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On the first-return integrals

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Abstract

Some pathological properties of the first-return integrals are explored. In particular it is proved that there exist Riemann improper integrable functions which are first-return recoverable almost everywhere, but not first-return integrable, with respect to each trajectory. It is also proved that the usual convergence theorems fail to be true for the first-return integrals.

Keywords: First-return integrals; First-return recovery

1. Introduction

We call *trajectory* in $[0, 1]$ any sequence $\bar{t} \equiv \{t_n\}$ of distinct points of $[0, 1]$, dense in $[0, 1]$. Given a trajectory $\bar{t}$ and an interval $J \subset [0, 1]$, we denote by $r(\bar{t}, J)$ the first element of $\bar{t}$ that belongs to J.

We call *partition* of $[0, 1]$ any finite collection of non-overlapping compact intervals $J_1, \dots, J_n$ such that $\bigcup_{i=1}^n J_i = [0, 1]$. Given a partition $\mathcal{P} = \{J_1, \dots, J_n\}$, we set $\text{mesh}(\mathcal{P}) = \sup_i |J_i|$.

Definition 1. (See [3].) A function $f : [0, 1] \to \mathbb{R}$ is said to be *first-return integrable with respect to a given trajectory* $\bar{t}$ on $[0, 1]$ if there exists a finite number A such that the following condition holds: for each $\varepsilon > 0$ there is a constant $\delta > 0$ such that

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$$\left| \sum_{J\in\mathcal{P}} f\big(r(\bar{t}, J)\big)|J| - A \right| < \varepsilon,$$

for every partition $\mathcal{P}$ of $[0,1]$ with $\mathrm{mesh}(\mathcal{P}) < \delta$.

In this case we write $A = (fr[\bar{t}\,])\int_0^1 f$.

Remark that for each function f (even not measurable) there is an equivalent function g which is first-return integrable with respect to a fixed trajectory $\bar{t}$. Namely the function g is defined as $g(x) = f(x)$ for $x \notin \bar{t}$ and $g(x) = 0$ for $x \in \bar{t}$.

M.J. Evans and P.D. Humke proved in [2, Theorem 2.3] that a function $f : [0,1] \to \mathbb{R}$ is Lebesgue measurable if and only if there exists a trajectory $\bar{t}$ such that, at almost each point $x \in [0,1]$, f is first-return recoverable with respect to $\bar{t}$, according to the following definition:

Definition 2. (See [2] and [3].) A function $f : [0,1] \to \mathbb{R}$ is said to be *first-return recoverable with respect to a given trajectory* $\bar{t}$ at $x \in [0,1]$ if

$$\lim_{k\to\infty} f(t_{n_k(x)}) = f(x),$$

where $\{t_{n_k(x)}\}_{k=1}^{\infty}$ is defined recursively via $t_{n_1(x)} = t_1$,

$$t_{n_{k+1}(x)} = \begin{cases} r(\bar{t}, (x - |t_{n_k(x)} - x|, x + |t_{n_k(x)} - x|)), & \text{if } x \neq t_{n_k(x)}, \\ t_{n_k(x)}, & \text{if } x = t_{n_k(x)}. \end{cases}$$

It is clear that each Riemann integrable function $f : [0,1] \to \mathbb{R}$ is first-return recoverable almost everywhere (br. a.e.) and first-return integrable with respect to each trajectory $\bar{t}$, with $(fr[\bar{t}\,])\int_0^1 f = \int_0^1 f$.

U.B. Darji and M.J. Evans proved in [1] that for each Lebesgue integrable function $f : [0,1] \to \mathbb{R}$ there exists a trajectory $\bar{t}$ such that f is first-return integrable on $[0,1]$ with respect to $\bar{t}$ and $(fr[\bar{t}\,])\int_0^1 f = (L)\int_0^1 f$. Moreover M.J. Evans and P.D. Humke proved that f is first-return recoverable with respect to the same trajectory $\bar{t}$ a.e. in $[0,1]$ (see [2, Theorem 2.1]).

