these maps are sent by  $e_4$  to identity 3-cells. Accordingly, the 3-cell

$$\tilde{\Gamma} := \gamma \xrightarrow{\rho_\gamma} [e_4(\gamma)] \xrightarrow{[\Gamma]} [e_4(\delta)] \xrightarrow{\rho_\delta^{-1}} \delta$$

of  $\mathcal{T}_4$  satisfies  $e_4(\tilde{\Gamma}) = \Gamma$ ; and it remains only to show that it is unique with this property. We shall do this by proving that, for every 3-cell  $\Delta : \gamma \Rightarrow \delta$  of  $\mathcal{T}_4$ , the following square commutes:

$$\begin{array}{ccc}
 \gamma & \xrightarrow{\Delta} & \delta \\
 \rho_\gamma \downarrow & & \downarrow \rho_\delta \\
 [e_4(\gamma)] & \xrightarrow{[e_4(\Delta)]} & [e_4(\delta)]
 \end{array} ; \tag{11}$$

as then  $e_4(\Delta) = \Gamma$  implies that  $\Delta = \rho_\delta^{-1} \circ [e_4(\Delta)] \circ \rho_\gamma = \rho_\delta^{-1} \circ [\Gamma] \circ \rho_\gamma = \tilde{\Gamma}$  as required. Since the 3-cells of  $\mathcal{T}_4$  are generated—albeit not freely—by those of the form  $[\Gamma], U, V, H, A, L$  and  $R$ , we may obtain commutativity in (11) by a structural induction on the form of  $\Delta$ . The commutativity is immediate when  $\Delta$  is one of the generating 3-cells listed above; and has been explicitly adjoined in all cases where  $\Delta$  is a derived 3-cell of  $\mathcal{T}_4$ , save for that where  $\Delta$  is a

unit or counit map for one of the adjoint equivalences  $l_f^* \dashv l_f$ ,  $r_f^* \dashv r_f$  or  $a_{fgh}^* \dashv a_{fgh}$ . As a representative sample of these cases, we show the square

$$\begin{array}{ccc}
 1_f & \xrightarrow{\eta_f} & l_f \circ l_f^* \\
 \rho \downarrow & & \downarrow \rho \\
 [1_{e_2(f)}] & \xrightarrow{[\eta_f]} & [l_{e_2(f)} \circ l_{e_2(f)}^*]
 \end{array}$$

to be commutative. Writing  $\tilde{\eta}_f$  for the 3-cell  $1_f \Rrightarrow [l_{e_2(f)} \circ l_{e_2(f)}^*]$  of Eq. (8) and  $L_f^*$  for the 3-cell  $l_f^* \Rrightarrow [l_{e_2(f)}^*]$  of (9), this is equally well to show that

$$\begin{array}{ccc}
 1_f & \xrightarrow{\eta_f} & l_f \circ l_f^* \\
 \tilde{\eta}_f \downarrow & & \downarrow \text{id} \circ L_f^* \\
 [l_{e_2(f)}] \circ [l_{e_2(f)}^*] & \xrightarrow{L_f^{-1} \circ \text{id}} & l_f \circ [l_{e_2(f)}^*]
 \end{array}$$

commutes; which follows by observing that  $L_f^*$  is the mate of  $L_f^{-1}$  under the adjunctions  $l_f^* \dashv l_f$  and  $[l_{e_2(f)}^*] \dashv [l_{e_2(f)}]$ .  $\square$

We may now assemble all of the above calculations to give an elementary description of the category **Tricat**. In order to give this without referencing the comonad  $\mathbf{Q}$ , we will first need to introduce some notation. For objects  $x, y$  of a tricategory  $\mathcal{T}$ , we define a *formal composite* of 1-cells  $f : x \dashrightarrow y$  by the following clauses:

- If  $x \in \mathcal{T}$  then  $I_x : x \dashrightarrow x$ ;
- If  $f : x \rightarrow y$  in  $\mathcal{T}$  then  $[f] : x \dashrightarrow y$ ;
- If  $f : x \dashrightarrow y$  and  $g : y \dashrightarrow z$  then  $g \otimes f : x \dashrightarrow z$ .

For each formal composite  $f : x \dashrightarrow y$  we recursively define its *realisation*  $|f| : x \rightarrow y$  by taking  $|[f]| = f$ ,  $|I_x| = I_x$  and  $|g \otimes f| = |g| \otimes |f|$ . Moreover, if given a second tricategory  $\mathcal{U}$  and a source- and target-preserving assignation  $F$  from the 0- and 1-cells of  $\mathcal{T}$  to those of  $\mathcal{U}$ , then we induce a mapping from formal composites  $x \dashrightarrow y$  to those  $Fx \dashrightarrow Fy$  by another recursion; we take  $F[f] = [Ff]$ ,  $F I_x = I_{Fx}$  and  $F(g \otimes f) = Fg \otimes Ff$ . We may now give our elementary restatement of the definition of **Tricat**; that it is in accordance with Definition 3.3 is a direct consequence of Propositions 3.5 and 3.9.

**Definition 3.10.** The category **Tricat** has as its objects, the tricategories of [14, Chapter 4]; whilst its maps  $F : \mathcal{T} \rightarrow \mathcal{U}$  are given by the following basic data:

- For each  $x \in \mathcal{T}$ , an object  $Fx \in \mathcal{U}$ ;
- For each  $f : x \rightarrow y$  of  $\mathcal{T}$ , a 1-cell  $Ff : Fx \rightarrow Fy$  of  $\mathcal{U}$ ;
- For each  $f, g : x \dashrightarrow y$  and  $\alpha : |f| \Rrightarrow |g|$  in  $\mathcal{T}$ , a 2-cell  $F_{f,g}(\alpha) : |Ff| \Rrightarrow |Fg|$  of  $\mathcal{U}$ ;

- For each  $f, g : x \dashrightarrow y$ , each  $\alpha, \beta : |f| \Rightarrow |g|$  and each  $\Gamma : \alpha \Rightarrow \beta$  in  $\mathcal{T}$ , a 3-cell  $F_{f,g}(\Gamma) : F_{f,g}(\alpha) \Rightarrow F_{f,g}(\beta)$  of  $\mathcal{U}$ ;

and the following coherence data:

- For each  $f, g, h : x \dashrightarrow y$ ,  $\alpha : |f| \Rightarrow |g|$  and  $\beta : |g| \Rightarrow |h|$  of  $\mathcal{T}$ , an invertible 3-cell  $V_{\alpha,\beta} : F_{g,h}(\beta) \circ F_{f,g}(\alpha) \Rightarrow F_{f,h}(\beta \circ \alpha)$  of  $\mathcal{U}$ ;
- For each  $f, g : x \dashrightarrow y$ ,  $h, k : y \dashrightarrow z$ ,  $\alpha : |f| \Rightarrow |g|$  and  $\beta : |h| \Rightarrow |k|$  of  $\mathcal{T}$ , an invertible 3-cell  $H_{\alpha,\beta} : F_{h,k}(\beta) \otimes F_{f,g}(\alpha) \Rightarrow F_{h \otimes f, k \otimes g}(\beta \otimes \alpha)$  of  $\mathcal{U}$ ;
- For each  $f : x \dashrightarrow y$  of  $\mathcal{T}$ , an invertible 3-cell  $U_f : 1_{|Ff|} \Rightarrow F_{f,f}(1_{|f|})$ ;
- For each  $f : x \dashrightarrow y$  in  $\mathcal{T}$ , invertible 3-cells  $L_f : l_{|Ff|} \Rightarrow F_{l_y \otimes f, f}(l_{|f|})$  and  $R_f : r_{|Ff|} \Rightarrow F_{f \otimes l_x, f}(r_{|f|})$  of  $\mathcal{U}$ ;
- For each  $f : x \dashrightarrow y$ ,  $g : y \dashrightarrow z$  and  $h : z \dashrightarrow w$  in  $\mathcal{T}$ , an invertible 3-cell  $A_{fgh} : a_{|Ff|, |Fg|, |Fh|} \Rightarrow F_{(h \otimes g) \otimes f, h \otimes (g \otimes f)}(a_{|f|, |g|, |h|})$  of  $\mathcal{U}$

subject to fourteen coherence axioms corresponding to the fourteen kinds of 3-cell equation adjoined in Definition 3.8. We give one of these axioms as a representative sample. Suppose given  $f, g : x \dashrightarrow y$  and  $\alpha : |f| \Rightarrow |g|$  in  $\mathcal{T}$ . Then we require that

$$\begin{array}{ccc}
 1_{|Fg|} \circ F_{f,g}(\alpha) & \xrightarrow{U_g \circ \text{id}} & F_{g,g}(1_{|g|}) \circ F_{f,g}(\alpha) & \xrightarrow{V_{\alpha, 1_{|g|}}} & F_{f,g}(1_{|g|} \circ \alpha) \\
 \cong \downarrow & & & & \downarrow F_{f,g}(\cong) \\
 F_{f,g}(\alpha) & \xlongequal{\hspace{10em}} & & & F_{f,g}(\alpha)
 \end{array}$$

should commute in  $\mathcal{U}$ .

The identities and composition of **Tricat** are given as follows. The identity homomorphism  $\mathcal{T} \rightarrow \mathcal{T}$  has all of its basic data given by identity assignments, and all of its coherence data given by identity 3-cells; whilst for homomorphisms  $F : \mathcal{T} \rightarrow \mathcal{U}$  and  $G : \mathcal{U} \rightarrow \mathcal{V}$ , their composite  $GF : \mathcal{T} \rightarrow \mathcal{V}$  has basic data given by

- $(GF)(x) = G(F(x))$ ;
- $(GF)(f) = G(F(f))$ ;
- $(GF)_{f,g}(\alpha) = G_{Ff, Fg}(F_{f,g}(\alpha))$ ;
- $(GF)_{f,g}(\Gamma) = G_{Ff, Fg}(F_{f,g}(\Gamma))$ ;

and coherence data obtained according to a common pattern which we illustrate with the case of  $U_f$ . Given  $f : x \dashrightarrow y$  in  $\mathcal{T}$ , we define the 3-cell  $U_f : 1_{|GFf|} \Rightarrow GF_{f,f}(1_{|f|})$  of  $\mathcal{V}$  to be the composite

$$1_{|GFf|} \xrightarrow{U_{Ff}} G_{Ff, Ff}(1_{|Ff|}) \xrightarrow{G_{Ff, Ff}(U_f)} G_{Ff, Ff}(F_{f,f}(1_{|f|})) = GF_{f,f}(1_{|f|}).$$

#### 4. Biased and unbiased trihomomorphisms

As promised above, we now give a comparison between the notion of trihomomorphism given in Definition 3.10 and the one already existing in the literature, a suitable reference for which

is [14, §3.3]. As observed above, the two notions cannot be isomorphic, since our trihomomorphisms admit a strictly associative composition, whereas those of [14] do not; at best, they form a bicategory (see [11] for the details). Closer inspection reveals that our homomorphisms are the richer structure: they explicitly assign to each two-dimensional pasting diagram of the domain tricategory a corresponding pasting diagram in the codomain. For the trihomomorphisms of [14] no such assignation is provided; and though one may be derived from the trihomomorphism data—as we shall see in Proposition 4.4 below—the derivation is non-canonical, and so only determined up to an invertible 3-cell. A similar phenomenon occurs in comparing the *unbiased* bicategories of [19, Chapter 1]—which incorporate specified composites for all possible one-dimensional pasting diagrams—with ordinary, or *biased*, bicategories—for which only nullary and binary composites are supplied. Again, from the latter we can derive the former; but again, in a non-canonical way that is determined only up to isomorphism. In recognition of this similarity, we adopt [19]’s terminology here, referring to the homomorphisms of Definition 3.10 as *unbiased homomorphisms*, and to those of [14, §3.3] as *biased homomorphisms*.

