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The finiteness dimension of local cohomology modules and its dual notion

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ABSTRACT

Let $\mathfrak{a}$ be an ideal of a commutative Noetherian ring R and M a finitely generated R-module. We explore the behavior of the two notions $f_{\mathfrak{a}}(M)$, the finiteness dimension of M with respect to $\mathfrak{a}$, and, its dual notion $q_{\mathfrak{a}}(M)$, the Artinianness dimension of M with respect to $\mathfrak{a}$. When $(R, \mathfrak{m})$ is local and $r := f_{\mathfrak{a}}(M)$ is less than $f_{\mathfrak{a}}^{\mathfrak{m}}(M)$, the $\mathfrak{m}$-finiteness dimension of M relative to $\mathfrak{a}$, we prove that $H_{\mathfrak{a}}^r(M)$ is not Artinian, and so the filter depth of $\mathfrak{a}$ on M does not exceed $f_{\mathfrak{a}}(M)$. Also, we show that if M has finite dimension and $H_{\mathfrak{a}}^i(M)$ is Artinian for all $i > t$, where t is a given positive integer, then $H_{\mathfrak{a}}^t(M)/\mathfrak{a}H_{\mathfrak{a}}^t(M)$ is Artinian. This immediately implies that if $q := q_{\mathfrak{a}}(M) > 0$, then $H_{\mathfrak{a}}^q(M)$ is not finitely generated, and so $f_{\mathfrak{a}}(M) \leq q_{\mathfrak{a}}(M)$.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring with identity and all modules are assumed to be unitary. Let M be a finitely generated R-module and $\mathfrak{a}$ an ideal of R. The notion $f_{\mathfrak{a}}(M)$, the finiteness dimension of M relative to $\mathfrak{a}$, is defined to be the least integer i such that $H_{\mathfrak{a}}^i(M)$ is not finitely generated if there exist such i's and $+\infty$ otherwise. Here $H_{\mathfrak{a}}^i(M)$ denotes the ith local cohomology module of M with respect to $\mathfrak{a}$. As a general reference for local cohomology, we refer the reader to the text book [6]. Hartshorne [12] has defined the notion $q_{\mathfrak{a}}(R)$ as the greatest integer i such that $H_{\mathfrak{a}}^i(R)$ is not Artinian. Dibaei and Yassemi [11] extended this notion to arbitrary R-modules, to the effect that for any R-module N they defined $q_{\mathfrak{a}}(N)$ as the greatest integer i such that $H_{\mathfrak{a}}^i(N)$ is not Artinian. Among other things, they showed that if M and N are two finitely generated R-modules such that M is supported in $\operatorname{Supp}_R N$, then $q_{\mathfrak{a}}(M) \leq q_{\mathfrak{a}}(N)$.

Our objective in this paper is to investigate the notions $f_{\mathfrak{a}}(M)$ and $q_{\mathfrak{a}}(M)$ more closely. Let M and $\mathfrak{a}$ be as above. By [1, Theorem 1.2], the R-module $\operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{f_{\mathfrak{a}}(M)}(M))$ is finitely generated. This easily leads to the conclusion that $H_{\mathfrak{a}}^{f_{\mathfrak{a}}(M)}(M)$ has finitely many associated primes; see [4,18]. In Section 3, we investigate the dual statements for $H_{\mathfrak{a}}^{q_{\mathfrak{a}}(M)}(M)$. We prove that $H_{\mathfrak{a}}^{q_{\mathfrak{a}}(M)}(M)/\mathfrak{a}H_{\mathfrak{a}}^{q_{\mathfrak{a}}(M)}(M)$ is Artinian. However, we give an example to show that the set of coassociated prime ideals of $H_{\mathfrak{a}}^{q_{\mathfrak{a}}(M)}(M)$ might be infinite; see Example 3.5. As an immediate application, we deduce that if $q := q_{\mathfrak{a}}(M) > 0$, then $H_{\mathfrak{a}}^q(M)$ is not finitely generated. In particular, if $q_{\mathfrak{a}}(M) > 0$, then it follows that $f_{\mathfrak{a}}(M) \leq q_{\mathfrak{a}}(M)$. This leads one to conjecture that $H_{\mathfrak{a}}^{f_{\mathfrak{a}}(M)}(M)$ is not Artinian. As can be seen easily, this is not true in general (see Example 2.6(i)). But, in this regard, we prove that if M is a module over a local ring $(R, \mathfrak{m})$ such that either $r := \operatorname{grade}(\mathfrak{a}, M) < \operatorname{depth} M$ or $r := f_{\mathfrak{a}}(M) < f_{\mathfrak{a}}^{\mathfrak{m}}(M)$, then $H_{\mathfrak{a}}^r(M)$ is not Artinian. In particular, in both cases we can immediately conclude that $f_{\mathfrak{a}}(M) \geq f - \operatorname{depth}(\mathfrak{a}, M)$. (For the definitions of the notions $f_{\mathfrak{a}}^{\mathfrak{m}}(M)$ and $f - \operatorname{depth}(\mathfrak{a}, M)$, see the paragraph preceding Theorem 2.5.)

