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Alternative Problems for Nonlinear Functional Equations

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1. INTRODUCTION

The branching or bifurcation of solutions of functional equations is an important subject in analysis and goes back to the time of Poincaré in his research in differential equations. One need only consult the book of Minorsky [*1*] to recognize its significance in applications. A simple version of such a problem is the following. Suppose $F(x, y)$ is an n-vector function of the n-vector x and the scalar y such that the Eq. $F(x, y) = 0$ has a unique solution $x = x^*(y)$ for $y < 0$ with $x^*(y) \to 0$ as $y \to 0$ and $F(0, 0) = 0$. Suppose also that the matrix $A = \partial F(0, 0)/\partial x$ is singular. One wishes to determine the nature of the solutions (if any exist) of the Eq. $F(x, y) = 0$ near $x = 0$, $y = 0$ for $y > 0$. In such a situation, there may exist more than one solution for each $y > 0$ and hence the term branching or bifurcation.

In this problem, one frequently replaces the original equation by an equivalent equation (the bifurcation or determining equation) of lower dimension. A general method for obtaining such equivalent equations is given in this paper and the problem of solving the equivalent equation is referred to as an alternative problem. To illustrate, suppose $F(x, y) = Ax - N(x, y)$ where $N(0, 0) = 0$ and N is continuous in x, y, locally

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Lipschitzian in x with the local Lipschitz constant approaching zero as $y \to 0$. If E is the projection operator taking n-dimensional space onto the range $\mathscr{R}(A)$ of A, then the equation

$$Ax - N(x, y) = 0 \tag{1}$$

is equivalent to the equation

$$\begin{aligned} &\text{(a)} \quad 0 = (I - E)\, N(x, y) \\ &\text{(b)} \quad Ax = EN(x, y). \end{aligned} \tag{2}$$

Furthermore, if M is a right inverse of A, then Eq. (2b) is equivalent to

$$x = x_0 + MEN(x, y), \tag{3}$$

where x_0 is an arbitrary element of the null space $\mathscr{N}(A)$ of A. From the above conditions on N and x_0 any fixed element in a bounded set, one can apply the contraction principle to (3) for $|y|$ small to obtain a function $x = x^*(x_0, y)$ which satisfies (3) and therefore (2b). The vector $x^*(x_0, y)$ will then be a solution of (1) if (2a) is satisfied; that is, if $(I - E)\, N(x^*(x_0, y), y) = 0$. If we regard y as fixed at some small value, then $x^*(x_0, y)$ is a function of the free parameter x_0 in $\mathscr{N}(A)$ and this equation can be considered as an equation in the unknown x_0.

In summary, we see that the solution of the n-dimensional equation (1) with the above smallness conditions of N can always be reduced to a discussion of the solution of a set of equations of the same dimension as the dimension of the null space of A. This has an obvious advantage and the advantage is even more significant when the theory is extended to infinite dimensional spaces (see Section 2). In specific applications in infinite dimensions, $\mathscr{N}(A)$ is very often finite dimensional and even when $\mathscr{N}(A)$ is infinite dimensional, it is still an improvement over the original problem since $\mathscr{N}(A)$ is a proper subspace.

Since our main interest in this paper is concerned with associating various alternative problems to functional equations the dependence of the functions upon parameters will be suppressed throughout.

The above remarks, even for the case of infinite dimensions, are implicit in many paper. One line of investigation was initiated by an early paper of Cesari [*2*] which was later extended and modified to apply to many problems in ordinary, partial and functional differential equations. See Cesari [*2–6*], Hale [*10–12*], Perello [*21*] for this approach as well as a more complete bibliography and list of applications. Concurrent with this development, analogous ideas for perturbation problems were being expressed by Friedrichs [*7*], Cronin [*8*], Bartle [*9*], Graves [*13*], Lewis [*14, 15*], Bass [*16, 17*], Nirenberg [*18*], Vainberg and Trenogin [*19*], Antosiewicz [*20*], and Rabinowitz [*22*].

In the papers [4, 5], Cesari injected a significant new idea when he observed that finite-dimensional alternative problems could always be associated with certain rather general types of equations (1) even when the nonlinearities $\mathcal{N}$ are not small. This was accomplished by proving that a certain operator could always be made a contraction. Of course, the dimension of the alternative problem will, in general, be greater than the dimension of $\mathcal{N}(A)$. The method of Cesari is generalized in Section 3 of this paper and the motivation for this extension follows from some very simple geometric ideas in Section 2.

Since Section 2 is rather technical, it is worthwhile to mention the main results for a special case. Suppose X, Z are Banach spaces, $A : X \to Z$ is a linear operator which has a bounded right inverse M and $\mathcal{N}(A), \mathcal{R}(A)$ admit projections by operators U, E, respectively. One may assume without loss of generality that $UM = 0$. If $N : X \to Z$ is a given operator, linear or not, then $UM = 0$ implies the equation

$$Ax = Nx, \tag{4}$$

is equivalent to the equation

$$\begin{aligned} &\text{(a)} \quad 0 = (I - E)\, Nx, \\ &\text{(b)} \quad x = Ux + MENx. \end{aligned} \tag{5}$$

Equations (5a), (5b) are the same as (2a), (3) except we are in infinite dimensions, have suppressed the dependence on parameters and made the observation that $x_0 = Ux$ since $UM = 0$.