Our goal in the remainder of this section will be to give a precise comparison between these two notions of homomorphism. We will define a 2-category of unbiased homomorphisms and a bicategory of biased homomorphisms, and prove these to be biequivalent. In each case, the 2-cells we consider are not the most general ones—those which between the biased homomorphisms are called *tritransformations*—since these do not admit a strictly associative composition. Instead we consider a restricted subclass of the tritransformations, those whose 1- and 2-cell components are all identity maps: these are the tricategorical *icons*<sup>2</sup> of [11], themselves a generalisation of the bicategorical icons of [18]. Since the 1- and 2-dimensional data for a tricategorical icon is trivial, it may be specified purely in terms of a collection of 3-cells satisfying axioms; and it is this which allows us to equip them with a strictly associative composition.

**Definition 4.1.** Let  $F, G : \mathcal{T} \rightarrow \mathcal{U}$  be unbiased homomorphisms. An *unbiased icon*  $\xi : F \Rightarrow G$  may exist only if  $F$  and  $G$  agree on 0- and 1-cells; and is then given by specifying, for every  $f, g : x \dashrightarrow y$  and  $\alpha : |f| \Rightarrow |g|$  in  $\mathcal{T}$ , a 3-cell  $\xi_{f,g}(\alpha) : F_{f,g}(\alpha) \Rrightarrow G_{f,g}(\alpha)$  of  $\mathcal{U}$ , subject to the following axioms.

- For each  $\Gamma : \alpha \Rrightarrow \beta : |f| \Rightarrow |g|$  of  $\mathcal{T}$ , the following diagram should commute in  $\mathcal{U}$ :

$$\begin{array}{ccc}
 F_{f,g}(\alpha) & \xrightarrow{F_{f,g}(\Gamma)} & F_{f,g}(\beta) \\
 \xi_{f,g}(\alpha) \downarrow & & \downarrow \xi_{f,g}(\beta) \\
 G_{f,g}(\alpha) & \xrightarrow{G_{f,g}(\Gamma)} & G_{f,g}(\beta);
 \end{array}$$

<sup>2</sup> In fact, the icons we consider are in [11] called *ico-icons*: with the unadorned name being reserved for a more general concept which we will not have use of here.

- For each  $f, g, h : x \dashrightarrow y$ ,  $\alpha : |f| \Rightarrow |g|$  and  $\beta : |g| \Rightarrow |h|$  of  $\mathcal{T}$ , the following diagram should commute in  $\mathcal{U}$ :

$$\begin{array}{ccc}
 F_{g,h}(\beta) \circ F_{f,g}(\alpha) & \xrightarrow{V_{\alpha,\beta}} & F_{f,h}(\beta \circ \alpha) \\
 \xi_{g,h}(\beta) \circ \xi_{f,g}(\alpha) \downarrow & & \downarrow \xi_{f,h}(\beta \circ \alpha) \\
 G_{g,h}(\beta) \circ G_{f,g}(\alpha) & \xrightarrow{V_{\alpha,\beta}} & G_{f,h}(\beta \circ \alpha);
 \end{array}$$

- For each  $f, g : x \dashrightarrow y$ ,  $h, k : y \dashrightarrow z$ ,  $\alpha : |f| \Rightarrow |g|$  and  $\beta : |h| \Rightarrow |k|$  of  $\mathcal{T}$ , the following diagram should commute in  $\mathcal{U}$ :

$$\begin{array}{ccc}
 F_{h,k}(\beta) \otimes F_{f,g}(\alpha) & \xrightarrow{H_{\alpha,\beta}} & F_{h \otimes f, k \otimes g}(\beta \otimes \alpha) \\
 \xi_{h,k}(\beta) \otimes \xi_{f,g}(\alpha) \downarrow & & \downarrow \xi_{h \otimes f, k \otimes g}(\beta \otimes \alpha) \\
 G_{h,k}(\beta) \otimes G_{f,g}(\alpha) & \xrightarrow{H_{\alpha,\beta}} & G_{h \otimes f, k \otimes g}(\beta \otimes \alpha);
 \end{array}$$

- For each  $f : x \dashrightarrow y$  in  $\mathcal{T}$ , the following diagrams should commute in  $\mathcal{U}$ :

$$\begin{array}{ccc}
 1_{|Ff|} \xrightarrow{U_f} F_{f,f}(1_{|f|}) & , & l_{|Ff|} \xrightarrow{L_f} F_{I_y \otimes f, f}(l_{|f|}) & , & r_{|Ff|} \xrightarrow{R_f} F_{f \otimes I_x, f}(r_{|f|}) \\
 \parallel & & \parallel & & \parallel \\
 1_{|Gf|} \xrightarrow{U_g} G_{f,f}(1_{|f|}) & , & l_{|Gf|} \xrightarrow{L_g} G_{I_y \otimes f, f}(l_{|f|}) & , & r_{|Gf|} \xrightarrow{R_g} G_{f \otimes I_x, f}(r_{|f|}); \\
 \downarrow \xi & & \downarrow \xi & & \downarrow \xi
 \end{array}$$

- For each  $f : x \dashrightarrow y$ ,  $g : y \dashrightarrow z$  and  $h : z \dashrightarrow w$  in  $\mathcal{T}$ , the following diagram should commute in  $\mathcal{U}$ :

$$\begin{array}{ccc}
 a_{|Ff|, |Fg|, |Fh|} & \xrightarrow{A_{fgh}} & F_{f \otimes (g \otimes h), (f \otimes g) \otimes h}(a_{|f|, |g|, |h|}) \\
 \parallel & & \downarrow \xi \\
 a_{|Gf|, |Gg|, |Gh|} & \xrightarrow{A_{fgh}} & G_{f \otimes (g \otimes h), (f \otimes g) \otimes h}(a_{|f|, |g|, |h|}).
 \end{array}$$

With the evident 2-cell composition, tricategories, unbiased homomorphisms and unbiased icons form a 2-category which we denote by **Tricat<sub>ub</sub>**.

We now give the corresponding notion of icon between biased homomorphisms. The definition is very similar to the one just given, and we have deliberately stated it in a way which facilitates easy comparison between the two. A more geometric statement of the axioms is given in [11, Definition 2].



**Definition 4.2.** Let  $F, G : \mathcal{T} \rightarrow \mathcal{U}$  be biased homomorphisms. A *biased icon*  $\xi : F \Rightarrow G$  may exist only if  $F$  and  $G$  agree on 0- and 1-cells; and is then given by specifying, for every  $\alpha : f \Rightarrow g$  in  $\mathcal{T}$ , a 3-cell  $\xi(\alpha) : F(\alpha) \Rrightarrow G(\alpha)$  of  $\mathcal{U}$ ; for every object  $x \in \mathcal{T}$ , an invertible 3-cell

$$\begin{array}{ccc} I_{Fx} & \xrightarrow{\iota_x^F} & FI_x \\ \parallel & \Downarrow M_x & \parallel \\ IG_x & \xrightarrow{\iota_x^G} & GI_x \end{array}$$

of  $\mathcal{U}$ ; and for each pair of 1-cells  $f : x \rightarrow y, g : y \rightarrow z$  of  $\mathcal{T}$ , an invertible 3-cell

$$\begin{array}{ccc} Fg \otimes Ff & \xrightarrow{\chi_{f,g}^F} & F(g \otimes f) \\ \parallel & \Downarrow \Pi_{f,g} & \parallel \\ Gg \otimes Gf & \xrightarrow{\chi_{f,g}^G} & G(g \otimes f) \end{array}$$

of  $\mathcal{U}$ , all subject to the following axioms.

- For each  $\Gamma : \alpha \Rightarrow \beta$  of  $\mathcal{T}$ , the following diagram should commute in  $\mathcal{U}$ :

$$\begin{array}{ccc} F(\alpha) & \xrightarrow{F(\Gamma)} & F(\beta) \\ \xi(\alpha) \downarrow & & \downarrow \xi(\beta) \\ G(\alpha) & \xrightarrow{F(\Gamma)} & G(\beta); \end{array}$$

- For each  $\alpha : f \Rightarrow g$  and  $\beta : g \Rightarrow h : x \rightarrow y$  of  $\mathcal{T}$ , the following diagram should commute in  $\mathcal{U}$ :

$$\begin{array}{ccc} F(\beta) \circ F(\alpha) & \xrightarrow{\cong} & F(\beta \circ \alpha) \\ \xi(\beta) \circ \xi(\alpha) \downarrow & & \downarrow \xi(\beta \circ \alpha) \\ G(\beta) \circ G(\alpha) & \xrightarrow{\cong} & G(\beta \circ \alpha); \end{array}$$

- For each  $\alpha : f \Rightarrow g : x \rightarrow y$  and  $\beta : h \Rightarrow k : y \rightarrow z$  of  $\mathcal{T}$ , the following diagram should commute in  $\mathcal{U}$ :

$$\begin{array}{ccc}
 \chi_{g,k}^F \circ (F(\beta) \otimes F(\alpha)) & \xrightarrow{\cong} & F(\beta \otimes \alpha) \circ \chi_{f,h}^F \\
 \Pi_{g,k} \circ (\xi(\beta) \otimes \xi(\alpha)) \downarrow & & \downarrow \xi(\beta \otimes \alpha) \circ \Pi_{f,h} \\
 \chi_{g,k}^G \circ (G(\beta) \otimes G(\alpha)) & \xrightarrow[\cong]{} & G(\beta \otimes \alpha) \circ \chi_{f,h}^G
 \end{array}$$

