Bounds of Hausdorff measure of the Sierpinski gasket

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Abstract

By a new method, we obtain the lower and upper bounds of the Hausdorff measure of the Sierpinski gasket, which can approach the Hausdorff measure of the Sierpinski gasket infinitely.

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0. Introduction

The computation and estimation of the Hausdorff dimension and measure of the fractal sets are important problems in fractal geometry. Generally, the computation of the Hausdorff dimension, especially the Hausdorff measure, is very difficult. As a referee has previously pointed out, “Hausdorff measure is an important notion in the study of fractals. However there are few concrete results about computation of Hausdorff measure even for some simple fractals. Part of reason is the difficulty of the problem.” For a self-similar set satisfying the open set condition, we know that its Hausdorff dimension equals its self-similar dimension [1]. However, there are not many results on the computation and estimation of Hausdorff measures for such fractal sets except for a few fractal sets on a line, like the Cantor set [2]. For the famous classical self-similar set, the Sierpinski gasket, its Hausdorff measure remains unknown. Nevertheless, efforts have been made in order to estimate the lower and upper bounds of its Hausdorff measure [3–8].
In this paper, we develop a new method of estimating the upper bounds and lower bounds of the Hausdorff measure of the Sierpinski gasket. We show that the Hausdorff measure of the Sierpinski gasket can be squeezed out by sequences of lower bounds and upper bounds. Precisely speaking, we show that:

**Theorem.** The Hausdorff measure of the Sierpinski gasket satisfies the estimation

\[ a_n e^{-\frac{16 \sqrt{3} n}{3}} s(\frac{1}{2})^n \leq H^s(S) \leq a_n, \quad \text{for } n \geq 1, \]

where \( a_n \) is defined in Proposition 1.2.

The above theorem provides us a way, at least in theory, to estimate the Hausdorff measure of the Sierpinski gasket as close as we want.

In the end of the paper, we give two conjectures about \( a_n \), for \( n \geq 3 \) and \( H^s(S) \).

1. **The Hausdorff measures of the self-similar sets**

Let \( D \subset \mathbb{R}^n \) be a nonempty set. \( E \subset \mathbb{R}^n \) is a self-similar set defined by \( m \) similar contracting maps \( S_i : D \to D \), with contracting ratios, \( 0 < c_i < 1 \) \( (i = 1, 2, \ldots, m) \) and satisfying open set condition, that is, there exists a nonempty open set \( U \) for which we have \( S_i[U] \cap S_j[U] = \emptyset \) for \( i \neq j \) and \( U \supseteq S_j[U] \) for all \( i \). Then \( \dim_H(E) = s, \ 0 < H^s(E) < +\infty \).

Where \( s \) satisfies \( \sum_{i=1}^{m} c_i^s = 1, \ \dim_H(E) \) and \( H^s(E) \) denote the Hausdorff dimension and measure of \( E \), respectively. If \( S_i[E] \cap S_j[E] = \emptyset, \ 0 < i < j \leq m \), we say that \( E \) satisfies strong separate condition (SSC). Let \( J_n = \{ (i_1i_2 \cdots i_n) : 1 \leq i_1, i_2, \ldots, i_n \leq m, \ n \geq 1 \} \) and \( E_{i_1i_2 \cdots i_n} = S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_n}(E) \) be self-similar \( E \). It is easy to know that \( E = \bigcup_{J_n} E_{i_1i_2 \cdots i_n} \).

**Proposition 1.1.** [7] Suppose that \( E \) is a self-similar set satisfying the open set condition, then for any measurable set \( U \), we have \( H^s(E \cup U) \leq |U|^s \), where \( s = \dim_H(E) \).

**Proposition 1.2.** Suppose that \( E \) is a self-similar set satisfying the open set condition. For \( n \geq 1, \ 1 \leq k \leq m^n, \) let \( \Delta_1, \Delta_2, \ldots, \Delta_k \in \{ E_{i_1i_2 \cdots i_n} : 1 \leq i_1, i_2, \ldots, i_n \leq m \} \) and \( \mu \) be the common self-similar probability measure on the \( E \), \( \mu(E_{i_1i_2 \cdots i_n}) = c_{i_1}^s c_{i_2}^s \cdots c_{i_n}^s \).