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To have the most generality, we will present our results for generalized local cohomology modules, the notion of which was introduced by Herzog [16] in 1974. For two R-modules M and N, the ith generalized local cohomology of M and N with respect to $\mathfrak{a}$ is defined by $H^i_{\mathfrak{a}}(M, N) := \varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N)$. Using the notion of generalized local cohomology modules, we can define the finiteness (resp. Artinianness) dimension of a pair (M, N) of finitely generated R-modules relative to the ideal $\mathfrak{a}$ by

$$f_{\mathfrak{a}}(M, N) := \inf\{i : H^i_{\mathfrak{a}}(M, N) \text{ is not finitely generated}\}$$

(resp.

$$q_{\mathfrak{a}}(M, N) := \sup\{i : H^i_{\mathfrak{a}}(M, N) \text{ is not Artinian}\}),$$

with the usual convention that the infimum (resp. supremum) of the empty set of integers is interpreted as $+\infty$ (resp. $-\infty$). We study the behavior of $f_{\mathfrak{a}}(M, N)$ and $q_{\mathfrak{a}}(M, N)$ under changing one of the $\mathfrak{a}$, M and N, when we fixed the two others. Correspondingly, in Section 2 we prove the following:

(i) if $\operatorname{Supp}_R M \subseteq \operatorname{Supp}_R N$, then $f_{\mathfrak{a}}(M, L) \geq f_{\mathfrak{a}}(N, L)$ for any finitely generated R-module L, and
(ii) if $\operatorname{pd} N < \infty$, then $f_{\mathfrak{a}}(M, N) \geq f_{\mathfrak{a}}(M, R) - \operatorname{pd} N$.

Finally, for any two finitely generated R-modules M and N, we establish the inequality

$$q_{\mathfrak{a}}(M, N) \leq \operatorname{Gpd}_N M + q_{\mathfrak{a}}(M \otimes_R N),$$

where $\operatorname{Gpd}_N M := \sup\{i : \operatorname{Ext}^i_R(M, N) \neq 0\}$.

2. The finiteness dimension of modules

The purpose of this section is to examine the behavior of the notion $f_{\mathfrak{a}}(M)$ more closely. Let us start this section by recording the following theorem.

Theorem 2.1. *Let L, M and N be finitely generated R-modules and $\mathfrak{a}$ an ideal of R. If $\operatorname{Supp}_R M \subseteq \operatorname{Supp}_R N$, then $f_{\mathfrak{a}}(M, L) \geq f_{\mathfrak{a}}(N, L)$. In particular, if $\operatorname{Supp}_R N = \operatorname{Supp}_R M$, then $f_{\mathfrak{a}}(M, L) = f_{\mathfrak{a}}(N, L)$.*

Proof. It is enough to show that $H^i_{\mathfrak{a}}(M, L)$ is finitely generated for all $i < f_{\mathfrak{a}}(N, L)$ and all finitely generated R-modules M such that $\operatorname{Supp}_R M \subseteq \operatorname{Supp}_R N$. To this end, we argue by induction on i. We have $H^0_a(M, L) \cong \operatorname{Hom}_R(M, \Gamma_{\mathfrak{a}}(L))$, and so the assertion is clear for $i = 0$. Now, assume that $i > 0$ and that the claim has been proved for $i - 1$. By Gruson's theorem (see e.g. [25, Theorem 4.1]), there is a chain

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\ell = M$$

of submodules of M such that each of the factors M_j/M_{j-1} is a homomorphic image of a direct sum of finitely many copies of N. In view of the long exact sequences of generalized local cohomology modules that are induced by the short exact sequences

$$0 \longrightarrow M_{j-1} \longrightarrow M_j \longrightarrow M_j/M_{j-1} \longrightarrow 0,$$