- For each  $f : x \rightarrow y$  in  $\mathcal{T}$ , the following diagrams should commute in  $\mathcal{U}$ :

$$\begin{array}{ccc}
 1_{Ff} \xrightarrow{\cong} F(1_f) & & l_{Ff} \xleftarrow{\gamma_f^F} F(l_f) \circ (\chi_{f,I_y}^F \circ (l_y^F \otimes 1_{Ff})) \\
 \parallel & & \parallel \\
 1_{Gf} \xrightarrow[\cong]{} G(1_f) & & l_{Gf} \xleftarrow[\gamma_f^G]{} G(l_f) \circ (\chi_{f,I_y}^G \circ (l_y^G \otimes 1_{Gf})) \\
 & & \downarrow \xi \circ (\Pi \circ (M \otimes \text{id})) \\
 & & \downarrow \xi \circ (\Pi \circ (M \otimes \text{id}))
 \end{array}$$

$$\text{and} \quad \begin{array}{ccc}
 r_{Ff} \xleftarrow{\delta_f^F} F(r_f) \circ (\chi_{I_x,f}^F \circ (1_{Ff} \otimes l_x^F)) & & \\
 \parallel & & \downarrow \xi \circ (\Pi \circ (\text{id} \otimes M)) \\
 r_{Gf} \xleftarrow[\delta_f^G]{} G(r_f) \circ (\chi_{I_x,f}^G \circ (1_{Gf} \otimes l_x^G)); & & 
 \end{array}$$

- For each  $f : x \rightarrow y, g : y \rightarrow z$  and  $h : z \rightarrow w$  in  $\mathcal{T}$ , the following diagram should commute in  $\mathcal{U}$ :

$$\begin{array}{ccc}
 (\chi_{g \otimes f,h}^F \circ (1_{Fh} \otimes \chi_{f_g}^F)) \circ a_{Ff,Fg,Fh} & \xleftarrow{\omega_{fgh}^F} & F(a_{fgh}) \circ (\chi_{f,g \otimes h}^F \circ (\chi_{gh}^F \otimes 1_{Ff})) \\
 (\Pi \circ (\text{id} \otimes \Pi)) \circ \text{id} \downarrow & & \downarrow \xi(a_{fgh}) \circ (\Pi \circ (\Pi \otimes \text{id})) \\
 (\chi_{g \otimes f,h}^G \circ (1_{Gh} \otimes \chi_{f_g}^G)) \circ a_{Gf,Gg,Gh} & \xleftarrow[\omega_{fgh}^G]{} & G(a_{fgh}) \circ (\chi_{f,g \otimes h}^G \circ (\chi_{gh}^G \otimes 1_{Gf})).
 \end{array}$$

It follows from [11, Section 2] that tricategories, biased homomorphisms and biased icons form a bicategory **Tricat<sub>b</sub>**.

We will now show **Tricat<sub>ub</sub>** and **Tricat<sub>b</sub>** to be biequivalent. First we show that every unbiased homomorphism  $\mathcal{T} \rightarrow \mathcal{U}$  gives rise to a biased homomorphism, and vice versa; then we show that

these assignments give rise to an equivalence of categories  $\mathbf{Tricat}_{\text{ub}}(\mathcal{T}, \mathcal{U}) \simeq \mathbf{Tricat}_{\text{b}}(\mathcal{T}, \mathcal{U})$ ; and finally, we show that these equivalences provide the local data for an identity-on-objects biequivalence  $\mathbf{Tricat}_{\text{ub}} \simeq \mathbf{Tricat}_{\text{b}}$ .

**Proposition 4.3.** *To each unbiased homomorphism  $F : \mathcal{T} \rightarrow \mathcal{U}$  we may assign a biased homomorphism  $F' : \mathcal{T} \rightarrow \mathcal{U}$  with the same action on 0- and 1-cells.*

**Proof.** Suppose given an unbiased homomorphism  $F : \mathcal{T} \rightarrow \mathcal{U}$ . In constructing the corresponding biased homomorphism  $F'$ , we will give only the data and omit verification of the coherence axioms, since these follow in a straightforward manner from the axioms for  $F$  and the tricategory axioms for  $\mathcal{U}$ . On 0- and 1-cells,  $F'$  agrees with  $F$ ; and on 2- and 3-cells is given by:

$$F'(\alpha : f \Rightarrow g) = F_{[f],[g]}(\alpha) \quad \text{and} \quad F'(\Gamma : \alpha \Rightarrow \beta : f \Rightarrow g) = F_{[f],[g]}(\Gamma).$$

The functoriality constraints for the homomorphisms of bicategories  $\mathcal{T}(x, y) \rightarrow \mathcal{U}(F'x, F'y)$  are given as follows:

- For each  $f : x \rightarrow y$  in  $\mathcal{U}$ , we take the constraint 3-cell  $1_{F'f} \cong F'(1_f)$  to be  $U_{[f]} : 1_{Ff} \Rightarrow F_{[f],[f]}(1_f)$ ;
- For each  $\alpha : f \Rightarrow g, \beta : g \Rightarrow h$  in  $\mathcal{U}$ , we take the constraint 3-cell  $F'(\beta) \circ F'(\alpha) \cong F'(\beta \circ \alpha)$  to be  $V_{\alpha,\beta} : F_{[g],[h]}(\beta) \circ F_{[f],[g]}(\alpha) \Rightarrow F_{[f],[h]}(\beta \circ \alpha)$ .

Next we provide the 2-cell components of the pseudo-natural transformations  $\chi_{f,g}$  and  $\iota_x$  and their adjoint inverses  $\chi_{f,g}^*$  and  $\iota_x^*$ . For each  $f : x \rightarrow y$  and  $g : y \rightarrow z$  in  $\mathcal{T}$  we take

$$\chi_{f,g} = F_{[g] \otimes [f], [g \otimes f]}(1_{g \otimes f}) : F'g \otimes F'f \Rightarrow F'(g \otimes f) \quad \text{and}$$

$$\chi_{f,g}^* = F_{[g \otimes f], [g] \otimes [f]}(1_{g \otimes f}) : F'(g \otimes f) \Rightarrow F'g \otimes F'f;$$

whilst for each  $x \in \mathcal{T}$  we take

$$\iota_x = F_{I_x, [I_x]}(1_{I_x}) : I_{F'x} \Rightarrow F'(I_x) \quad \text{and}$$

$$\iota_x^* = F_{[I_x], I_x}(1_{I_x}) : F'(I_x) \Rightarrow I_{F'x}.$$

Given 2-cells  $\alpha : f \Rightarrow g : x \rightarrow y$  and  $\beta : h \Rightarrow k : y \rightarrow z$  in  $\mathcal{T}$ , we obtain the corresponding pseudo-naturality 3-cell for  $\chi$  as the composite:

$$\begin{aligned}
 & \chi_{g,k} \circ (F'(\beta) \otimes F'(\alpha)) \\
 & \quad \downarrow = \\
 & F_{[k] \otimes [g], [k \otimes g]}(1_{k \otimes g}) \circ (F_{[h], [k]}(\beta) \otimes F_{[f], [g]}(\alpha)) \\
 & \quad \downarrow \text{id} \circ H \\
 & F_{[k] \otimes [g], [k \otimes g]}(1_{k \otimes g}) \circ F_{[h] \otimes [f], [k] \otimes [g]}(\beta \otimes \alpha) \\
 & \quad \downarrow V \\
 & F_{[h] \otimes [f], [k \otimes g]}(1_{k \otimes g} \circ (\beta \otimes \alpha)) \\
 & \quad \downarrow F_{[h] \otimes [f], [k \otimes g]}(\cong) \\
 & F_{[h] \otimes [f], [k \otimes g]}((\beta \otimes \alpha) \circ 1_{h \otimes f}) \\
 & \quad \downarrow V^{-1} \\
 & F_{[h \otimes f], [k \otimes g]}(\beta \otimes \alpha) \circ F_{[h] \otimes [f], [h \otimes f]}(1_{h \otimes f}) = F'(\beta \otimes \alpha) \circ \chi_{f,h}.
 \end{aligned}$$

We next require unit and counit isomorphisms for the adjoint equivalences  $\chi^* \dashv \chi$  and  $\iota^* \dashv \iota$ . So given  $f : x \rightarrow y$  and  $g : y \rightarrow z$  in  $\mathcal{T}$ , we obtain the isomorphism  $1_{F'(g \otimes f)} \cong \chi_{f,g} \circ \chi_{f,g}^*$  as the following composite:

$$\begin{aligned}
 & 1_{F'(g \otimes f)} = 1_{F(g \otimes f)} \\
 & \quad \downarrow U \\
 & F_{[g \otimes f], [g \otimes f]}(1_{g \otimes f}) \\
 & \quad \downarrow F_{[g \otimes f], [g \otimes f]}(\cong) \\
 & F_{[g \otimes f], [g \otimes f]}(1_{g \otimes f} \circ 1_{g \otimes f}) \\
 & \quad \downarrow V^{-1} \\
 & F_{[g] \otimes [f], [g \otimes f]}(1_{g \otimes f}) \circ F_{[g \otimes f], [g] \otimes [f]}(1_{g \otimes f}) = \chi_{f,g} \circ \chi_{f,g}^*;
 \end{aligned}$$

the other three cases are dealt with similarly. It remains only to give the invertible modifications  $\gamma$ ,  $\delta$  and  $\omega$  witnessing the coherence of the functoriality constraints  $\chi$  and  $\iota$ . The same

argument pertains in each case, and so we give it only for  $\gamma$ . Here, for each  $f : x \rightarrow y$  in  $\mathcal{T}$ , we must give an invertible 3-cell

$$\begin{array}{ccc}
 & F'I_y \otimes Ff & \xrightarrow{\chi_{f,I_y}} F'(I_y \otimes f) \\
 \iota_y \otimes 1_{F'f} \nearrow & & \Downarrow \gamma_f \\
 I_{F'y} \otimes F'f & \xrightarrow{l_{F'f}} & F'f;
 \end{array}$$

and we obtain this as the composite:

$$\begin{aligned}
 & F_{[I_y \otimes f], [f]}(l_f) \circ (F_{[I_y] \otimes [f], [I_y \otimes f]}(1_{I_y \otimes f}) \circ (F_{I_y, [I_y]}(1_{I_y}) \otimes 1_{Ff})) \\
 & \quad \downarrow \text{id} \circ (\text{id} \circ (\text{id} \otimes U)) \\
 & F_{[I_y \otimes f], [f]}(l_f) \circ (F_{[I_y] \otimes [f], [I_y \otimes f]}(1_{I_y \otimes f}) \circ (F_{I_y, [I_y]}(1_{I_y}) \otimes F_{[f], [f]}(1_f))) \\
 & \quad \downarrow \text{id} \circ (\text{id} \otimes H) \\
 & F_{[I_y \otimes f], [f]}(l_f) \circ (F_{[I_y] \otimes [f], [I_y \otimes f]}(1_{I_y \otimes f}) \circ F_{I_y \otimes [f], [I_y] \otimes [f]}(1_{I_y} \otimes 1_f)) \\
 & \quad \downarrow V \circ (\text{id} \circ V) \\
 & F_{I_y \otimes [f], [f]}(l_f \circ (1_{I_y \otimes f} \circ (1_{I_y} \otimes 1_f))) \\
 & \quad \downarrow F_{I_y \otimes [f], [f]}(\cong) \\
 & F_{I_y \otimes [f], [f]}(l_f) \\
 & \quad \downarrow L_f^{-1} \\
 & l_{Ff}. \quad \square
 \end{aligned}$$

**Proposition 4.4.** *To each biased homomorphism  $F : \mathcal{T} \rightarrow \mathcal{U}$  we may assign an unbiased homomorphism  $F' : \mathcal{T} \rightarrow \mathcal{U}$  with the same action on 0- and 1-cells.*

**Proof.** Let there be given a biased homomorphism  $F : \mathcal{T} \rightarrow \mathcal{U}$ . We first define, for every  $f : x \dashrightarrow y$  in  $\mathcal{T}$ , an adjoint equivalence 2-cell  $\kappa_f : |Ff| \Rrightarrow F|f|$  in  $\mathcal{U}$ . We do this by recursion on the form of  $f$ .