The problem whether a first-return recoverable function is first-return integrable with respect to the same trajectory was solved in [2, Theorem 2.2] for bounded and measurable functions. It was also proved that for such functions the value of the first-return integrals coincide with the value of the Lebesgue integral. Concerning the case of unbounded and measurable functions, M. Csörnyei, U.B. Darji, M.J. Evans and P.D. Humke constructed in [3] a trajectory $\bar{t}$ and a function $f : [0,1] \to [0,+\infty)$ such that $f(x) = 0$ for $x \notin \bar{t}$ and such that f is first-return recoverable a.e. and first-return integrable on $[0,1]$ both with respect to $\bar{t}$, but $(fr[\bar{t}\,])\int_0^1 f > 0$.

In this paper we prove that:

Theorem 1. *There exist Riemann improper integrable functions which are first-return recoverable a.e. with respect to a generic trajectory $\bar{t}$, but not first-return integrable with respect to $\bar{t}$.*

Theorem 2. *Monotone convergence theorem, dominated convergence theorem and Fatou's Lemma fail to be true for the first-return integrals.*

Moreover, we give a convergence theorem, based on the following notion of first-return equi-integrability:

Definition 3. A sequence of function $\{f_n : [0,1] \to \mathbb{R}\}$ is said to be *first-return equi-integrable with respect to a given trajectory* $\bar{t}$ on $[0,1]$ if each function f_n is first-return integrable with respect to $\bar{t}$ on $[0,1]$ and for each $\varepsilon > 0$ there is a constant $\delta > 0$ (independent of n) such that

$$\sup_n \left| \sum_{J \in \mathcal{P}} f_n\big(r(\bar{t}, J)\big)|J| - \big(fr[\bar{t}\,]\big)\int_0^1 f_n \right| < \varepsilon, \tag{1}$$

for each partition $\mathcal{P}$ of $[0,1]$ with $\mathrm{mesh}(\mathcal{P}) < \delta$.

Theorem 3. *Let* $\bar{t}$ *be a trajectory, and let* $\{f_n\}$ *be a sequence of functions defined on* $[0,1]$ *and convergent pointwisely to* f. *If* $\{f_n\}$ *is first-return equi-integrable with respect to* $\bar{t}$ *on* $[0,1]$*, then* f *is first-return integrable with respect to* $\bar{t}$ *on* $[0,1]$ *and*

$$\lim_{n\to\infty} \big(fr[\bar{t}\,]\big)\int_0^1 f_n = \big(fr[\bar{t}\,]\big)\int_0^1 f. \tag{2}$$

2. Proof of Theorem 1

Let $\{a_n\}$ be a strictly decreasing sequence in $(0,1]$ such that $a_1 = 1$ and such that there exists $\lambda > 0$ with

$$a_n < \lambda(a_n - a_{n+1}) \quad \text{for each } n \in \mathbb{N}. \tag{3}$$

Moreover, let $\{c_n\}$ be a decreasing sequence of positive numbers such that

$$\sum_{n=1}^{\infty} c_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} (c_{2n} - c_{2n+1}) < \infty. \tag{4}$$

We define

$$f(x) = \begin{cases} (-1)^n \frac{c_n}{a_n - a_{n+1}}, & x \in (a_{n+1}, a_n], \\ 0, & x = 0. \end{cases} \tag{5}$$

It is clear that f is Riemann improper integrable, and not Lebesgue integrable, on $[0,1]$ with

$$\int_0^1 f = \sum_{n=1}^{\infty} (-1)^n c_n.$$

Let $\bar{t} \equiv \{t_n\}$ be a generic trajectory. It is easy to check that f is first-return recoverable with respect to $\bar{t}$ a.e. in $[0,1]$.

We show that f is not first-return integrable with respect to $\bar{t}$ on $[0,1]$.

If $n < m$ and J_1, J_2 are non-overlapping subintervals of $[0,1]$ such that $r(\bar{t}, J_1) = t_n$, $r(\bar{t}, J_2) = t_m$, then we set $r(\bar{t}, J_1) \prec r(\bar{t}, J_2)$.