Theorem 2 of Section 2 states the following: If $W : X \to X$ is any projection operator with range in $\mathcal{R}(M)$, then the operator $P = U + W$ is a projection and there exists a projection $Q : Z \to Z$ with $\mathcal{R}(Q) \subset \mathcal{R}(E)$ such that equation (4) is equivalent to the equations,

$$\begin{aligned} &\text{(a)} \quad 0 = (I - Q)\, Nx, \\ &\text{(b)} \quad x = Px + MQNx. \end{aligned} \tag{6}$$

For $W = 0$, Eqs. (6) reduce to Eqs. (5) and for $W \neq 0$, $\mathcal{N}(A) = \mathcal{R}(U)$ is a proper subspace of $\mathcal{R}(P)$. The relation $\mathcal{R}(Q) \subset \mathcal{R}(E)$ implies that the bound of MQN and its local Lipschitz constant cannot be any larger than the corresponding constants of MEN. Therefore, for an appropriate choice of W one might be able to apply the contraction principle to obtain a solution $x^*(x_0)$ of the equation $x = x_0 + MQNx$ for $x_0 \in \mathcal{R}(P)$ fixed in a bounded set. This function $x^*(x_0)$ satisfies $Px^*(x_0) = x_0$ and is a solution of (4) if $(I - Q)\, Nx^*(x_0) = 0$. This can be considered as an alternative problem for (4) and is an equation for x_0.

To show that (6) is equivalent to (4) one uses only the fact that A, M, P, Q satisfy certain relations; namely, $MQA = I - P$ (Lemma 4) and $(I - Q)\, A = AP$ (Lemma 5). Once this observation is made, one can take these as postulates and obtain a generalization of the method of Cesari [4] which is given in Section 3. Section 4 gives an indication of how to apply index theory to determine when the alternative problem has a solution. Section 5.1 is devoted to the explicit calculation of the operator E in (5) for linear boundary value problems. In Section 5.2 a procedure is given for constructing the subspace W and thus the operatoıs P, Q that occur in (6).

2. Alternative Problems for a Special Case

If X, Z are Banach spaces and B is an operator which takes a subset of X into Z, we shall let $\mathscr{D}(B) \subset X$, $\mathscr{R}(B) \subset Z$ denote the domain and range respectively of B. If E is a projection operator defined in a Banach space Z, we shall denote $\mathscr{R}(E)$ as Z_E and the symbol Z_E shall always denote a subspace of Z which is obtained through a projection operator E in this way. The symbol I will denote the identity operator. If $Z = Z_E \oplus Z_{I-E}$, where Z_E, Z_{I-E} are closed subspaces of Z, we will say (Z_E, Z_{I-E}) splits Z or simply Z_E splits Z, or the projection E splits Z. Also, E^c denotes $I - E$.

Let X, Z be Banach spaces; let $N : X \to Z$ be an operator which may be linear or nonlinear; let $A : \mathscr{D}(A) \subset X \to Z$ be a linear operator which may have a nontrivial null space and may have $\mathscr{R}(A)$ deficient in Z; and let $F = A - N$.

Lemma 1. *If A has a bounded right inverse M and $\mathscr{R}(A)$ admits projection by E, then x^* is a solution of $Fx = 0$ if and only if x^* satisfies $Ax - AMENx = 0$, $E^cNx = 0$.*

Proof. If x^* satisfies $Fx = 0$, then $Ax^* - Nx^* = 0$, which implies $Nx^* \in \mathscr{R}(A)$. Since AME is the identity on $\mathscr{R}(A)$ this last statement may be read $Ax^* - AMENx^* = 0$, $E^cNx^* = 0$. Since the argument is reversible, this proves the lemma.

Lemma 2. *If N is continuous in a neighborhood of zero, $N = 0$ and there exists a function $\eta(\rho)$, $\rho \geqslant 0$, continuous at $\rho = 0$, such that $\eta(0) = 0$ and*

$$\| MENx_1 - MENx_2 \| \leqslant \eta(\rho) \| x_1 - x_2 \|,$$

for $\| x_1 \|, \| x_2 \| \leqslant \rho$, then there exists a function $x^* = x^*(x_0)$ continuous in a neighborhood of the origin in $X \times \mathscr{N}(A)$ such that

$$x^*(x_0) - x_0 - MENx^*(x_0) = 0;$$

that is, $x^*(x_0)$ satisfies $Ax - AMENx = 0$.

Proof. This is an obvious application of the contraction principle.

Combining Lemmas 1 and 2, we obtain:

THEOREM 1. (An Alternative Problem). *Suppose the conditions of Lemmas 1 and 2 are satisfied and $x^*(x_0)$ is the function defined in Lemma 2. The Eq. $Fx = 0$ has a solution if and only if there exist an $x_0 \in \mathcal{N}(A)$ such that*

$$E^c N x^*(x_0) = 0. \tag{7}$$

As we have remarked in the introduction, the Lipschitz constant $\eta(\rho)$ in Lemma 2 may depend upon another parameter, say y, and can be made small enough to apply the contraction principle by taking y small. In any case, an important step in obtaining the above alternative problem of Theorem 1 is the smallness of the Lipschitz constant of MEN. Since we have no control on the norm of M and E, smallness must come from N and this forces us to deal with small nonlinearities. To rid ourselves of this restriction we will have to look closer at our operators M and E and see what generalizations can be made to control the Lipschitz norm of MEN other than from the nonlinearity. In these problems, A is of course specified and thus M is specified "modulo" $\mathcal{N}(A)$. Now suppose there is a projection Q in Z_1 such that $Z_Q \subset Z_E = \mathcal{R}(A)$. In general $\| MQ \| \leqslant \| M\varphi \|$. If we knew that the equation $Ax = Nx$ was equivalent to $Q^c Nx = 0$, $x = x_0 + MQNx$ for x_0 in some subspace of X and $\| MQ \| < \| ME \|$, then the Lipschitz constant and bound of MQN are less than the corresponding constants for MEN. In fact, if we can find a sequence of such projections Q_n where $Z_{Qn} \subset \mathcal{R}(A)$ and $\| MQ_n \| \to 0$ $(n \to \infty)$ we will be able to deal with arbitrarily large non-linearities. Now replacing E by Q in our theory means our new alternative problem will have the form $Q^c Nx^*(x_0) = 0$.