- If  $f = [g]$  for some  $g : x \rightarrow y$ , then we take  $\kappa_f = \kappa_f^* = 1_{Fg} : Fg \Rrightarrow Fg$ ;
- If  $f = I_x$  for some  $x$ , then we take  $\kappa_f = \iota_x : I_{F_x} \Rrightarrow FI_x$  and  $\kappa_f^* = \iota_x^*$ ; and
- If  $f = h \otimes g$  for some  $g : x \dashrightarrow z$  and  $h : z \dashrightarrow y$ , then we take  $\kappa_f$  to be

$$|F(h \otimes g)| = |Fh| \otimes |Fg| \xrightarrow{\kappa_h \otimes \kappa_g} F|h| \otimes F|g| \xrightarrow{\chi_{|g|, |h|}} F(|h| \otimes |g|) = F(|h \otimes g|);$$

and give its adjoint inverse  $\kappa_f^*$  dually.

We now define the unbiased homomorphism  $F'$ . To simplify notation, we allow binary compositions to associate to the right, and assert 0-dimensional composition  $\otimes$  to bind more tightly than 1-dimensional composition  $\circ$ . As demanded by the proposition, the basic data for  $F'$  will agree with that for  $F$  on 0- and 1-cells; whilst on 2- and 3-cells it is given by

$$F'_{f,g}(\alpha) = \kappa_g^* \circ F\alpha \circ \kappa_f \quad \text{and} \quad F'_{f,g}(\Gamma) = \kappa_g^* \circ F\Gamma \circ \kappa_f.$$

The coherence data for  $F'$  is given as follows. The invertible 3-cell  $V_{\alpha,\beta} : F'_{g,h}(\beta) \circ F'_{f,g}(\alpha) \Rightarrow F'_{f,h}(\beta \circ \alpha)$  is obtained as the chain of isomorphisms:

$$\begin{aligned} & (\kappa_h^* \circ F\beta \circ \kappa_g) \circ (\kappa_g^* \circ F\alpha \circ \kappa_f) \\ & \cong (\kappa_h^* \circ F\beta) \circ (\kappa_g \circ \kappa_g^*) \circ (F\alpha \circ \kappa_f) \\ & \cong (\kappa_h^* \circ F\beta) \circ (F\alpha \circ \kappa_f) \\ & \cong \kappa_h^* \circ F\beta \circ F\alpha \circ \kappa_f \\ & \cong \kappa_h^* \circ F(\beta \circ \alpha) \circ \kappa_f; \end{aligned}$$

the invertible 3-cell  $H_{\alpha,\beta} : F_{h,k}(\beta) \otimes F_{f,g}(\alpha) \Rightarrow F_{h\otimes f,k\otimes g}(\beta \otimes \alpha)$  by the chain of isomorphisms:

$$\begin{aligned} & (\kappa_k^* \circ F\beta \circ \kappa_h) \otimes (\kappa_g^* \circ F\alpha \circ \kappa_f) \\ & \cong (\kappa_k^* \otimes \kappa_g^*) \circ (F\beta \otimes F\alpha) \circ (\kappa_h \otimes \kappa_f) \\ & \cong (\kappa_k^* \otimes \kappa_g^*) \circ (\chi_{|g|,|k|}^* \circ F(\beta \otimes \alpha) \circ \chi_{|f|,|h|}) \circ (\kappa_h \otimes \kappa_f) \\ & \cong (\kappa_k^* \otimes \kappa_g^* \circ \chi_{|g|,|k|}^*) \circ F(\beta \otimes \alpha) \circ (\chi_{|f|,|h|} \circ \kappa_h \otimes \kappa_f) \\ & = \kappa_{k\otimes g}^* \circ F(\beta \otimes \alpha) \circ \kappa_{h\otimes f} \end{aligned}$$

(where from the second to the third line we apply pseudonaturality of  $\chi$ ); and the invertible 3-cell  $U_f : 1_{|Ff|} \Rightarrow F_{f,f}(1_{|f|})$  by the chain of isomorphisms:

$$1_{|Ff|} \xrightarrow{\cong} \kappa_f^* \circ \kappa_f \xrightarrow{\cong} \kappa_f^* \circ 1_{F|f|} \circ \kappa_f \xrightarrow{\cong} \kappa_f^* \circ F(1_{|f|}) \circ \kappa_f = F_{f,f}(1_{|f|}).$$

It remains to give the invertible 3-cells  $L_f$ ,  $R_f$  and  $A_{fgh}$ . As these three cases are very similar, we give details only for  $L_f : l_{|Ff|} \Rightarrow F_{I_y \otimes f, f}(l_{|f|})$ ; which is obtained by the following chain of isomorphisms:

$$\begin{aligned} l_{|Ff|} & \cong \kappa_f^* \circ l_{F|f|} \circ 1 \otimes \kappa_f \\ & \cong \kappa_f^* \circ (F(l_{|f|}) \circ \chi_{|f|, I_y} \circ \iota_y \otimes 1) \circ 1 \otimes \kappa_f \\ & \cong \kappa_f^* \circ F(l_{|f|}) \circ (\chi_{|f|, I_y} \circ \iota_y \otimes \kappa_f) \\ & = F_{I_y \otimes f, f}(l_{|f|}), \end{aligned}$$

where for the first isomorphism we apply pseudonaturality of  $l$ , and for the second we use the inverse of the coherence 3-cell

$$\begin{array}{ccc}
 & FI_y \otimes F|f| & \xrightarrow{\chi_{|f|, I_y}} F(I_y \otimes |f|) \\
 \nearrow \iota_y \otimes 1 & & \Downarrow \gamma_{|f|} \\
 I_{F_y} \otimes F|f| & \xrightarrow{l_{F|f|}} & F|f| \\
 & & \searrow F(l_{|f|})
 \end{array}$$

The fourteen coherence axioms for  $F'$  now all follow from the coherence theorem for biased homomorphisms [14, Chapter 11].  $\square$

**Proposition 4.5.** *The assignments of Propositions 4.3 and 4.4 induce an equivalence of categories  $\mathbf{Tricat}_{\text{ub}}(\mathcal{T}, \mathcal{U}) \simeq \mathbf{Tricat}_b(\mathcal{T}, \mathcal{U})$ .*

**Proof.** We begin by making the assignment of Proposition 4.3 into a functor. So suppose given an unbiased icon  $\xi : F \Rightarrow G$ ; we produce from it a biased icon  $\xi' : F' \Rightarrow G'$  as follows. We take its basic data to be given by:

- $\xi'(\alpha : f \Rightarrow g) = \xi_{[f], [g]}(\alpha) : F'(\alpha) \Rightarrow G'(\alpha)$ ;
- $\Pi_{f,g} = \xi_{[g] \otimes [f], [g] \otimes [f]}(1_{g \otimes f}) : \chi_{f,g}^{F'} \Rightarrow \chi_{f,g}^{G'}$ ; and
- $M_x = \xi_{I_x, [I_x]}(1_{I_x}) : \iota_x^{F'} \Rightarrow \iota_x^{G'}$ .

The 3-cells  $\Pi_{f,g}$  and  $M_x$  are invertible, with the 3-cell  $\Pi_{f,g}^{-1}$  being given as the mate under adjunction of the 3-cell  $\xi_{[g \otimes f], [g] \otimes [f]}(1_{g \otimes f}) : \chi_{f,g}^{F'} \Rightarrow \chi_{f,g}^{G'}$ , and  $M_x^{-1}$  being the mate under adjunction of  $\xi_{[I_x], I_x}(1_{I_x}) : \iota_x^{F'} \Rightarrow \iota_x^{G'}$ ; whilst the biased icon axioms for  $\xi'$  follow immediately from the unbiased icon axioms for  $\xi$ . It is easy to see that the assignment  $\xi \mapsto \xi'$  is functorial, and so we obtain a functor  $(-)' : \mathbf{Tricat}_{\text{ub}}(\mathcal{T}, \mathcal{U}) \rightarrow \mathbf{Tricat}_b(\mathcal{T}, \mathcal{U})$ .

We next make the assignment of Proposition 4.4 functorial. So given a biased icon  $\xi : F \Rightarrow G$  we must produce an unbiased icon  $\xi' : F' \Rightarrow G'$ . We first define, for every  $f : x \dashrightarrow y$  in  $\mathcal{T}$ , invertible 3-cells

$$\begin{array}{ccc}
 |Ff| \xrightarrow{\kappa_f^F} F|f| & & F|f| \xrightarrow{\kappa_f'^F} |Ff| \\
 \parallel \Downarrow \phi_f \parallel & \text{and} & \parallel \Downarrow \phi_f' \parallel \\
 |Gf| \xrightarrow{\kappa_f^G} G|f| & & G|f| \xrightarrow{\kappa_f'^G} |Gf|
 \end{array}$$

where  $\kappa_f^F, \kappa_f^G, \kappa_f'^F$  and  $\kappa_f'^G$  are defined as in the proof of Proposition 4.4. In fact, it suffices to give  $\phi_f$ , since we may then obtain  $\phi_f'$  as the mate under adjunction of  $(\phi_f)^{-1}$ . We define  $\phi_f$  by recursion on the form of  $f$ :

- If  $f = [g]$  for some  $g : x \rightarrow y$ , then we take  $\phi_f = \text{id} : 1_{Fg} \Rightarrow 1_{Gg}$ ;
- If  $f = I_x$  for some  $x$ , then we take  $\phi_f = M_x : \iota_x^F \Rightarrow \iota_x^G$ ; and

- If  $f = h \otimes g$  for some  $g : x \dashrightarrow z$  and  $h : z \dashrightarrow y$ , then we take  $\phi_f$  to be

$$\kappa_{h \otimes g}^F = \chi_{|g|, |h|}^F \circ (\kappa_h^F \otimes \kappa_g^F) \xrightarrow{\Pi_{|g|, |h|} \circ (\phi_h \otimes \phi_g)} \chi_{|g|, |h|}^G \circ (\kappa_h^G \otimes \kappa_g^G) = \kappa_{h \otimes g}^G.$$

Now for a biased icon  $\xi : F \Rightarrow G$ , the corresponding unbiased icon  $\xi' : F' \Rightarrow G'$  has 3-cell components  $\xi'_{f,g}(\alpha)$  given by

$$F'_{f,g}(\alpha) = \kappa_g^{*F} \circ F(\alpha) \circ \kappa_f^F \xrightarrow{\phi_g^{*\xi(\alpha)} \circ \phi_f} \kappa_g^{*G} \circ G(\alpha) \circ \kappa_f^G = G'_{f,g}(\alpha).$$

The unbiased icon axioms for  $\xi'$  follow by straightforward diagram chasing. Moreover, it is easy to see that the assignment  $\xi \mapsto \xi'$  respects composition and so we obtain a functor  $(-)' : \mathbf{Tricat}_b(\mathcal{T}, \mathcal{U}) \rightarrow \mathbf{Tricat}_{ub}(\mathcal{T}, \mathcal{U})$ .