Let \n\[ b_k = \min_{\Delta_i \in \{ E_{i_1 \cdots i_k} \}} \left\{ \frac{|\bigcup_{i=1}^{k} \Delta_i|^s}{\mu(\bigcup_{i=1}^{k} \Delta_i)} \right\}, \]

where the minimum is taken for all possible union of \( k \) elements of \( \{ E_{i_1i_2 \cdots i_n} \} \) and \( a_n = \min_{1 \leq k \leq m^n} (b_k). \) If there exists a constant \( A > 0 \) such that \( a_n \geq A(n = 1, 2, \ldots) \), then \( \dim_H(E) \geq A \).

**Proof.** By [1, p. 33], we can get the same values for Hausdorff measure and dimension if in the definition of \( H^s_\delta(E) \) we use \( \delta \)-cover of just open set. So in the mass distribution principle of [1], we can replace any sets by any open sets. For any open set \( V \), let \( F_n = \bigcup_{E_{i_1i_2 \cdots i_n} \subset V} E_{i_1i_2 \cdots i_n} \). It is obvious that \( F_n \subset F_{n+1}, \ \bigcup_{n=1}^{+\infty} F_n = E \cap V \). By the property of measure \( \mu \) and the definitions of \( a_n, b_k \), we get
\[
\mu(V) = \mu(E \cap V) = \mu \left( \bigcup_{n=1}^{+\infty} F_n \right) = \lim_{n \to +\infty} \mu(F_n)
\]
\[
= \lim_{n \to +\infty} \mu \left( \bigcup_{E_{i_{12} \cdots i_{n}} \subset V} E_{i_{12} \cdots i_{n}} \right) \leq \frac{1}{a_n} \left| \bigcup_{E_{i_{12} \cdots i_{n}} \subset V} E_{i_{12} \cdots i_{n}} \right|^s \leq \frac{1}{a_n} |V|^s \leq \frac{1}{A} |V|^s.
\]

By the mass distribution principle of [1], we have \(H^s(E) \geq A\). \(\square\)

**Proposition 1.3.** For \(n \geq 1\), the \(a_n\) decreases and \(\lim_{n \to +\infty} a_n = H^s(E)\).

**Proof.** Suppose that

\[
a_n = \min_{1 \leq k \leq m^n} \min_{\Delta_i \in \{E_{i_{12} \cdots i_n}\}} \left\{ \frac{\left| \bigcup_{i=1}^k \Delta_i \right|^s}{\mu(\bigcup_{i=1}^k \Delta_i)} \right\} = \frac{|U_{k_0}|}{\mu(U_{k_0})},
\]

where the \(U_{k_0}\) is the union of some \(k_0\) elements of \(\{E_{i_{12} \cdots i_n}\}\).

By Proposition 1.1, we get \(H^s(E \cap U_{k_0}) \leq |U_{k_0}|^s\).

So

\[
\mu(U_{k_0}) H^s(E) \leq |U_{k_0}|^s.
\]

Therefore

\[
H^s(E) \leq \frac{|U_{k_0}|^s}{\mu(U_{k_0})} = a_n.
\]

Next we prove that \(a_n\) decreases. Because \(U_{k_0}\) is the union of some \(k_0\) elements of \(\{E_{i_{12} \cdots i_n}\}\), \(U_{k_0}\) is the union of some \(k_0m\) elements of \(\{E_{i_{12} \cdots i_{n+1}}\}\). By the definition of \(a_n\), we obtain

\[
a_{n+1} \leq \frac{\{|U_{k_0}|^s\}}{\mu(U_{k_0})} = a_n.
\]

Let \(L = \lim_{n \to +\infty} a_n\), so \(a_n \geq L\). By Proposition 1.2, \(H^s(E) \geq L\).

In (1.1), let \(n \to +\infty\). We get \(H^s(E) \leq L\). So \(H^s(E) = \lim_{n \to +\infty} a_n = L\). \(\square\)

**Corollary 1.1.** If \(c_1 = c_2 = \cdots = c_m = c\), then \(b_k = \min_{\Delta_i \in \{E_{i_{12} \cdots i_n}\}} \left( \frac{\left| \bigcup_{i=1}^k \Delta_i \right|^s}{k^m c^m} \right)\), where the minimum is taken for all possible union of \(k\) elements of \(\{E_{i_{12} \cdots i_n}\}\). \(a_n = \min_{1 \leq k \leq m^n} b_k\).

Then for \(n \geq 1\), \(a_n\) decreases and \(\lim_{n \to +\infty} a_n = H^s(E)\).