Of course, the new alternative problem will, in general, be a function of x_0 from a subspace of X which properly contains $\mathcal{N}(A)$. To apply this type of procedure, one will therefore be forced to enlarge the space in which x_0 is allowed to vary. If Q is to replace E, then $\mathcal{N}(A)$ must be replaced by a space that is a function of the chosen Q.

We now provide a systematic decomposition of $Z_E = \mathcal{R}(A)$ which will induce such a projection Q. In what follows we assume that $\mathcal{N}(A) = X_U$ and $\mathcal{R}(A) = Z_E$ split X and Z respectively (U was not required for Theorem 1).

The bounded right inverse M of A can be assumed to satisfy $UM = 0$ since, if not, we could replace M by $(I - U) M$. This simply removes the "degeneracy" mentioned in the last remark. This allows us to write $x = MAx + Ux$ for any $x \in \mathcal{D}(A)$. In fact, if x belongs to $\mathcal{D}(A)$, then $Ax = z$ belongs to $\mathcal{R}(A)$ which implies $x = Mz + x_0$ for x_0 in $\mathcal{N}(A)$. But the condition $UM = 0$ implies $x_0 = Ux$.

LEMMA 3. *Let X_W be any subspace in $\mathscr{R}(M)$ which splits X. The following conclusions are then valid:*

(a) $UW = WU = 0$.

(b) $P = U + W$ *is a projection in* X.

(c) *The preimage in Z_E under M of X_W and X_{I-W} splits Z_E inducing a projection $J : Z_E \to Z_{E_J}$, $X_W = M(Z_{E_J})$, and $Q = (I - J)E$.*

(d) $Q = (I - J)E$ *is a projection and* $X_{p^c} = M(Z_Q)$.

Proof. (a) We observe that $UW = 0$ since $\mathscr{R}(W) \subset \mathscr{R}(M)$ and $UM = 0$ by hypothesis. Also U, W projections imply $UW = 0$ is equivalent to $X_W \subset X_{I-U}$ and this is equivalent to $X_U \subset X_{I-W}$, which is equivalent to $WU = 0$. Assertion (b) follows immediately from (a). (c) The continuity and linearity of M imply the existence of a projection J splitting Z_E and we may choose it so that $X_W = M(Z_{E_J})$. (d) By direct computations one shows that $Q = (I - J)E$ is a projection operator. It is clear that $X_{p^c} = M(Z_Q)$ since $X_W = M(Z_{E_J})$.

The accompanying figure may be helpful in visualizing the above construction.

LEMMA 4. *If P, Q are the projection operators defined in Lemma 3, then*

$$x = MQAx + Px, \tag{8}$$

for any $x \in \mathscr{D}(A)$.

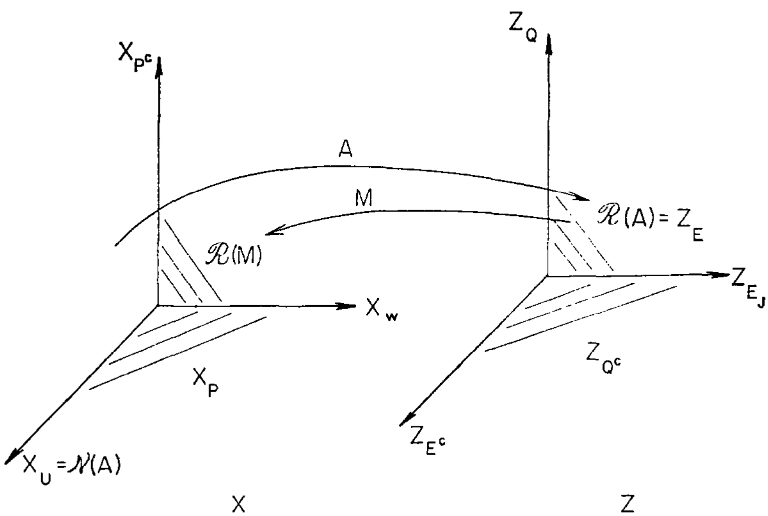

FIG. 1

Proof. Since $x = MAx + Ux$ for x in $\mathscr{D}(A)$ and Q is a projection, we can write

$$x = MQAx + M(I - Q)\,Ax + Ux.$$

From the preceding lemma, $MQAx \in X_{I-W}$ since $X_{p^c} \subset X_{W^c}$, $M(I-Q)Ax \in X_W$ since $(I - Q)\,Z_E = Z_{E_J}$. Operating on this latter relation with W and using the fact from Lemma 3 that $WU = 0$, we obtain $Wx = WM(I - Q)\,Ax = M(I - Q)\,Ax$. This proves the lemma.

LEMMA 5. *If P, Q are the projection operators defined in Lemma 3, then $(I - Q)\,A = AP$ for any $x \in \mathscr{D}(A)$.*

Proof. Applying A to (8) and using the fact that M is a right inverse of A, we obtain $Ax = QAx + APx$ which is the conclusion of the lemma.

With the aid of these three lemmas, we can state the following:

THEOREM 2. *Suppose A has a bounded right inverse M and $\mathscr{R}(A)$, $\mathscr{N}(A)$ admit projections. Suppose the subspace X_W in $\mathscr{R}(M)$ splits X, and let the projection operators P, Q be defined by Lemma 3. Then the equation $Ax - Nx = 0$ has a solution if and only if*

$$\begin{aligned} x &= Px + MQNx \\ Q^c(Ax - N)\,x &= 0. \end{aligned} \tag{9}$$

Proof. Let P, Q be the operators given in Lemma 3. If $F = A - N$, then $Fx = 0$ is equivalent to $QFx = 0, Q^cFx = 0$. But $QFx = 0$ is equivalent to $QAx - QNx = 0$. If $QAx - QNx = 0$, then formula (8) of Lemma 4 implies $MQNx = MQAx = (I - P)\,x$. Thus, $x = Px + MQNx$. Conversely, if $(I - P)\,x = MQNx$ then $A(I - P)\,x = AMQNx = QNx$. But, this relation and Lemma 5 imply $QAx = QNx$ or $QFx = 0$. Since we are assuming $Q^cFx = 0$, we have proved the theorem.