It remains to show that the two functors just defined are quasi-inverse to each other. Firstly, for each unbiased homomorphism  $F : \mathcal{T} \rightarrow \mathcal{U}$  we must provide an invertible unbiased icon  $\eta_F : F \Rightarrow F''$ , naturally in  $F$ . To this end we define, for each  $f : x \dashrightarrow y$  in  $\mathcal{T}$ , isomorphic 3-cells

$$\begin{aligned} \theta_f : F_{f, [|f|]}(1_{|f|}) &\Rightarrow \kappa_f^{F'} : |Ff| \Rightarrow F|f| \quad \text{and} \\ \theta_f^* : F_{[|f|], f}(1_{|f|}) &\Rightarrow \kappa_f^{*F'} : F|f| \Rightarrow |Ff|. \end{aligned}$$

We do this by recursion on the form of  $f$ . If  $f = [g]$  then we take

$$\theta_f = \theta_f^* = U_{[g]}^{-1} : F_{[g], [g]}(1_g) \Rightarrow 1_{Fg};$$

if  $f = I_x$  for some  $x$ , then we may take both  $\theta_f$  and  $\theta_f^*$  to be identity cells; and if  $f = h \otimes g$  for some  $g : x \dashrightarrow z$  and  $h : z \dashrightarrow y$ , then we take  $\theta_f$  to be given by the composite

$$\begin{aligned} F_{h \otimes g, [|h \otimes g|]}(1) &\cong F_{h \otimes g, [|h| \otimes |g|]}(1 \circ 1) \\ &\cong F_{[|h|] \otimes [|g|], [|h| \otimes |g|]}(1) \circ F_{h \otimes g, [|h|] \otimes [|g|]}(1) \\ &\cong F_{[|h|] \otimes [|g|], [|h| \otimes |g|]}(1) \circ F_{h, [|h|]}(1) \otimes F_{g, [|g|]}(1) \\ &\cong \chi_{|f|, |g|}^{F'} \circ \kappa_h^{F'} \otimes \kappa_g^{F'} \\ &= \kappa_{h \otimes g}^{F'} \end{aligned}$$



and give  $\theta_f^*$  dually. We now define the unbiased icon  $\eta_F : F \Rightarrow F''$  to have components  $(\eta_F)_{f,g}(\alpha)$  given by

$$\begin{array}{c}
 F_{f,g}(\alpha) \\
 \downarrow F_{f,g}(\cong) \\
 F_{f,g}(1_{|g|} \circ (\alpha \circ 1_{|f|})) \\
 \downarrow (1 \circ U^{-1}) \cdot U^{-1} \\
 F_{[|g|],g}(1_{|g|}) \circ (F_{[|f|],[|g|]}(\alpha) \circ F_{f,[|f|]}(1_{|f|})) \\
 \downarrow \theta_g^* \circ (\text{id} \circ \theta_f) \\
 \kappa_g^{F'} \circ (F'(\alpha) \circ \kappa_f^{F'}) \\
 \downarrow = \\
 F''_{f,g}(\alpha).
 \end{array}$$

With some effort we may check the icon axioms for  $\eta_F$ ; whilst the naturality of  $\eta_F$  in  $F$  is almost immediate. To conclude the proof, we must provide for each biased homomorphism  $F : \mathcal{T} \rightarrow \mathcal{U}$  an invertible biased icon  $\epsilon_F : F'' \Rightarrow F$ , naturally in  $F$ . Given such an  $F$ , it is clear that  $F''$  agrees with it on 0- and 1-cells; whilst on 2-cell data we have that:

$$\begin{aligned}
 F''(\alpha : f \Rightarrow g) &= F'_{[f],[g]}(\alpha) = 1_{Fg} \circ (F\alpha \circ 1_{Ff}); \\
 \chi_{f,g}^{F''} &= F'_{[g] \otimes [f], [g \otimes f]}(1_{g \otimes f}) = 1_{Fg} \otimes 1_{Ff} \circ \chi_{f,g}^F \circ F(1_{g \otimes f}) \circ 1_{F(g \otimes f)}; \quad \text{and} \\
 \iota_x^{F''} &= F'_{I_x, [I_x]}(1_{I_x}) = \iota_x^F \circ 1_{I_x} \circ 1_{I_x}.
 \end{aligned}$$

Thus we may take each of  $\epsilon_F(\alpha) : F''(\alpha) \Rightarrow F(\alpha)$ ,  $\Pi_{f,g} : \chi_{f,g}^{F''} \Rightarrow \chi_{f,g}^F$  and  $M_x : \iota_x^{F''} \Rightarrow \iota_x^F$  to be given by the appropriate bicategorical coherence constraint. The icon axioms for  $\epsilon_F$  follow from coherence for biased trihomomorphisms; whilst naturality of  $\epsilon_F$  in  $F$  is again almost immediate.  $\square$

**Theorem 4.6.** *The bicategories  $\mathbf{Tricat}_{\text{ub}}$  and  $\mathbf{Tricat}_{\text{b}}$  are biequivalent.*

**Proof.** We will show the functors  $(-)' : \mathbf{Tricat}_{\text{ub}}(\mathcal{T}, \mathcal{U}) \rightarrow \mathbf{Tricat}_{\text{b}}(\mathcal{T}, \mathcal{U})$  to provide the local structure of an identity-on-objects homomorphism of bicategories  $\mathbf{Tricat}_{\text{ub}} \rightarrow \mathbf{Tricat}_{\text{b}}$ . The result then follows by observing this homomorphism to be biessentially surjective on objects (trivially) and locally an equivalence (by Proposition 4.5); and so a biequivalence. The only data we lack for the homomorphism  $\mathbf{Tricat}_{\text{ub}} \rightarrow \mathbf{Tricat}_{\text{b}}$  are its functoriality constraint 2-cells. So we must provide for each tricategory  $\mathcal{T}$ , a biased icon  $e_{\mathcal{T}} : 1_{\mathcal{T}} \Rightarrow (1_{\mathcal{T}})' : \mathcal{T} \rightarrow \mathcal{T}$ ; and for each pair of unbiased homomorphisms  $F : \mathcal{T} \rightarrow \mathcal{U}$  and  $G : \mathcal{U} \rightarrow \mathcal{V}$ , a biased icon  $m_{F,G} : G' \circ F' \Rightarrow (G \circ F)' : \mathcal{T} \rightarrow \mathcal{V}$ . For the former, it is not hard to check that  $(-)'$  in fact preserves identities strictly, so that we may take  $e_{\mathcal{T}}$  to be an identity icon. For the latter, we

observe that  $G' \circ F'$  and  $(G \circ F)'$  agree on 0- and 1-cells as required; whilst on 2-cells, their respective data is given as follows. For  $\alpha : f \Rightarrow g$  in  $\mathcal{T}$ , we have

$$(G' \circ F')(\alpha) = G'(F'(\alpha)) = G_{[Ff],[Fg]}(F_{[f],[g]}(\alpha)) \quad \text{and}$$

$$(G \circ F)'(\alpha) = (G \circ F)_{[f],[g]}(\alpha) = G_{[Ff],[Fg]}(F_{[f],[g]}(\alpha));$$

so that we may take  $m_{F,G}(\alpha)$  to be an identity 3-cell. Next, for  $x \in \mathcal{T}$  we have

$$\iota_x^{G' \circ F'} = G'(\iota_x^{F'}) \circ \iota_{F'x}^{G'} = G_{[I_{F'x}],[I_{F'x}]}(F_{I_x,[I_x]}(1_{I_x})) \circ G_{I_{F'x},[I_{F'x}]}(1_{I_{F'x}}) \quad \text{and}$$

$$\iota_x^{(G \circ F)'} = (G \circ F)_{I_x,[I_x]}(1_{I_x}) = G_{I_{F'x},[I_{F'x}]}(1_{I_x});$$

so that we may take  $M_x : \iota_x^{G' \circ F'} \cong \iota_x^{(G \circ F)'}$  to be the 3-cell

$$\begin{array}{c} G_{[I_{F'x}],[I_{F'x}]}(F_{I_x,[I_x]}(1_{I_x})) \circ G_{I_{F'x},[I_{F'x}]}(1_{I_{F'x}}) \\ \downarrow \nu \\ G_{I_{F'x},[I_{F'x}]}(F_{I_x,[I_x]}(1_{I_x}) \circ 1_{I_{F'x}}) \\ \downarrow G(\cong) \\ G_{I_{F'x},[I_{F'x}]}(F_{I_x,[I_x]}(1_{I_x})). \end{array}$$

Finally, for  $f : x \rightarrow y$  and  $g : y \rightarrow z$  in  $\mathcal{T}$ , we have that

$$\begin{aligned} \chi_{f,g}^{G' \circ F'} &= G'(\chi_{f,g}^{F'}) \circ \chi_{F'f,F'g}^{G'} \\ &= G_{[Fg \circ Ff],[F(g \circ f)]}(F_{[g] \otimes [f],[g \circ f]}(1_{g \circ f})) \circ G_{[Fg] \otimes [Ff],[Fg \circ Ff]}(1_{Fg \circ Ff}) \quad \text{and} \\ \chi_{f,g}^{(G \circ F)'} &= (G \circ F)_{[g] \otimes [f],[g \circ f]}(1_{g \circ f}) \\ &= G_{[Fg] \otimes [Ff],[F(g \circ f)]}(F_{[g] \otimes [f],[g \circ f]}(1_{g \circ f})); \end{aligned}$$

so that we may take  $\Pi_{f,g} : \chi_{f,g}^{G' \circ F'} \cong \chi_{f,g}^{(G \circ F)'}$  to be the 3-cell

$$\begin{array}{c} G_{[Fg \circ Ff],[F(g \circ f)]}(F_{[g] \otimes [f],[g \circ f]}(1_{g \circ f})) \circ G_{[Fg] \otimes [Ff],[Fg \circ Ff]}(1_{Fg \circ Ff}) \\ \downarrow \nu \\ G_{[Fg] \otimes [Ff],[F(g \circ f)]}(F_{[g] \otimes [f],[g \circ f]}(1_{g \circ f}) \circ 1_{Fg \circ Ff}) \\ \downarrow G(\cong) \\ G_{[Fg] \otimes [Ff],[F(g \circ f)]}(F_{[g] \otimes [f],[g \circ f]}(1_{g \circ f})). \end{array}$$