2. **The Hausdorff measure of the Sierpinski gasket**

Take an equilateral triangle (including its inside) with side length 1 in the Euclidean plane \(R^2\). Call it \(S_0\) and delete all but the three corner equilateral triangles (including their boundary) with side length \(\frac{1}{2}\) to obtain \(S_1\) (see Fig. 1). Continue in this way, replacing at the each equilateral triangle of \(S_{n-1}\) by the three corner equilateral triangles with side length \(\frac{1}{2}\) to get \(S_n\). We obtain \(S_0 \supset S_1 \supset \cdots \supset S_n \supset \cdots\).

The nonempty set \(S = \bigcap_{n=0}^{+\infty} S_n\) is called the Sierpinski gasket. For each \(n \geq 0\), \(S_n\) consists of \(3^n\) equilateral triangles with side length \(2^{-n}\). Any one of such equilateral triangles is called a \(2^{-n}\)-basic equilateral triangle. The Hausdorff dimension of \(S\) is \(s = \dim_H(S) = \log_2 3\).

**Theorem.** The Hausdorff measure of the Sierpinski gasket satisfies the estimation

\[
a_n e^{-\frac{16\sqrt{3}}{\pi} \left( \frac{1}{2} \right)^n} \leq H^s(S) \leq a_n, \quad \text{for } n \geq 1.
\]
By a simple calculation, we can get that $a_1 = 1$, $a_2 = \frac{3^{\frac{3}{2}}}{6} \approx 0.9508$ (see Fig. 5). By the inequality, $|e^{-|x|} - 1| \leq |x|$, for $x \geq 0$, we have an error bound of our estimation,

\[
\text{error} \leq d_n|e^{-\frac{16\sqrt{3}}{9}s(\frac{1}{2})} - 1| \leq 0.9508|e^{-\frac{16\sqrt{3}}{9}s(\frac{1}{2})} - 1|, \quad \text{for } n \geq 1.
\]

**Proof.** Let $n \geq 1$, $1 \leq k \leq 3^n$, $\Delta_1, \Delta_2, \ldots, \Delta_k \in S_n$.

**Case 1.** If

\[
\left( \bigcup_{i=1}^{k} \Delta_i \right) \cap \Delta_j^0 \neq \emptyset \quad (j = 1, 2, 3),
\]

$\Delta_j^0$ for $j = 1, 2, 3$, is defined by Fig. 2, let $|\bigcup_{i=1}^{k} \Delta_i| = d$, by [5, Proposition 1.3] (see Fig. 2), we have

\[
\left| \bigcup_{i=1}^{k} \Delta_i \right| \geq \left| \left( \bigcup_{i=1}^{k} \Delta_i \right) \cap \Delta_{EFD} \right| \geq \frac{1}{4}.
\]

For each $\Delta_i$, there exists $\Delta_j^{n-1} \in S_{n-1}$ such that $\Delta_i \subset \Delta_j^{n-1}$ ($j = 1, 2, \ldots, k_{n-1}$) and $\Delta_1^{n-1}, \Delta_2^{n-1}, \ldots, \Delta_{k_{n-1}}^{n-1}$ are different from each other. It is easy to know that

\[
k \leq 3k_{n-1} \quad \text{and} \quad \left| \bigcup_{j=1}^{k_{n-1}} \Delta_j^{n-1} \right| \leq d + 2\left( \frac{1}{2} \right)^n = d + \left( \frac{1}{2} \right)^{n-1}.
\]
So
\[
\frac{|\bigcup_{i=1}^{k} \Delta_i|}{|\bigcup_{j=1}^{k-1} \Delta_{j-1}|} \geq \frac{d}{d + \left(\frac{1}{2}\right)^{n-1}} \geq \frac{\frac{1}{4}}{\frac{1}{4} + \left(\frac{1}{2}\right)^{n-1}} = \frac{1}{1 + 4\left(\frac{1}{2}\right)^{n-1}}.
\]
Therefore
\[
\frac{|\bigcup_{i=1}^{k} \Delta_i|^s}{k(\frac{1}{3})^n} \geq \left(\frac{1}{1 + 4\left(\frac{1}{2}\right)^{n-1}}\right)^s \frac{|\bigcup_{j=1}^{k-1} \Delta_{j-1}|^s}{k\left(\frac{1}{3}\right)^{n-1}} \geq \left(\frac{1}{1 + 4\left(\frac{1}{2}\right)^{n-1}}\right)^s \frac{|\bigcup_{j=1}^{k-1} \Delta_{j-1}|^s}{k_{n-1}\left(\frac{1}{3}\right)^{n-1}}.
\]
(2.2)