The operators P, Q in (9) are constructed according to Lemma 3. To solve Eqs. (9), we can proceed as before. More specifically, we can assume that x_0 in X_P is fixed and first solve the equation $x = x_0 + MQNx$ for $x = x^*(x_0)$ as a function of x_0. One can attempt to do this by successive approximations making smallness hypotheses on MQN as was made on MEN in Lemma 2. We reemphasize that more freedom is now available because of the arbitrariness in the choice of the operator Q. The alternative problem becomes now the equation $Q^c(A - N)\,x^*(x_0) = 0$. We do not dwell on this case further but proceed to a more general discussion.

3. The Alternative Problem for a General Case

In this section, we discuss the alternative problem in a more general setting. The idea is simply to abstract the properties obtained in the previous section by the special decomposition. Let X, Z be Banach spaces; let $N : \mathscr{D}(N) \subset X \to Z$ be an operator which may be linear or nonlinear; let $A : \mathscr{D}(A) \subset X \to Z$ be a linear operator, and let $F = A - N$. We make the following hypotheses:

H_1 : There exist projection operators P, Q splitting X, Z respectively, such that $(I - Q) A = AP$.

H_2 : There exists a linear map M such that $M_{Z_Q} \to X_{(I-P)}$ and satisfies

(i) $MQAx = (I - P) x, \qquad x \in \mathscr{D}(A)$

(ii) $AMQNx = QNx, \qquad x \in \mathscr{D}(A)$

H_3 : All fixed points of $T = P + MQN$ belong to $\mathscr{D}(A)$.

Remark 3. For the operators A, N satisfying the conditions of the previous section, it was demonstrated that the hypotheses H_1–H_3 are nonvacuous. In fact, it was shown that there are many operators P, Q which satisfy H_1 since H_1 is just Lemma 5 and the operators P, Q defined by Lemma 3 depend upon the choice of a rather arbitrary subspace of X. Hypothesis H_2(ii) corresponds to the existence of a right inverse for A. Hypothesis H_2(i) is equation (8) in Lemma 4. Hypothesis H_3 was automatically satisfied for the particular P, Q constructed and the bounded right inverse considered. If A, N are as specified above and M, P, Q exist so that H_1–H_3 are satisfied, it is not true that P, Q can be obtained by the construction in the previous section. In fact, it was assumed there that A had a bounded right inverse and $\mathscr{R}(A), \mathscr{N}(A)$ admitted projections. There is no way to deduce these properties from H_1–H_3. We note that H_1–H_3 for $P = Q^c$ are the axioms that Cesari [*4*] uses in his treatment of this problem. Quite obviously then, the advantage of the preceding discussion lies in its purely natural geometric development.

Lemma 6. *If* H_1, H_2, H_3 *are satisfied, then the equation* $Fx = 0$ *has a solution* $x \in \mathscr{D}(A) \cap \mathscr{D}(N)$ *if and only if* $x = Tx$, $Q^cFx = 0$.

Proof. $Fx = 0$ implies $QFx = 0, Q^cFx = 0$. Therefore $QAx - QNx = 0$ and H_2(i) implies $MQAx = MQNx = (I - P) x$ or $x = Px - MQNx = Tx$. Conversely, $x = Tx$ implies $(I - P) x = MQNx$. H_3 implies $x \in \mathscr{D}(A)$ and H_2(ii) implies $A(I - P) x = AMQNx = QNx$. But this fact together with H_1 implies $QFx = 0$. By hypothesis, $Q^cFx = 0$ and the lemma is proved.

We now introduce a fourth hypothesis to take the place of the smallness conditions on MEN in Lemma 2.

H_4: There exist a constant μ and a continuous nondecreasing function $\alpha(\rho)$, $0 \leqslant \rho < \infty$, such that

$$\| MQNx_1 - MQNx_2 \| \leqslant \alpha(\rho) \| x_1 - x_2 \|,$$

$$\| MQNx_1 \| \leqslant \alpha(\rho) \| x_1 \| + \mu \qquad \text{for} \quad \| x_1 \|, \| x_2 \| < \rho.$$

For the next lemma we need two definitions. For any positive constants c, d with $c < d$ let

$$V(c) = \{x \in X_P : \| x \| \leqslant c\}$$

$$S(\tilde{x}, c, d) = \{x \in X : Px = \tilde{x}, \tilde{x} \in V(c), \| x \| \leqslant d\}.$$

LEMMA 7. *If hypothesis* H_4 *is satisfied and* $\alpha(d) < 1$, $\alpha(d)\, d < d - c - \mu$, *then there exists a unique continuous function* $G : V(c) \to X$, $G\tilde{x} \in S(\tilde{x}, c, d)$ *such that* $TG\tilde{x} = G\tilde{x}$ *for each* $\tilde{x}$ *in* V.

Proof. For any $\tilde{x} \in V$ and $x \in X$ let $H(x, \tilde{x}) = \tilde{x} + MQNx$. It is easy to see that the hypotheses on $\alpha(d)$, d and c imply $H(\cdot, \tilde{x}) : S(\tilde{x}, c, d) \to S(\tilde{x}, c, d)$ and is a contraction. Therefore, there is a unique fixed point of $H(\cdot, \tilde{x})$ in $S(\tilde{x}, c, d)$ and it is obviously continuous in $\tilde{x}$. Since $\tilde{x} = P\tilde{x}$, this fixed point is also a fixed point of T. This proves the lemma. (Cf. Lemma 2).