We set $\mathbb{N}_1 = \{n \in \mathbb{N}\colon r(\bar{t}, [a_{n+1}, a_n]) \prec r(\bar{t}, [a_{n+2}, a_{n+1}])\}$ and $\mathbb{N}_2 = \mathbb{N} \setminus \mathbb{N}_1$. At least one of the series $\sum_{n\in\mathbb{N}_1} c_n$, $\sum_{n\in\mathbb{N}_2} c_n$ is divergent; then, without loss of generality, we can assume that $\sum_{n\in\mathbb{N}_1} c_n = +\infty$.

We also set $\mathbb{N}_1^+ = \{n \in \mathbb{N}_1\colon n \text{ is even}\}$ and $\mathbb{N}_1^- = \{n \in \mathbb{N}_1\colon n \text{ is odd}\}$. At least one of the series $\sum_{n\in\mathbb{N}_1^+} c_n$, $\sum_{n\in\mathbb{N}_1^-} c_n$ is divergent; then, without loss of generality, we can assume that $\sum_{n\in\mathbb{N}_1^+} c_n = +\infty$.

Given $\varepsilon, M > 0$, let N be an integer such that

$$c_N < \varepsilon/\lambda. \tag{6}$$

Take $0 < \delta < 2\max\{a_n - a_{n+1}\colon n > N\}$ and take two even numbers $m > N$ and $p > 0$ such that

$$a_m < \delta; \tag{7}$$

$$\sum_{n\in\mathbb{N}_1^+\cap\{m,\dots,m+p-2\}} c_n > M + c_1 + \sum_{n=1}^{\infty}(c_{2n} - c_{2n+1}) + \varepsilon. \tag{8}$$

Define a partition $\mathcal{P}$ of $[0,1]$ with $\mathrm{mesh}(\mathcal{P}) < \delta$ such that

(i) $[0, a_{m+p}] \in \mathcal{P}$;
(ii) $r(\bar{t}, [0, a_{m+p}]) \in [a_{m+p+1}, a_{m+p}]$;
(iii) for each $n \in \mathbb{N}_1^+ \cap \{m,\dots,m+p-2\}$, it is $[a_{n+2}, a_n] \in \mathcal{P}$;
(iv) for each $J \in \mathcal{P}' = \mathcal{P} \setminus \{[0, a_{m+p}], [a_{n+2}, a_n], n \in \mathbb{N}_1^+ \cap \{m,\dots,m+p-2\}\}$, there is $s \in \mathbb{N}$ such that $J \subset [a_{s+1}, a_s]$.

Then

$$\sum_{J\in\mathcal{P}} f\big(r(\bar{t}, J)\big)|J| = f\big(r\big(\bar{t}, [0, a_{m+p}]\big)\big)\cdot a_{m+p} + \sum_{J\in\mathcal{P}'} f\big(r(\bar{t}, J)\big)|J| + \sum_{n\in\mathbb{N}_1^+\cap\{m,\dots,m+p-2\}} \frac{c_n}{a_n - a_{n+1}}(a_n - a_{n+2}).$$

Now remark that, for $s = 1,\dots,m+p-1$ and for $s \in \mathbb{N}_2^+ \cap \{m,\dots,m+p-2\}$, where $\mathbb{N}_2^+ = \{n \in \mathbb{N}_2\colon n \text{ is even}\}$, it is $\bigcup\{J \in \mathcal{P}\colon J \subset [a_{s+1}, a_s]\} = [a_{s+1}, a_s]$. Hence by (5) we have

$$\begin{aligned}
&\sum_{J\in\mathcal{P}} f\big(r(\bar{t}, J)\big)|J| \\
&\quad = \frac{c_{m+p}}{a_{m+p} - a_{m+p+1}}\cdot a_{m+p} - c_1 \\
&\qquad + \sum_{n=1}^{m/2-1}(c_{2n} - c_{2n+1}) + \sum_{n\in\mathbb{N}_2^+\cap\{m,\dots,m+p-2\}}(c_{2n} - c_{2n+1}) \\
&\qquad + \sum_{n\in\mathbb{N}_1^+\cap\{m,\dots,m+p-2\}} \frac{c_n}{a_n - a_{n+1}}(a_n - a_{n+2}).
\end{aligned}$$

Thus, by (3), (6), and (8), we have

$$\begin{aligned}
&\left|\sum_{J\in\mathcal{P}} f\big(r(\bar{t}, J)\big)|J|\right| \\
&\quad \geqslant \sum_{n\in\mathbb{N}_1^+\cap\{m,\dots,m+p-2\}} c_n - c_1 - \sum_{n=1}^{\infty}(c_{2n} - c_{2n+1}) - \lambda c_{m+p} \\
&\quad > M.
\end{aligned}$$

By the arbitrariness of δ and M this implies that f is not first-return integrable with respect to $\bar{t}$ on $[0,1]$.