It is similar to the \(a_n\) of Corollary 1.1. Let
\[
a_n^{(1)} = \min_{3 \leq k \leq 3^n} \min_{\Delta_i \in S_k} \left\{ \frac{|\bigcup_{i=1}^{k} \Delta_i|^s}{k(\frac{1}{3})^n} : \text{for each } l = 1, 2, 3, \left(\bigcup_{i=1}^{k} \Delta_i\right) \cap \Delta_l^0 \neq \emptyset \right\}.
\]

It is obvious that \(a_n \leq a_n^{(1)}\). From (2.2), we have \(a_n^{(1)} \geq \left(\frac{1}{1 + 4\left(\frac{1}{2}\right)^{n-1}}\right)^s a_n^{(1)}\).

Therefore, for any \(l \geq 1\),
\[
a_{l+n}^{(1)} \geq \left(\frac{1}{1 + 4\left(\frac{1}{2}\right)^{l+n-1}}\right)^s a_{l+n-1}^{(1)} \geq \left(\frac{1}{1 + 4\left(\frac{1}{2}\right)^{l+n-2}}\right)^s \cdots \left(\frac{1}{1 + 4\left(\frac{1}{2}\right)^{n}}\right)^s a_n^{(1)}.
\]

Take logarithm on two sides and use inequality, \(\ln(1 + x) < x, x > 0\), we get
\[
\ln a_{l+n}^{(1)} \geq -s \left[ \ln \left(1 + 4\left(\frac{1}{2}\right)^{l+n-1}\right) + \ln \left(1 + 4\left(\frac{1}{2}\right)^{l+n-2}\right) + \cdots + \ln \left(1 + 4\left(\frac{1}{2}\right)^{n}\right) \right] + \ln a_n^{(1)}
\]
\[
\geq -s \left[ 4\left(\frac{1}{2}\right)^{l+n-1} + 4\left(\frac{1}{2}\right)^{l+n-2} + \cdots + 4\left(\frac{1}{2}\right)^{n} \right] + \ln a_n^{(1)}.
\]

It is similar to the proof of Proposition 1.3. It is easy to get that \(\{a_n^{(1)}\}\) decreases. Suppose 
\(\lim_{n \to \infty} a_n^{(1)} = \alpha_1\) and let \(l \to +\infty\). We have
\[
\ln \alpha_1 \geq -s \frac{4\left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} + \ln a_n^{(1)} = \ln \left(a_n^{(1)} e^{-8s\left(\frac{1}{2}\right)^n}\right).
\]

So \(\alpha_1 \geq a_n^{(1)} e^{-8s\left(\frac{1}{2}\right)^n} \geq a_n e^{-8s\left(\frac{1}{2}\right)^n}\).

Case 2. If \(\bigcup_{i=1}^{k} \Delta_i\) only intersects two basic equilateral triangles of \(S_1\), with no loss generality, suppose \(\bigcup_{i=1}^{k} \Delta_i\) intersects \(\Delta_0^0, \Delta_2^0, \Delta_3^0\) (see Fig. 2).

(a) If
\[
\left(\bigcup_{i=1}^{k} \Delta_i\right) \cap (\Delta_{11} \cup \Delta_{12}) \neq \emptyset \quad \text{and} \quad \left(\bigcup_{i=1}^{k} \Delta_i\right) \cap \Delta_3^0 \neq \emptyset \quad \text{or}
\]

\[
\left( \bigcup_{i=1}^{k} \Delta_i \right) \cap (\Delta_{21} \cup \Delta_{23}) \neq \emptyset \quad \text{and} \quad \left( \bigcup_{i=1}^{k} \Delta_i \right) \cap \Delta_2^0 \neq \emptyset,
\]
by the symmetry, we can suppose that
\[
\left( \bigcup_{i=1}^{k} \Delta_i \right) \cap (\Delta_{11} \cup \Delta_{12}) \neq \emptyset \quad \text{and} \quad \left( \bigcup_{i=1}^{k} \Delta_i \right) \cap \Delta_0^{0} \neq \emptyset.
\]

At that time, \(|\bigcup_{i=1}^{k} \Delta_i| \geq \frac{1}{4} \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8} \). It is similar to \(a_{n}^{(1)}\) of case 1.