THEOREM 3. *Suppose* H_1–H_4 *are satisfied,* $\alpha(d)$, d *and* c *satisfy the conditions of Lemma 7 and* G *is the function in that lemma. The equation* $Fx = 0$ *has a solution if there exists an* $\tilde{x}$ *in* V *such that*

$$Q^c FG\tilde{x} = 0. \tag{10}$$

Conversely, if there exists an x *such that* $Fx = 0$, $\| x \| < d$, $\| Px \| < c$ *then* $x = GPx$ *and* $\tilde{x} = Px$ *is a solution of* (10).

Proof. The first part of the theorem follows directly from Lemma 7 and the sufficiency of Lemma 6. For the second part, the necessity of Lemma 6 implies that the equation $x = Tx$ has a unique solution $G\tilde{x}$. Therefore, $x = FPx$ and $\tilde{x} = Px$ must also satisfy (10).

It should be emphasized that our continuity and smallness hypotheses are only on the product MQN. This permits certain simple problems in partial differential equations to be discussed [*6*], [*12*].

4. A Topological Method for Solving the Determining Equations

In this section, we give a procedure for finding an $\tilde{x} \in V(c)$ such that $Q^cFG\tilde{x} = 0$. The idea is in the paper of Cesari [4] and our results are a slight generalization since the spaces are not assumed to be separable Hilbert spaces.

Let $\dim(X_P) = m$, $\dim(Z_{Q^c}) = n$ and let E_m, E_n be two auxiliary Euclidean spaces. Let $\{\varphi_1, ..., \varphi_m\}$ be a unit basis for X_P, and $\varphi_1^*, ..., \varphi_n^*$ a unit basis for Z_{Q^c}. Define $v \in E_m$ as $v = (v_1, ..., v_m)$, $| v | = \sum_{i=1}^m | v_i |$ and similarly $u \in E_n$ as $u = (u_1, ..., u_n)$, $| u | = \sum_{i=1}^n | u_i |$. We define the following mappings:

(i) $\beta : E_n \to Z_{Q^c}$, $\beta(u) = \sum_{i=1}^n u_i\varphi_i^*$. Clearly β is an isomorphism.

(ii) $\phi : E_m \to X_P$, $\phi(v) = \sum_{i=1}^m v_i\varphi_i$. The map ϕ is an isomorphism. If $\Gamma = \phi^{-1}V(c)$, then we may assume Γ is the unit ball simply by introducing a different equivalent norm on the finite dimensional space E_m.

(iii) $\psi : \mathscr{D}(A) \cap \mathscr{D}(N) \to E_n$, $\psi = \beta^{-1}Q^cF$. If N is continuous, then ψ is continuous. In fact, this will be the case if Q^cA is continuous. But $Q^cA = AP$ and AP linear with $APX = APX_P$ finite dimensional imply AP continuous.

(iv) $K : \Gamma \to E_n$, $K = \psi\phi$; $\mathscr{K} = \Gamma \to E_n$, $\mathscr{K} = \psi G\phi$.

Explanation of K and $\mathscr{K}$

K takes a point in $V(c)$, puts it into the function $F = A - N$ and looks at its projection in Z_Qc.

$\mathscr{K}$ takes a point in $V(c)$ into the fixed point of T defined by G in Lemma 7, puts it into the function F and looks at its projection in Z_Qc. Therefore, equivalent to solving our problem is finding a $\bar{v} \in \Gamma$ such that $\mathscr{K}(\bar{v}) = 0$.

Let μ, μ_0 be the topological degrees of $\mathscr{K}$, K, respectively, with respect to the origin O of E_m.

If $\partial\Gamma$ denotes the boundary of Γ, then $C_0 = K |_{\partial\Gamma} : \partial\Gamma \to E_m$, $C = \mathscr{K} |_{\partial\Gamma} : \partial\Gamma \to E_m$ are singular $(m - 1)$-cycles whose topological order with respect to the origin of E_m are still μ_0, μ. Define

$$\| C, C_0 \| = \inf_{A, A_0} \max_{v \in \partial\Gamma} \| A(v) - A_0(v) \|$$

where A, A_0 are all possible parameterizations of C_0 and C. Letting $L = \operatorname{dist}(0, C_0)$ and $\Lambda = \max_{v \in \partial\Gamma} \| K(v) - \mathscr{K}(v) \|$, we know $\| C, C_0 \| < L$ implies $\mu = \mu_0$ and also $\| C, C_0 \| \leqslant \Lambda$. Therefore, we have the following:

THEOREM 4. *Under all of the above hypotheses, $\mu_0 \neq 0$ and $\Lambda < L$ imply there exists a $\bar{v} \in \Gamma$ such that $\mathscr{K}(\bar{v}) = 0$.*

Proof. The hypotheses tell us $\mu \neq 0$ and the result is immediate.

5. APPLICATIONS

5.1. *Linear Boundary Value Problems*

We now show how continuous linear side conditions can lead to an abstract alternative problem. Let X, Z be Banach spaces. Let $G : X \to Z$ be given by $Gx = Bx - Nx$ with N satisfying the conditions of Lemma 2. Suppose $\mathscr{R}(B) = Z$ and B has a bounded right inverse M. We can now apply Theorem 1 to the function G with $E = I$ since $\mathscr{R}(B) = Z$. Thus, the function $x^*(x_0)$, x_0 in $\mathscr{N}(B)$, given in Lemma 2, is a solution of $Gx = 0$. Now impose side conditions in the form $\Gamma x = 0$ where Γ is a continuous linear map $\Gamma : X \to V$, a B-space. Let $\tilde{X} = \mathscr{N}(\Gamma), F = G|_{\tilde{X}}, A = B|_{\tilde{X}}$. If $F\tilde{x} = A\tilde{x} - N\tilde{x}$, then solving $Gx = 0$, subject to $\Gamma x = 0$, is equivalent to solving $F\tilde{x} = 0$. Since $A = B|_{\tilde{X}}$ we will in general expect $\mathscr{R}(A)$ to be a proper subspace in Z. If $\mathscr{R}(A) = Z_E$, where E is a projection, then the alternative problem $E^c F\tilde{x}^*(\tilde{x}_0) = 0$ is equivalent to $F\tilde{x} = 0$. Here $\tilde{x}^*(\tilde{x}_0)$, $\tilde{x}_0$ in $\mathscr{N}(A)$, is the function described in Lemma 2.