Finally, by straightforward diagram chasing we can verify in succession: the icon axioms for  $m_{F,G}$ ; naturality of  $m_{F,G}$  in  $F$  and  $G$ ; and the pentagon and triangle axioms for  $e_{\mathcal{T}}$  and

$m_{F,G}$ . This completes the definition of the homomorphism  $\mathbf{Tricat}_{ub} \rightarrow \mathbf{Tricat}_b$  and hence the proof.  $\square$

### 5. Homomorphisms of weak $\omega$ -categories

We now turn to our second application of the techniques described in Section 2, for which we shall develop a notion of homomorphism between the weak  $\omega$ -categories of Michael Batanin. These weak  $\omega$ -categories are defined as algebras for suitable finitary monads on the category of globular sets; and as such, the naturally-arising morphisms between them are those which preserve all of the  $\omega$ -categorical operations on the nose. Whilst in [2, Definition 8.8], Batanin suggests a way of weakening these maps to obtain a notion of homomorphism, it is not made clear how the homomorphisms he describes should be composed, or even that they may be composed at all. The description that we shall now give of a *category* of homomorphisms between weak  $\omega$ -categories is therefore a useful contribution towards the goal of describing the totality of structure formed by (algebraic) weak  $\omega$ -categories and the weak higher cells between them.

We begin by briefly recalling Batanin’s definition of weak  $\omega$ -category: see [2] or [19] for the details, or [3] for a more modern treatment. As stated above, weak  $\omega$ -categories in this sense are algebras for certain finitary monads on the category of globular sets, where a *globular set* is a presheaf over the category  $\mathbf{G}$  generated by the graph

$$0 \begin{array}{c} \xrightarrow{\sigma} \\ \xrightarrow{\tau} \end{array} 1 \begin{array}{c} \xrightarrow{\sigma} \\ \xrightarrow{\tau} \end{array} 2 \begin{array}{c} \xrightarrow{\sigma} \\ \xrightarrow{\tau} \end{array} 3 \begin{array}{c} \xrightarrow{\sigma} \\ \xrightarrow{\tau} \end{array} \cdots,$$

subject to the equations  $\sigma\sigma = \tau\sigma$  and  $\sigma\tau = \tau\tau$ , and where the finitary monads in question are the contractible globular operads of [2]. A *globular operad* is a monad  $P$  on  $[\mathbf{G}^{op}, \mathbf{Set}]$  equipped with a cartesian monad morphism  $\kappa : P \rightarrow T$ , where  $T$  is the monad for strict  $\omega$ -categories, and where to call  $\kappa$  *cartesian* is to assert that all of its naturality squares are pullbacks. By Lemma 6.8 and Proposition 6.11 of [3], any given monad  $P$  admits at most one such augmentation  $\kappa$ , so that for a monad on  $[\mathbf{G}^{op}, \mathbf{Set}]$  to be a globular operad is a property, not extra structure.<sup>3</sup>

Since the identity monad on  $[\mathbf{G}^{op}, \mathbf{Set}]$  is a globular operad, it is clear that not every globular operad embodies a sensible theory of weak  $\omega$ -categories. Those which do are characterised by [2, Definition 8.1] in terms of a property of *contractibility*. We will not recall the definition here, because we will not need to: our development makes sense for an arbitrary globular operad, and it will be convenient to work at this level of generality. Thus, for the remainder of this section, we let  $P$  be a fixed globular operad.

**Definition 5.1.** We write  $\omega\text{-Cat}_s$  for the category of  $P$ -algebras and  $P$ -algebra morphisms, refer to its objects as *weak  $\omega$ -categories*, and to its morphisms as *strict homomorphisms*.

The monad  $T$  for strict  $\omega$ -categories is finitarily monadic (see [19]), and this together with the existence of a cartesian  $\kappa : P \rightarrow T$  implies that  $P$  is also finitary. Hence  $\omega\text{-Cat}_s$  is a locally finitely presentable category, and so in order to apply the machinery of Section 2, it remains only

<sup>3</sup> Note that this is by contrast with the situation for plain operads, as noted in [20].

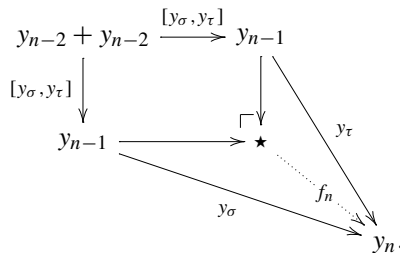
to distinguish in  $\omega\text{-Cat}_s$  a set of maps describing the basic  $n$ -cells together with the inclusions of their boundaries. In what follows we write

$$\omega\text{-Cat}_s \begin{array}{c} \xleftarrow{K} \\ \xrightarrow[\perp]{} \\ \xrightarrow{V} \end{array} [\mathbf{G}^{\text{op}}, \mathbf{Set}]$$

for the free/forgetful adjunction induced by  $P$ .

**Definition 5.2.** The *generating cofibrations*  $\{\iota_n : \partial_n \rightarrow 2_n\}_{n \in \mathbb{N}}$  of  $\omega\text{-Cat}_s$  are the images under  $K$  of the set of morphisms  $\{f_n\}_{n \in \mathbb{N}}$  of  $[\mathbf{G}^{\text{op}}, \mathbf{Set}]$  defined as follows (where we write  $y$  for the Yoneda embedding  $\mathbf{G} \rightarrow [\mathbf{G}^{\text{op}}, \mathbf{Set}]$ ):

- $f_0$  is the unique map  $0 \rightarrow y_0$ ;
- $f_1$  is the map  $[y_\sigma, y_\tau] : y_0 + y_0 \rightarrow y_1$ ;
- $f_n$  (for  $n \geq 2$ ) is the map induced by the universal property of pushout in the following diagram:



**Definition 5.3.** We define  $\mathbf{Q} : \omega\text{-Cat}_s \rightarrow \omega\text{-Cat}_s$  to be the universal cofibrant replacement comonad for the generating cofibrations of Definition 5.2, and define the category  $\omega\text{-Cat}_s$  of weak  $\omega$ -categories and  $\omega$ -homomorphisms to be the co-Kleisli category of this comonad.

We shall now give an explicit description of the comonad  $\mathbf{Q}$  in terms of *computads*. Computads were introduced in [24] as a tool for presenting free higher-dimensional categories. In the context of strict  $\omega$ -categories they have been studied extensively under the name of *polygraph*: see [8,21]. For the weak  $\omega$ -categories under consideration here, the appropriate notion of computad is due to Batanin [1]. In the definition, we make use of the functors

$$B_n := \omega\text{-Cat}_s(\partial_n, -) : \omega\text{-Cat}_s \rightarrow \mathbf{Set} \quad \text{and}$$

$$E_n := \omega\text{-Cat}_s(2_n, -) : \omega\text{-Cat}_s \rightarrow \mathbf{Set}$$

and the natural transformation  $\rho_n := \omega\text{-Cat}_s(\iota_n, -) : E_n \Rightarrow B_n$ .

**Definition 5.4.** For each integer  $n \geq -1$ , we define the category  $n\text{-Cptd}$  of  $n$ -computads, together with a free/forgetful adjunction

$$\omega\text{-Cat}_s \begin{matrix} \xleftarrow{F_n} \\ \perp \\ \xrightarrow{U_n} \end{matrix} n\text{-Cptd},$$

by induction on  $n$ . For the base case  $n = -1$ , we define  $(-1)\text{-Cptd}$  to be the terminal category,  $U_{-1}$  to be the unique functor into it, and  $F_{-1}$  to be the functor picking out the initial weak  $\omega$ -category. For the inductive step, given  $n \geq 0$  we define an  $n$ -computad to be given by an  $(n - 1)$ -computad  $C$ , a set  $X$ , and a function

$$x : X \rightarrow B_n F_{n-1} C.$$

A morphism of  $n$ -computads  $(C, X, x) \rightarrow (C', X', x')$  is given by a morphism  $f : C \rightarrow C'$  of  $(n - 1)$ -computads and a map of sets  $g : X \rightarrow X'$  making the diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ x \downarrow & & \downarrow x' \\ B_n F_{n-1} C & \xrightarrow{B_n F_{n-1} f} & B_n F_{n-1} C' \end{array}$$

commute. In other words, the category  $n\text{-Cptd}$  is just the comma category  $\mathbf{Set} \downarrow B_n F_{n-1}$ . The functor  $U_n : \omega\text{-Cat}_s \rightarrow n\text{-Cptd}$  sends  $\mathcal{A}$  to the triple  $(U_{n-1} \mathcal{A}, X_{(\mathcal{A}, n)}, x_{(\mathcal{A}, n)})$  where  $X_{(\mathcal{A}, n)}$  and  $x_{(\mathcal{A}, n)}$  are obtained from a pullback diagram

$$\begin{array}{ccc} X_{(\mathcal{A}, n)} & \xrightarrow{u_{(\mathcal{A}, n)}} & E_n \mathcal{A} \\ x_{(\mathcal{A}, n)} \downarrow & & \downarrow (\rho_n)_{\mathcal{A}} \\ B_n F_{n-1} U_{n-1} \mathcal{A} & \xrightarrow{B_n \epsilon_{n-1} \mathcal{A}} & B_n \mathcal{A}; \end{array} \tag{12}$$

here  $\epsilon_{n-1}$  denotes the counit of the adjunction  $F_{n-1} \dashv U_{n-1}$ . To complete the definition, we must exhibit a left adjoint  $F_n$  for  $U_n$ . The value of this at an  $n$ -computad  $D = (C, X, x)$  is given by taking the following pushout in  $\omega\text{-Cat}_s$ :

$$\begin{array}{ccc} X \cdot \partial_n & \xrightarrow{\bar{x}} & F_{n-1} C \\ X \cdot \iota_n \downarrow & & \downarrow \psi_D \\ X \cdot 2_n & \xrightarrow{\phi_D} & F_n D, \end{array} \tag{13}$$

where the map  $\bar{x}$  is obtained as the transpose of  $x : X \rightarrow B_n F_{n-1} C$  under the adjunction  $(-) \cdot \partial_n \dashv B_n : \omega\text{-Cat}_s \rightarrow \mathbf{Set}$ . The adjointness  $F_n \dashv U_n$  follows by direct calculation.