Let
\[
a_{n}^{(2)} = \min_{2 \leq k \leq 2 \cdot 3^{n-1}} \min_{\Delta_i \in S_n} \left\{ \left| \bigcup_{i=1}^{k} \Delta_i \right| : \left( \bigcup_{i=1}^{k} \Delta_i \right) \text{satisfies (2.3)} \right\}.
\]

Like case 1, it is easy to prove that \(\{a_{n}^{(2)}\}\) decreases. Suppose that \(\lim_{n \to \infty} a_{n}^{(2)} = \alpha_2\), we have
\[
a_{n}^{(2)} \geq \left( \frac{\frac{\sqrt{3}}{8}}{\frac{\sqrt{3}}{8} + (\frac{1}{2})^{n-1}} \right)^{s} a_{n-1}^{(2)} = \left( \frac{1}{1 + \frac{8\sqrt{3}}{3} (\frac{1}{2})^{n-1}} \right)^{s} a_{n-1}^{(2)}.
\]
So \(\alpha_2 \geq a_{n}^{(2)} e^{-\frac{16\sqrt{3}}{3} s (\frac{1}{2})^{n}} \geq a_{n} e^{-\frac{16\sqrt{3}}{3} s (\frac{1}{2})^{n}} \).

(b) If
\[
\left( \bigcup_{i=1}^{k} \Delta_i \right) \subset (\Delta_{13} \cup \Delta_{22}),
\]
by the similarity, there exist a positive integer \(t\) and \(\Delta'_2, \Delta'_3 \in S_t\) (see Figs. 2, 3) such that
\[
2^t \Delta'_2 = \Delta_2^0, \quad 2^t \Delta'_3 = \Delta_3^0, \quad \bigcup_{i=1}^{k} \Delta_i \subset \Delta'_2 \cup \Delta'_3 \quad \text{and}
\]
\[
\left( \bigcup_{i=1}^{k} \Delta_i \right) \cap (\Delta'_{11} \cup \Delta'_{12}) \neq \emptyset, \quad \left( \bigcup_{i=1}^{k} \Delta_i \right) \cap \Delta'_3 \neq \emptyset \quad \text{or}
\]
\[
\left( \bigcup_{i=1}^{k} \Delta_i \right) \cap (\Delta'_{21} \cup \Delta'_{23}) \neq \emptyset, \quad \left( \bigcup_{i=1}^{k} \Delta_i \right) \cap \Delta'_2 \neq \emptyset.
\]
Because \(\frac{\left| \bigcup_{i=1}^{k} \Delta_i \right|}{k} = \frac{\left| \bigcup_{i=1}^{k} 2^t \Delta'_i \right|}{k} \), \(\bigcup_{i=1}^{k} 2^t \Delta_i\) satisfies case (b). Note that \(\Delta_{13} \cup \Delta_{22}\) contains \(2 \cdot 3^{n-2}\) elements of \(S_n\). It is similar to \(a_{n}^{(2)}\) of case (a).

Let
\[
a_{n}^{(3)} = \min_{2 \leq k \leq 2 \cdot 3^{n-2}} \min_{\Delta_i \in S_n} \left\{ \left| \bigcup_{i=1}^{k} \Delta_i \right|^s : \left( \bigcup_{i=1}^{k} \Delta_i \right) \subset (\Delta_{13} \cup \Delta_{22}) \right\}.
\]
3. Two conjectures about Sierpinski gasket

For the Sierpinski gasket, by a simple calculation and the definition of $a_n$ in Corollary 1.1, it is easy to get $a_1 = 1$ (see Fig. 4), $a_2 = \frac{3\sqrt{3}}{2} \approx 0.9508$ (see Fig. 5). When $n$ is bigger than 2, a good and efficient algorithm of $a_n$ still needs to be found. We can only make the following conjectures.

The $\{a_n^{(3)}\}$ decreases and suppose that $\lim_{n \to \infty} a_n^{(3)} = \alpha_3$, we have $\alpha_3 \geq a_n^{(3)} e^{-\frac{16\sqrt{3}}{3}s(\frac{1}{2})^n} \geq a_n e^{-\frac{16\sqrt{3}}{3}s(\frac{1}{2})^n}$.