Our problem is to give an explicit characterization of $\mathscr{R}(A)$ and the corresponding projection operator E. Once E is obtained Lemma 3 can be used to obtain other alternative problems. We shall give two methods for computing E, the first being essentially due to Antosiewicz [*20*], although we state a stronger result, and the second makes use of the familiar Fredholm alternative $\mathscr{R}(A) = {}^{\perp}\mathscr{N}(A^*)$.

LEMMA 9. *Let X, Z, V be Banach spaces; let $B : X \to Z$ be a linear map with a bounded right inverse $M : Z \to X$; let $\Gamma : X \to V$ be a bounded linear map and define $X_0 = \mathscr{N}(B)$, $V_0 = \Gamma(X_0)$, $\tilde{X} = \mathscr{N}(\Gamma)$, $A = B|_{\tilde{X}}$. Then $\mathscr{R}(A) = (\Gamma M)^{-1}(V_0)$.*

Proof. If $z \in \mathscr{R}(A)$, there is an x in $\mathscr{D}(A)$ such that $Ax = z$. Since $\mathscr{D}(A) \subset \mathscr{D}(B)$, $Bx = z$ and so $x = Mz + x_0$, $x_0 \in \mathscr{N}(B)$. Therefore, $\Gamma Mz = \Gamma(-x_0)$ and, thus, $\mathscr{R}(A) \subset (\Gamma M)^{-1}(V_0)$. The argument is clearly reversible.

If V_0 is closed in V then $\mathscr{R}(A)$ is closed. This follows from the continuity of ΓM and the lemma. Certainly this is a necessary condition for $\mathscr{R}(A)$ to admit projection.

Now assume $\mathscr{R}(A)$ does admit a projection E and let $\nu = I - E$; $\Phi = \Gamma M$, $V = V_0 \oplus V_0^c$, $\tau : V \to V_0^c$, a projection. We know $\mathscr{R}(A) = \nu^{-1}(0)$ and, by Lemma 9, $z \in \mathscr{R}(A)$ if and only if $\Phi z \in V_0$, and $\Phi z \in V_0$ is equivalent to $\tau\Phi z = 0$. So we have $\nu z = 0$ if and only if $\tau\Phi z = 0$. Therefore writing $\nu z = (\Psi\tau\Phi) z$, we wish to determine a continuous function Ψ such that $\nu^2 = \nu$. This latter relation is equivalent to $\Psi\tau\Phi\Psi\tau\Phi = \Psi\tau\Phi$ which will be satisfied if Ψ is a right inverse of Φ on $\mathscr{R}(\tau\Phi)$ since $\tau^2 = \tau$. We will have $\mathscr{D}(\Psi) \supset \mathscr{R}(\tau\Phi)$ if $\mathscr{R}(\tau\Phi) \subset \mathscr{R}(\Phi)$. This is true if $V_0{}^c \subset \mathscr{R}(\Phi)$ or $V_0 \subset \mathscr{R}(\Phi)$. We quote a Lemma to establish the existence of the right inverse Ψ for Φ. The proof may be found in [*18*, *Chap. 6*, *Lemma 4*].

LEMMA 10. *If Φ is a closed linear operator, $\mathscr{N}(\Phi)$ admits a projection, and $\mathscr{R}(\Phi)$ is closed, then Φ has a bounded right inverse Ψ defined in $\mathscr{R}(\Phi)$.*

This lemma and the above discussion enable us to obtain sufficient conditions for the existence of and a representation theorem for the projection E onto $\mathscr{R}(A)$. We state this result as:

THEOREM 5. *If the notation is the same as in Lemma 9 and if*

T_1 : *$(V_0, V_0{}^c)$ splits V, defining a projection $\tau : V \to V_0{}^c$,*

T_2 : *$\mathscr{N}(\Gamma M)$ admits a projection and $\mathscr{R}(\Gamma M)$ is closed,*

T_3 : *either V_0 or $V_0{}^c$ is a subset of $\mathscr{R}(\Gamma M)$,*

then $\mathscr{R}(A)$ admits a projection. This projection operator E is defined by $E = I - \Psi\tau\Phi$, where $\Phi = \Gamma M$ and Ψ is a bounded inverse for Φ.

THEOREM 6. *Suppose T_1–T_3 are satisfied and $\mathscr{N}(A^*)$ of the adjoint operator A^* for A is finite dimensional. Choose $\{z_j\}, \{z_j^*\}$ such that $\{z_j^*\}$ is a basis for $\mathscr{N}(A^*)$; $\langle z_j, z_k^*\rangle = 0$ for $j \neq k$ and $= 1$ for $j = k$. Then E can be chosen as $E = I - \sum \langle \cdot, z_j^*\rangle z_j$.*

The proof of this theorem is easily seen by remembering that $\mathscr{R}(A) = {}^{\perp}\mathscr{N}(A^*)$.

Remark. If $\bar{A}^*$ is a formal adjoint, the restriction of A^* to some weak*-dense manifold $N \subset \mathscr{D}(A^*)$, then the theorem applies using $\bar{A}^*$ throughout since ${}^{\perp}\mathscr{N}(\bar{A}^*) = {}^{\perp}\mathscr{N}(A^*)$.