Possible choices of sequences $\{a_n\}$ and $\{c_n\}$ satisfying conditions (3) and (4) are $a_n = \frac{1}{2^{n-1}}$ and $c_n = \frac{1}{n}$, $n = 1, 2, \ldots$, respectively.

3. Proof of Theorem 2

Let $\bar{t}$ and f be defined as in [3, §4]. For each natural n we define $f_n(x) = 0$ if $x \notin \bar{t}$ or $x = t_k$ with $k > n$, and we define $f_n(t_k) = f(t_k)$ if $k \leqslant n$. It is clear that f_n is first-return integrable with respect to $\bar{t}$, and that $(fr[\bar{t}\,])\int_0^1 f_n = 0$ for $n \in \mathbb{N}$. Then the claim follows immediately by condition $(fr[\bar{t}\,])\int_0^1 f > 0$, proved in [3, §4.4].

4. Proof of Theorem 3

Let ε and δ be as in Definition 3, and fix a partition $\mathcal{P}$ with $\operatorname{mesh}(\mathcal{P}) < \delta$. Since $\lim_n f_n(r(\bar{t}, J)) = f(r(\bar{t}, J))$, for each $J \in \mathcal{P}$ there exists a natural number N such that

$$\bigl|f_n\bigl(r(\bar{t}, J)\bigr) - f\bigl(r(\bar{t}, J)\bigr)\bigr| < \frac{\varepsilon}{p} \tag{9}$$

(where p is the cardinality of $\mathcal{P}$), for $n > N$ and $J \in \mathcal{P}$. Then, by (1) and (9), for $n, m > N$ we have

$$\begin{aligned}
&\left|\bigl(fr[\bar{t}\,]\bigr)\int_0^1 f_n - \bigl(fr[\bar{t}\,]\bigr)\int_0^1 f_m\right| \\
&\quad\leqslant \left|\sum_{J\in\mathcal{P}} f_n\bigl(r(\bar{t}, J)\bigr)|J| - \bigl(fr[\bar{t}\,]\bigr)\int_0^1 f_n\right| + \left|\sum_{J\in\mathcal{P}} f_m\bigl(r(\bar{t}, J)\bigr)|J| - \bigl(fr[\bar{t}\,]\bigr)\int_0^1 f_m\right| \\
&\qquad + \left|\sum_{J\in\mathcal{P}} f_n\bigl(r(\bar{t}, J)\bigr)|J| - \sum_{J\in\mathcal{P}} f_m\bigl(r(\bar{t}, J)\bigr)|J|\right| \\
&\quad\leqslant \varepsilon + \varepsilon + 2\varepsilon = 4\varepsilon.
\end{aligned}$$

Therefore the sequence $\{(fr[\bar{t}\,])\int_0^1 f_n\}$ is convergent, ε being arbitrary.

Let $A = \lim_n (fr[\bar{t}\,])\int_0^1 f_n$.

Now we show that f is first-return integrable with respect to $\bar{t}$ on $[0, 1]$ and $(fr[\bar{t}\,])\int_0^1 f = A$. To this aim let ε and δ be as in Definition 3. Then, by (1), for each partition $\mathcal{P}$ with $\operatorname{mesh}(\mathcal{P}) < \delta$ we have

$$\left|\sum_{J\in\mathcal{P}} f\bigl(r(\bar{t}, J)\bigr)|J| - A\right| = \lim_n \left|\sum_{J\in\mathcal{P}} f_n\bigl(r(\bar{t}, J)\bigr)|J| - \bigl(fr[\bar{t}\,]\bigr)\int_0^1 f_n\right| \leqslant \varepsilon.$$

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