Case 3. If $\bigcup_{i=1}^k \Delta_i$ only intersects one basic equilateral triangle of $S_1$, by the similarity, there exists a positive integer $i_0$ and $i_1, i_2, \ldots, i_0 \in \{1, 2, 3\}$ such that $\bigcup_{i=1}^k \Delta_i \subset E_{i_0} \circ f_{i_2} \circ \cdots \circ f_{i_0}(S)$ and $\bigcup_{i=1}^k \Delta_i$ intersects at least two of $E_{i_1} \circ E_{i_2} \circ \cdots \circ E_{i_0}$, $E_{i_1} \circ E_{i_2} \circ \cdots \circ E_{i_0}$, $E_{i_1} \circ E_{i_2} \circ \cdots \circ E_{i_0}$. Therefore $f_{i_0}^{-1} \circ \cdots \circ f_{i_2}^{-1} \circ f_{i_0}^{-1} \left( \bigcup_{i=1}^k \Delta_i \right)$ intersects at least two of $f_1(S), f_2(S), f_3(S)$. Note that

$$\frac{|\bigcup_{i=1}^k \Delta_i|^s}{\mu(\bigcup_{i=1}^k \Delta_i)} = \frac{|f_{i_0}^{-1} \circ \cdots \circ f_{i_2}^{-1} \circ f_{i_0}^{-1} \left( \bigcup_{i=1}^k \Delta_i \right)|^s}{\mu((\bigcup_{i=1}^k f_{i_1}^{-1} \circ \cdots \circ f_{i_0}^{-1}(\Delta_i)))}$$

and $f_{i_0}^{-1} \circ \cdots \circ f_{i_2}^{-1} \circ f_{i_0}^{-1} \left( \bigcup_{i=1}^k \Delta_i \right)$ satisfies case 1 or case 2. It is similar to $a_n^{(1)}$ of case 1.

Let

$$a_n^{(4)} = \min_{1 \leq k \leq 3^{n-1}} \min_{i_1, i_2, \ldots, i_k} \left\{ \frac{|\bigcup_{i=1}^k \Delta_i|^s}{k(\frac{1}{2})^n} : \bigcup_{i=1}^k \Delta_i \subset \Delta_{i_1}^{0}, i_1 \in \{1, 2, 3\} \right\}.$$  

The $\{a_n^{(4)}\}$ decreases and suppose that $\lim_{n \to \infty} a_n^{(4)} = \alpha_4$, we have $\alpha_4 \geq a_n^{(4)} \min\{e^{-\frac{16\sqrt{3}}{3}s(\frac{1}{2})^n}, e^{-8s(\frac{1}{2})^n}\} \geq a_n e^{-\frac{16\sqrt{3}}{3}s(\frac{1}{2})^n}$.

From cases 1–3, we obtain

$$H^s(S) \leq \lim_{n \to \infty} a_n = \lim_{n \to \infty} \min\{a_n^{(1)}, a_n^{(2)}, a_n^{(3)}, a_n^{(4)}\} = \min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \geq \min\{a_n e^{-8s(\frac{1}{2})^n}, a_n e^{-\frac{16\sqrt{3}}{3}s(\frac{1}{2})^n}, a_n e^{-\frac{16\sqrt{3}}{3}s(\frac{1}{2})^n}, a_n e^{-\frac{16\sqrt{3}}{3}s(\frac{1}{2})^n}\} = a_n e^{-\frac{16\sqrt{3}}{3}s(\frac{1}{2})^n}.$$  

So, $a_n e^{-\frac{16\sqrt{3}}{3}s(\frac{1}{2})^n} \leq H^s(S) \leq a_n$, for $n \geq 1$.  \qed
Conjecture 1. For the Sierpinski gasket, 

\[ a_3 = \frac{7^6}{24} \approx 0.91047736 \quad (\text{see Fig. 6}), \quad a_3 = \frac{13^6}{66} \approx 0.88319434 \quad (\text{see Fig. 7}), \]

\[ a_5 = \frac{25^6}{192} \approx 0.85592100617, \quad a_6 = \frac{49^6}{570} \approx 0.83769501528, \]

\[ a_7 = \frac{97^6}{1698} \approx 0.830033293836, \quad a_8 = \frac{193^6}{5082} \approx 0.825227465852. \]

Conjecture 2. The Hausdorff measure of the Sierpinski gasket satisfies

\[ 0.779355 \approx a_8 e^{-\frac{16}{3} \frac{1}{s} \frac{1}{y^2}} \leq H^x(S) \leq a_8. \]

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References