EXAMPLE. Consider the n-vector problem

$$\dot{x}(t) = L(t)\, x + f(t, x), \qquad t \in I = [0, \omega] \tag{11}$$

$$x(0) = x(\omega). \tag{12}$$

For this particular case, the various operators and spaces used in our abstract formulation of the problem become $X = Z = C(R^n, I)$, $V = R^n$, $B = d/dt - L$, $\Gamma x = x(0) - x(\omega)$, $N = f(\cdot, x)$. The space $\tilde{X}$ is $\mathscr{N}(\Gamma)$ which are the T-periodic continuous functions and $A = B\,|_{\tilde{X}}$.

Problem (11), (12) is equivalent to

$$Gx = 0 \tag{11'}$$

$$\Gamma x = 0, \tag{12'}$$

where $G : X \to Z$; $Gx = Bx - Nx$. If H is the principal matrix solution for $\dot{x} = L(t)\,x$, then H spans $X_0 = \mathcal{N}(B)$. The operator B has a bounded right inverse M provided by the variation of constants formula. Indeed,

$$Mz(t) = H(t) \int_0^t H^{-1}(s)\, z(s)\, ds.$$

We note that $\mathcal{R}(B) = Z$ and now show that $\mathcal{R}(A)$ admits projection. According to Theorem 4 and the hypotheses T_1–T_3, this follows if either $V_0 = \Gamma(X_0)$ or $V_0{}^c$ is a subset of $\mathcal{R}(\Gamma M)$. We show that $\mathcal{R}(\Gamma M) = V$ so that this is certainly true. Indeed

$$\mathcal{R}(\Gamma M) \supset \text{span}\, H(\omega) \int_0^T H^{-1}(s)\, H(s)\, ds = \text{span}\, H(\omega) = R^n = V.$$

We now construct E according to Theorem 5. We have

$$\Phi = \Gamma M = H(\omega) \int_0^\omega H^{-1}(s)(\cdot)\, ds$$

and, by inspection, $\Psi : R^n \to Z$ is given by $(\Psi a)(t) = H(t)\, H^{-1}(\omega)\, a/T$. It remains to find a representation for τ, the projection of V onto $V_0{}^c$. We know that $V_0 = \Gamma(X_0) = \text{span}[I - H(\omega)]$ and $R^n = \mathcal{N}[I - H^*(\omega)] \oplus \mathcal{R}[I - H(\omega)]$ If $\alpha = (\alpha_1, \ldots, \alpha_p)$ is an orthonormal basis for $\mathcal{N}[I - H^*(\omega)]$, $\gamma = (\gamma_{p+1}, \ldots, \gamma_n)$ is an orthonormal basis for $\mathcal{R}[I - H(\omega)]$, then $\tau = \text{diag}(I_p, 0)$, relative to the coordinate system (α, γ). By standard linear algebra $\tau = (\alpha, \gamma)\, \text{diag}(I_p, 0)(\alpha, \gamma)'$ relative to the basis used to express (α, γ) where we use the symbol $'$ for transpose. Thus $\tau = \alpha\alpha'$. From Theorem 5, $E = I - \Psi\tau\Phi$, which written out in full is

$$E = I - H(t)\, H^{-1}(\omega)\, \alpha\alpha' H(\omega) \int_0^\omega H^{-1}(s)(\cdot)\, ds/\omega.$$

This problem can also be approached by using the remark following Theorem 6. Let $\bar{A}^* z = \dot{z}L + \ddot{z}$ for all z in $\mathcal{D}(\bar{A}^*) = \{z^*$ which are of bounded variation and in $C^2(R^n, I)$, $\dot{z}^*(0) = \dot{z}^*(\omega)\}$. It is clear that $\mathcal{D}(\bar{A}^*)$ is weak*-dense in Z^*. Direct calculation (integration by parts) shows that $A^* z = \bar{A}^* z$ for z in $\mathcal{D}(\bar{A}^*)$. The homogeneous adjoint equation is $\dot{y} = -yL$, where $y = \dot{z}^*$. Thus $z^* \in \mathcal{N}(\bar{A}^*)$ implies there exists a^* such that

$a^*[H^{-1}(\omega) - I] = 0$. If β is a basis for the solutions of this equation, then $K = \beta H^{-1}$ is a basis for $\mathscr{N}(\bar{A}^*)$. Let $z_j^* = \beta_j H^{-1}$ be the jth row of K, z_k = kth column of $K' (\int_0^\omega KK')^{-1}$. Direct calculation shows

$$\left\langle K, K' \left(\int_0^\omega KK'\right)^{-1} \right\rangle = I$$

and therefore $\langle z_j^*, z_k \rangle$ is the Kronecker delta. The operator E can then be chosen as

$$E = I - K' \left(\int_0^\omega KK'\right)^{-1} \int_0^\omega K(s)(\cdot)\, ds. \tag{13}$$

If one returns to the theory in Section 3 to obtain the alternative problem (bifurcation equations) in Theorem 1, one observes this is the same as given in Lewis [*14*].

5.2. *Large Nonlinearities*

We wish to show the above theory enables us to deal with large nonlinearities. In particular, we show how a certain type of problem leads in a natural way to a decomposition of the space which yields the projection operators of Lemma 3 and the corresponding contraction operators in Cesari's papers [*4*, *5*]. We consider the following problem. Let $(X, \| \ \|) = (C([a, b], R^n), \| \ \|_\infty)$, $\Gamma : \mathscr{D}(\Gamma) \subset X \to R^n$, a bounded linear operator and let $S = \mathscr{N}(\Gamma)$, a Banach space. Given $A : \mathscr{D}(A) \subset S \to S$, a linear operator and $N : S \to S$, a continuous nonlinear operator, we search for an $x \in S$ such that $(A - N)x = 0$. In this particular problem we make the assumption that the associated linear problem $Ax = -\lambda x$, $x \in S$ has a complete set of unit eigenfunctions $\{\varphi_n\}$ with corresponding eigenvalues $|\lambda_1| \leqslant |\lambda_2| \leqslant \cdots \leqslant |\lambda_m| \leqslant \cdots$ where $|\lambda_m| \neq 0$ for some m. Therefore, we know that for any $x \in S$, $x \sim \sum C_j \varphi_j$ where $C_i = \int_a^b \varphi_i(t)\, x(t)\, dt$.