For each natural number  $n$ , we have a functor  $W_n : (n + 1)\text{-Cptd} \rightarrow n\text{-Cptd}$ , sending  $(C, X, x)$  to  $C$ ; and the category  $\omega\text{-Cptd}$  of  $\omega$ -computads is defined to be the limit of the diagram

$$\dots \xrightarrow{W_1} 0\text{-Cptd} \xrightarrow{W_0} (-1)\text{-Cptd}.$$

For each  $n \in \mathbb{N}$  we have  $W_n U_n = U_{n-1}$ , so that the  $U_n$ 's form a cone over this diagram; and we write  $U : \omega\text{-Cat}_s \rightarrow \omega\text{-Cptd}$  for the induced comparison functor. It now follows by a straightforward calculation that  $U$  has a left adjoint  $F$ , whose value at an object  $(C_n)$  of  $\omega\text{-Cptd}$  is given by the colimit of the diagram

$$F_{-1}C_{-1} \xrightarrow{\psi_{C_0}} F_0C_0 \xrightarrow{\psi_{C_1}} F_1C_1 \rightarrow \dots$$

where the maps  $\psi_{C_i}$  are given as in (13).

We now show that the comonad  $FU$  generated by the adjunction  $F \dashv U : \omega\text{-Cat}_s \rightarrow \omega\text{-Cptd}$  is isomorphic to the universal cofibrant replacement comonad  $\mathbf{Q}$ . In order to do this, we will first need some auxiliary definitions and results. Given a natural number  $n$ , we define a morphism of globular sets  $f : X \rightarrow Y$  to be  $n$ -bijeuctive if  $f_0, \dots, f_n$  are invertible, and  $n$ -fully faithful if the square

$$\begin{array}{ccc} X_{i+1} & \xrightarrow{f_{i+1}} & Y_{i+1} \\ (s,t) \downarrow & & \downarrow (s,t) \\ X_i \times_{X_{i-1}} X_i & \xrightarrow{f_i \times f_{i-1}} & Y_i \times_{Y_{i-1}} Y_i \end{array}$$

is a pullback for all  $i \geq n$ . We extend this notation by declaring every morphism of  $[\mathbf{G}^{\text{op}}, \mathbf{Set}]$  to be  $(-1)$ -bijeuctive, and only the isomorphisms to be  $(-1)$ -fully faithful.

**Proposition 5.5.** *For each integer  $n \geq -1$ , there is an orthogonal factorisation system on  $\omega\text{-Cat}_s$  whose left and right classes comprise those maps  $f$  such that  $\forall f$  is  $n$ -bijeuctive, respectively  $n$ -fully faithful.*

**Proof.** The case  $n = -1$  is trivial; so assume  $n \geq 0$ . It's easy to show that the  $n$ -bijeuctive and  $n$ -fully faithful maps form an orthogonal factorisation system on  $[\mathbf{G}^{\text{op}}, \mathbf{Set}]$ ; what we must show is that this lifts to  $\omega\text{-Cat}_s$ . Since this latter is the category of algebras for the monad  $P$  on  $[\mathbf{G}^{\text{op}}, \mathbf{Set}]$ , it suffices for this to show that the functor  $P$  preserves  $n$ -bijeuctive morphisms. Indeed, if this is the case, then we may factorise a  $P$ -algebra map  $f : (X, x) \rightarrow (Y, y)$  as follows. First we let

$$f = X \xrightarrow{g} Z \xrightarrow{h} Y$$

be the ( $n$ -bijeuctive,  $n$ -fully faithful) factorisation of  $f$ . Now consider the square

$$\begin{array}{ccc}
 PX & \xrightarrow{g \cdot x} & Z \\
 Pg \downarrow & & \downarrow h \\
 PZ & \xrightarrow{y \cdot Ph} & Y.
 \end{array}$$

It is certainly commutative; and since  $Pg$  is  $n$ -bijeuctive and  $h$  is  $n$ -fully faithful, we induce a unique morphism  $z : PZ \rightarrow Z$  making both squares commute. It’s now easy to verify using the uniqueness of diagonal fillers, that this makes  $Z$  into a  $P$ -algebra, and  $g$  and  $h$  into  $P$ -algebra maps. Thus we have verified the factorisation property; and the lifting property may be verified similarly.

Thus to complete the proof it suffices to show that  $P$  preserves  $n$ -bijeuctive maps. But if  $f : X \rightarrow Y$  is  $n$ -bijeuctive, then by direct examination, so is  $Tf$  (where we recall that  $T$  is the monad for strict  $\omega$ -categories). Now by virtue of the cartesian  $\kappa : P \rightarrow T$ , the map  $Pf$  is a pull-back of the  $n$ -bijeuctive  $Tf$ , and hence itself  $n$ -bijeuctive.  $\square$

**Proposition 5.6.** *For any  $n \in \mathbb{N}$  and  $n$ -computad  $D = (C, X, x)$ , the map  $\psi_D : F_{n-1}C \rightarrow F_n D$  of Eq. (13) is  $(n - 1)$ -bijeuctive.*

**Proof.** The case  $n = 0$  is trivial; so suppose  $n \geq 1$ . In this case, the map  $\psi_D$  is a pushout of a coproduct of copies of  $\iota_n : \partial_n \rightarrow 2_n$ , and so—by standard properties of orthogonal factorisation systems—will be  $(n - 1)$ -bijeuctive so long as  $\iota_n$  is. But we defined  $\iota_n$  to be the image under the free functor  $K$  of the map  $f_n \in [\mathbf{G}^{\text{op}}, \mathbf{Set}]$ , and so the result follows by observing that  $K$  preserves  $(n - 1)$ -bijeuctives (because  $P$  does), and that  $f_n$  is  $(n - 1)$ -bijeuctive by direct examination.  $\square$

With these preliminaries in place, we may now prove our main result.

**Proposition 5.7.** *The comonad  $Q$  is isomorphic to the comonad generated by the adjunction  $F \dashv U : \omega\text{-Cat}_s \rightarrow \omega\text{-Cptd}$ .*

**Proof.** Let there be given a weak  $\omega$ -category  $\mathcal{A}$ . We will use Proposition 2.6 to show that the counit morphism  $\epsilon_{\mathcal{A}} : FU\mathcal{A} \rightarrow \mathcal{A}$  provides a universal cofibrant replacement of  $\mathcal{A}$ . Thus we must equip  $\epsilon_{\mathcal{A}}$  with a choice of liftings against the generating cofibrations which makes it into an initial object of  $\mathbf{AAF}/\mathcal{A}$ , the category of algebraic acyclic fibrations into  $\mathcal{A}$ .

We first observe that to equip a strict homomorphism  $f : \mathcal{X} \rightarrow \mathcal{A}$  with a choice of liftings against the generating cofibrations is to give, for each  $n \in \mathbb{N}$ , a section of the function  $((\rho_n)_{\mathcal{X}}, E_n f) : E_n \mathcal{X} \rightarrow B_n \mathcal{X} \times_{B_n \mathcal{A}} E_n \mathcal{A}$ . Thus to equip  $\epsilon_{\mathcal{A}} : FU\mathcal{A} \rightarrow \mathcal{A}$  with a choice of liftings is to give functions

$$k_n : B_n FU\mathcal{A} \times_{B_n \mathcal{A}} E_n \mathcal{A} \rightarrow E_n FU\mathcal{A} \tag{14}$$

for each  $n \in \mathbb{N}$  such that  $((\rho_n)_{FU\mathcal{A}}, E_n \epsilon_{\mathcal{A}}) \circ k_n$  is the identity. Now,  $FU\mathcal{A}$  is obtained as the following colimit:

$$\begin{array}{ccccc}
 F_{-1}U_{-1}\mathcal{A} & \xrightarrow{\psi_{U_0\mathcal{A}}} & F_0U_0\mathcal{A} & \xrightarrow{\psi_{U_1\mathcal{A}}} & F_1U_1\mathcal{A} & \longrightarrow & \dots \\
 & \searrow \alpha_{-1} & \downarrow \alpha_0 & \swarrow \alpha_1 & & & \\
 & & FU\mathcal{A} & & & & 
 \end{array}$$

and  $\epsilon_{\mathcal{A}} : FU\mathcal{A} \rightarrow \mathcal{A}$  as the unique map with  $\epsilon_{\mathcal{A}} \cdot \alpha_n = (\epsilon_n)_{\mathcal{A}}$  for all  $n \geq -1$ . Given  $n \in \mathbb{N}$ , we have by Proposition 5.6 that  $\psi_{U_m\mathcal{A}}$  is  $(n - 1)$ -bijective for each  $m \geq n$ , from which it follows by standard properties of orthogonal factorisation systems that  $\alpha_{n-1}$  is also  $(n - 1)$ -bijective. Moreover, the functor  $B_n : \omega\text{-Cat}_s \rightarrow \mathbf{Set}$  sends  $(n - 1)$ -bijectives to isomorphisms, so that  $B_n \alpha_{n-1} : B_n F_{n-1} U_{n-1} \mathcal{A} \rightarrow B_n FU\mathcal{A}$  is an isomorphism: and so composing the pullback square (12) with this map yields a pullback square

$$\begin{array}{ccc}
 X_{(\mathcal{A},n)} & \xrightarrow{u_{(\mathcal{A},n)}} & E_n \mathcal{A} \\
 B_n \alpha_{n-1} \circ x_{(\mathcal{A},n)} \downarrow & & \downarrow (\rho_n)_{\mathcal{A}} \\
 B_n FU\mathcal{A} & \xrightarrow{B_n \epsilon_{\mathcal{A}}} & B_n \mathcal{A}.
 \end{array}$$

So to give  $k_n$  as in (14) is equally well to give  $k'_n : X_{(\mathcal{A},n)} \rightarrow E_n FU\mathcal{A}$  such that

$$(\rho_n)_{FU\mathcal{A}} \circ k'_n = B_n \alpha_{n-1} \circ x_{(\mathcal{A},n)} \quad \text{and} \quad E_n \epsilon_{\mathcal{A}} \circ k'_n = u_{(\mathcal{A},n)}; \tag{15}$$

and we may obtain such a  $k'_n$  as the transpose of the composite

$$X_{(\mathcal{A},n)} \cdot 2_n \xrightarrow{\phi_{U_n\mathcal{A}}} F_n U_n \mathcal{A} \xrightarrow{\alpha_n} FU\mathcal{A},$$

under the adjunction  $(-) \cdot 2_n \dashv E_n : \omega\text{-Cat}_s \rightarrow \mathbf{Set}$ . A straightforward calculation now verifies the equalities in (15).