Since

$$\mathscr{R}(A) = \Big\{\varphi : \varphi \sim \sum_{\lambda_j \neq 0} C_j \varphi_j\Big\}, \qquad \mathscr{N}(A) = \Big\{\varphi : \varphi \sim \sum_{\lambda_j = 0} C_j \varphi_j\Big\},$$

the projection operators E, U onto $\mathscr{R}(A), \mathscr{N}(A)$, respectively, are given by

$$E\varphi \sim \sum_{\lambda_j \neq 0} C_j \varphi_j, \qquad U\varphi \sim \sum_{\lambda_j = 0} C_j \varphi_j.$$

Therefore $E = I - U$. If we define $M\varphi - \sum_{\lambda_j \neq 0} C_j \lambda_j^{-1} \varphi_j$ on $\mathscr{R}(A)$, M serves as a bounded right inverse for A. For a given integer n, we now choose

$$X_W = \left\{ \bar{\varphi} = W_n\varphi \sim \sum_{i=1,\lambda_i \neq 0}^{n} C_i\varphi_i \right\} \subset M\mathscr{R}(A).$$

Therefore we have

$$P^n\varphi = U\varphi + W^n\varphi \sim \sum_{i=1}^{n} C_i\varphi_i + \sum_{j=n+1}^{\infty} C_j\varphi_j .$$

Then $J_n\varphi = W_n\varphi$ or $(I - J_n) = (I - W_n) = (I - P_n + U)$ and

$$Q_n = (I - J_n)\, E = (I - P_n + U)(I - U) = (I - P_n).$$

Therefore if we only consider $n \geqslant m$, we have $Q_n{}^c\varphi = P_n\varphi = \sum_{i=1}^{n} C_i\varphi_i$.

Now suppose $\tilde{x}$ is a given element of X with $P_n\tilde{x} = \tilde{x}$. From the preceding theory, it follows that each fixed point $x^* = x^*(\tilde{x})$ with $P_nx^* = \tilde{x}$ of $T_n = \tilde{x} + MQ_nN$ lies in $\mathscr{D}(A)$ and solves our problem if and only if $Q_n{}^c(A - N)\, x = 0$.

Suppose $\bar{\mu}, k_n$ are constants and $\bar{\alpha}(\rho)$ is a continuous nondecreasing function on $[0, \infty)$ such that

$$\| MQ_nx \| \leqslant k_n \| x \| \qquad \text{for all} \quad x$$

and

$$\| Nx_1 \| \leqslant \bar{\alpha}(\rho)\| x_1 \| + \bar{\mu},$$
$$\| Nx_1 - Nx_2 \| \leqslant \bar{\alpha}(\rho)\| x_1 - x_2 \| \qquad \text{for} \quad \| x_1 \|, \| x_2 \| \leqslant \rho.$$

If $\| \tilde{x} \| \leqslant c$, then we can apply the contraction principle to the set

$$\{x \in X : \| x \| \leqslant d \leqslant \rho, P_nx = \tilde{x}\}$$

provided

$$k_n\bar{\alpha}(d) < 1,\ k_n\bar{\alpha}(d)\, d + k_n\bar{\mu} \leqslant d - c.$$

If $d > c$, then the smaller k_n, the larger the nonlinearities which may be considered. We now compute k_n.

Assume we have lower bounds $| \mu_i | \leqslant | \lambda_i |$ for our eigenvalues and that $\sum_{i=1}^{\infty} | \mu_i |^{-2}$ converges. Then we note that

$$| MQ_nx(t)| \leqslant \sum_{i=n+1}^{\infty} | C_i | \, | \mu_i |^{-1} \leqslant \left(\sum_{i=n+1}^{\infty} | \mu_i |^{-2} \right)^{1/2} \left(\sum_{i=1}^{\infty} | C_i |^2 \right)^{1/2}.$$

If for the moment we view S as embedded in the Hilbert space L_2, we see

$$\left(\sum_{i=1}^{\infty} | C_i |^2 \right)^{1/2} = \left(\int_a^b x^2(t)\, dt \right)^{1/2} \leqslant \| x \|_\infty \sqrt{b - a}.$$

Therefore, the $k_n \to 0$ as $n \to \infty$ and, thus, there exists an N_0 such that the above inequalities used in the contraction principle can be satisfied for $n \geqslant N_0$ regardless of the non-linearities. If $x_n^*(\tilde{x})$, $n \geqslant N_0$ is the unique fixed point of T_n, then $x_n^*(\tilde{x})$ is a solution of $Ax = Nx$ if and only if $\tilde{x}$ is a solution of $P_n(A - N)\, x_n^*(\tilde{x}) = 0$ since $Q_n{}^c = P_n$. For each $n \geqslant N_0$, this is a finite dimensional alternative problem for the equation $Ax = Nx$. This method of decomposition was given by Cesari [*4*, *5*]. A detailed treatment of the periodic case was done by Knobloch [*23*].

Added in proof. The paper of J. S. Locker to appear in Transactions of the American Mathematical Society was brought to our attention after this paper was in press. Locker also obtained the generalization of Cesari's method given in Section 3, but the motivation seems to arise in a more specific manner.

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