Thus we have equipped  $\epsilon_{\mathcal{A}}$  with a choice of liftings  $(k_n)$  against the generating cofibrations; it remains to show that this makes  $(\epsilon_{\mathcal{A}}, k_n)$  into an initial object of  $\mathbf{AAF}/\mathcal{A}$ . So suppose that  $f : \mathcal{X} \rightarrow \mathcal{A}$  is a strict homomorphism with a choice of liftings  $j_n : B_n \mathcal{X} \times_{B_n \mathcal{A}} E_n \mathcal{A} \rightarrow E_n \mathcal{X}$ . We shall define a morphism  $\beta : FU\mathcal{A} \rightarrow \mathcal{X}$  satisfying  $f \cdot \beta = \epsilon$ . To do so is equally well to give a cocone

$$\begin{array}{ccccc}
 F_{-1}U_{-1}\mathcal{A} & \xrightarrow{\psi_{U_0\mathcal{A}}} & F_0U_0\mathcal{A} & \xrightarrow{\psi_{U_1\mathcal{A}}} & F_1U_1\mathcal{A} & \longrightarrow & \dots \\
 & \searrow \beta_{-1} & \downarrow \beta_0 & \swarrow \beta_1 & & & \\
 & & \mathcal{X} & & & & 
 \end{array} \tag{16}$$

satisfying  $f \cdot \beta_n = \epsilon_n$  for all  $n \geq -1$ . We do so by recursion on  $n$ . For the base case, we take  $\beta_{-1}$  to be the unique map from the initial object  $F_{-1}U_{-1}\mathcal{A}$ . For the inductive step, let  $n \geq 0$  and



suppose that we have already defined  $\beta_{n-1}$  satisfying  $f \cdot \beta_{n-1} = \alpha_{n-1}$ . By virtue of the pushout diagram (13) and the requirement that (16) should be a cocone, to give  $\beta_n : F_n U_n \mathcal{A} \rightarrow \mathcal{X}$  is equally well to give a morphism  $b_n : X_{(\mathcal{A},n)} \cdot 2_n \rightarrow \mathcal{X}$  making the square

$$\begin{array}{ccc} X_{(\mathcal{A},n)} \cdot \partial_n & \xrightarrow{\overline{x_{(\mathcal{A},n)}}} & F_{n-1} U_{n-1} \mathcal{A} \\ X \cdot \iota_n \downarrow & & \downarrow \beta_{n-1} \\ X_{(\mathcal{A},n)} \cdot 2_n & \xrightarrow{b_n} & \mathcal{X} \end{array}$$

commute; which, taking transposes under adjunction, is equally well to give a morphism  $b'_n : X_{(\mathcal{A},n)} \rightarrow E_n \mathcal{X}$  making

$$\begin{array}{ccc} X_{(\mathcal{A},n)} & \xrightarrow{x_{(\mathcal{A},n)}} & B_n F_{n-1} U_{n-1} \mathcal{A} \\ b'_n \downarrow & & \downarrow B_n \beta_{n-1} \\ E_n \mathcal{X} & \xrightarrow{(\rho_n)_{\mathcal{X}}} & B_n \mathcal{X} \end{array} \tag{17}$$

commute. To do this, we consider the following diagram:

$$\begin{array}{ccc} X_{(\mathcal{A},n)} & \xrightarrow{u_{(\mathcal{A},n)}} & E_n \mathcal{A} \\ B_n \beta_{n-1} \circ x_{(\mathcal{A},n)} \downarrow & & \downarrow (\rho_n)_{\mathcal{A}} \\ B_n \mathcal{X} & \xrightarrow{B_n f} & B_n \mathcal{A} \end{array} \tag{18}$$

It commutes by (12) and the condition that  $f \cdot \beta_{n-1} = \epsilon_{n-1}$ , and so we induce a map  $X_{(\mathcal{A},n)} \rightarrow B_n \mathcal{X} \times_{B_n \mathcal{A}} E_n \mathcal{A}$  by universal property of pullback. We now define  $b'_n$  to be the composite of this with  $j_n : B_n \mathcal{X} \times_{B_n \mathcal{A}} E_n \mathcal{A} \rightarrow E_n \mathcal{X}$ . Commutativity in (18), together with the fact that  $j_n$  is a section now imply both that (17) is commutative and that  $f \cdot \beta_n = \alpha_n$  as required. This completes the construction of  $\beta : FU\mathcal{A} \rightarrow \mathcal{X}$ ; and further calculation now shows that this map preserves the choices of liftings for  $\epsilon_{\mathcal{A}}$  and for  $f$ , and moreover, that it is the unique morphism  $FU\mathcal{A} \rightarrow \mathcal{X}$  over  $\mathcal{A}$  with this property.

Therefore, by Proposition 2.6, we have shown that the functor and counit part of the universal cofibrant replacement comonad coincide (up to isomorphism) with the functor and counit part of the comonad induced by the adjunction  $F \dashv U : \omega\text{-Cat}_s \rightarrow \omega\text{-Cptd}$ . To show that the same is true for the multiplication is a long but straightforward calculation using Proposition 3.5 which we omit.  $\square$

We end the paper with some brief remarks on higher cells. We have a functor  $D : \mathbf{G} \rightarrow \omega\text{-Cat}$  obtained as the composite

$$\mathbf{G} \xrightarrow{y_{(-)}} \hat{\mathbf{G}} \xrightarrow{\text{free}} T\text{-Alg} \xrightarrow{\kappa^*} P\text{-Alg} = \omega\text{-Cat}_s \hookrightarrow \omega\text{-Cat}.$$

Observe that  $\omega\text{-Cat}$  has products—because  $\omega\text{-Cat}_s$  has them and the inclusion map preserves them, being a right adjoint—so that, as in [2, Definition 8.9], we may define an  $m$ -cell from  $\mathcal{A}$  to  $\mathcal{B}$  to be a homomorphism  $\mathcal{A} \times Dm \rightarrow \mathcal{B}$ . Whilst it is unclear how one should compose such  $m$ -cells in general, there is one form of composition we do have: namely, that along a 0-cell boundary.

**Proposition 5.8.** *There is a category  $\omega\text{-Cat}_m$  whose objects are weak  $\omega$ -categories and whose morphisms  $\mathcal{A} \rightarrow \mathcal{B}$  are  $m$ -cells from  $\mathcal{A}$  to  $\mathcal{B}$ .*

**Proof.** Take the co-Kleisli category of the comonad  $(-)\times Dm$  on  $\omega\text{-Cat}$ .  $\square$

**Corollary 5.9.**  *$\omega\text{-Cat}$  is enriched over the cartesian monoidal category of globular sets.*

**Proof.** The hom object from  $\mathcal{A}$  to  $\mathcal{B}$  is the globular set  $[\mathcal{A}, \mathcal{B}]$  with

$$[\mathcal{A}, \mathcal{B}]_m := \omega\text{-Cat}(\mathcal{A} \times Dm, \mathcal{B});$$

whilst composition and identities at dimension  $m$  are given as in  $\omega\text{-Cat}_m$ .  $\square$

## Acknowledgments

The author thanks the organisers of PSSL 85, Nice, and of Category Theory 2008, Calais, at which material from this paper was presented; and also an anonymous referee for helpful comments on an earlier draft of this paper. He acknowledges the support of a Research Fellowship of St John's College, Cambridge and of a Marie Curie Intra-European Fellowship, Project No. 040802.

## References

- [1] M. Batanin, Computads for finitary monads on globular sets, in: Higher category theory, Evanston, IL, 1997, in: *Contemp. Math.*, vol. 230, American Mathematical Society, 1998, pp. 37–57.
- [2] M. Batanin, Monoidal globular categories as a natural environment for the theory of weak  $n$ -categories, *Adv. Math.* 136 (1) (1998) 39–103.
- [3] M. Batanin, M. Weber, Algebras of higher operads as enriched categories, preprint, 2008, available at <http://arxiv.org/abs/0803.3594>.
- [4] J. Beck, Distributive laws, in: Seminar on Triples and Categorical Homology Theory, ETH, Zürich, 1966/67, in: *Lecture Notes in Math.*, vol. 80, Springer, 1969, pp. 119–140.
- [5] J. Bénabou, Introduction to bicategories, in: Reports of the Midwest Category Seminar, in: *Lecture Notes in Math.*, vol. 47, Springer, 1967, pp. 1–77.
- [6] R. Blackwell, G.M. Kelly, A.J. Power, Two-dimensional monad theory, *J. Pure Appl. Algebra* 59 (1) (1989) 1–41.
- [7] A.K. Bousfield, Constructions of factorization systems in categories, *J. Pure Appl. Algebra* 9 (2–3) (1977) 207–220.
- [8] A. Burroni, Higher-dimensional word problems with applications to equational logic, *Theoret. Comput. Sci.* 115 (1) (1993) 43–62.
- [9] P.J. Freyd, Aspects of topoi, *Bull. Austral. Math. Soc.* 7 (1972) 1–76.
- [10] R. Garner, Understanding the small object argument, *Appl. Categ. Structures* 17 (3) (2009) 247–285.
- [11] R. Garner, N. Gurski, The low-dimensional structures tricategories form, *Math. Proc. Cambridge Philos. Soc.* 146 (3) (2009) 551–589.
- [12] R. Gordon, J. Power, R. Street, Coherence for tricategories, *Mem. Amer. Math. Soc.* 117 (558) (1995).
- [13] M. Grandis, W. Tholen, Natural weak factorization systems, *Arch. Math.* 42 (4) (2006) 397–408.
- [14] N. Gurski, An algebraic theory of tricategories, PhD thesis, University of Chicago, 2006.

- [15] K. Hess, P.-E. Parent, J. Scott, Co-rings over operads characterize morphisms, preprint, 2006, available at <http://arxiv.org/abs/math/0505559>.
- [16] G.M. Kelly, A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on, *Bull. Austral. Math. Soc.* 22 (1) (1980) 1–83.
- [17] S. Lack, A Quillen model structure for bicategories, *K-Theory* 33 (3) (2004) 185–197.
- [18] S. Lack, S. Paoli, 2-nerves for bicategories, *K-Theory* 38 (2) (2008) 153–175.
- [19] T. Leinster, Operads in higher-dimensional category theory, *Theory Appl. Categ.* 12 (3) (2004) 73–194.
- [20] T. Leinster, Are operads algebraic theories? *Bull. London Math. Soc.* 38 (2) (2006) 233–238.
- [21] F. Métayer, Cofibrant objects among higher-dimensional categories, *Homology, Homotopy Appl.* 10 (1) (2008) 181–203.
- [22] D.G. Quillen, *Homotopical Algebra*, Lecture Notes in Math., vol. 43, Springer-Verlag, 1967.
- [23] A. Radulescu-Banu, *Cofibrance and completion*, PhD thesis, MIT, 1999.
- [24] R. Street, Limits indexed by category-valued 2-functors, *J. Pure Appl. Algebra* 8 (2) (1976) 149–181.