$j = 1, \ldots, \ell$, it suffices to treat only the case $\ell = 1$. So, we have an exact sequence

$$0 \longrightarrow K \longrightarrow \bigoplus_{i=1}^{t} N \longrightarrow M \longrightarrow 0,$$

where $t \in \mathbb{N}$ and K is a finitely generated R-module. This induces the long exact sequence

$$\cdots \longrightarrow H^{i-1}_{\mathfrak{a}}(K, L) \longrightarrow H^i_a(M, L) \longrightarrow H^i_{\mathfrak{a}}\left(\bigoplus_{i=1}^{t} N, L\right) \longrightarrow \cdots.$$

By the induction hypothesis, $H^{i-1}_{\mathfrak{a}}(K, L)$ is finitely generated. Also, $H^i_a(\oplus_{i=1}^t N, L) \cong \oplus_{i=1}^t H^i_a(N, L)$ is finitely generated, because $i < f_{\mathfrak{a}}(N, L)$. Hence $H^i_{\mathfrak{a}}(M, L)$ is finitely generated. □

Corollary 2.2. *Let $\mathfrak{a}$ be an ideal of R and L, M and N finitely generated R-modules.*

(i) *If $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is an exact sequence, then for any finitely generated R-module C, we have $f_{\mathfrak{a}}(M, C) = \inf\{f_{\mathfrak{a}}(L, C), f_{\mathfrak{a}}(N, C)\}$.*
(ii) $f_{\mathfrak{a}}(M) = \inf\{f_{\mathfrak{a}}(C, M) : C$ *is a finitely generated R-module*$\}$.
(iii) $f_{\mathfrak{a}}(M, L) = \inf\{f_{\mathfrak{a}}(R/\mathfrak{p}, L) : \mathfrak{p} \in \operatorname{Supp}_R M\}$.
(iv) *If $\operatorname{Supp}_R(\frac{M}{\Gamma_{\mathfrak{a}}(M)}) \subseteq \operatorname{Supp}_R(\frac{N}{\Gamma_{\mathfrak{a}}(N)})$, then $f_{\mathfrak{a}}(M, L) \geq f_{\mathfrak{a}}(N, L)$.*

Proof. (i) is clear.

(ii) Since $H^i_{\mathfrak{a}}(R, M) \cong H^i_{\mathfrak{a}}(M)$, it follows that $f_{\mathfrak{a}}(R, M) = f_{\mathfrak{a}}(M)$. Now, the claim is clear by Theorem 2.1.

(iii) Set $K := \oplus_{\mathfrak{p}\in \mathrm{Ass}_R M} R/\mathfrak{p}$. Then K is finitely generated and $\mathrm{Supp}_R K = \mathrm{Supp}_R M$. So, by (i) and Theorem 2.1, we deduce that

$$\begin{aligned} f_{\mathfrak{a}}(M, L) &= f_{\mathfrak{a}}(K, L) \\ &= \inf\{f_{\mathfrak{a}}(R/\mathfrak{p}, L) : \mathfrak{p} \in \mathrm{Ass}_R M\} \\ &= \inf\{f_{\mathfrak{a}}(R/\mathfrak{p}, L) : \mathfrak{p} \in \mathrm{Supp}_R M\}. \end{aligned}$$

(iv) [8, Lemma 2.11] implies that $H^i_{\mathfrak{a}}(C, L) \cong \mathrm{Ext}^i_R(C, L)$ for all $\mathfrak{a}$-torsion R-modules C and all i. So, the exact sequence $0 \longrightarrow \Gamma_{\mathfrak{a}}(M) \longrightarrow M \longrightarrow \frac{M}{\Gamma_{\mathfrak{a}}(M)} \longrightarrow 0$ implies the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^{i-1}_R(\Gamma_{\mathfrak{a}}(M), L) \longrightarrow H^i_{\mathfrak{a}}\left(\frac{M}{\Gamma_{\mathfrak{a}}(M)}, L\right) \longrightarrow H^i_{\mathfrak{a}}(M, L) \longrightarrow \mathrm{Ext}^i_R(\Gamma_{\mathfrak{a}}(M), L) \longrightarrow \cdots .$$

This yields that $f_{\mathfrak{a}}(M, L) = f_{\mathfrak{a}}(\frac{M}{\Gamma_{\mathfrak{a}}(M)}, L)$ and similarly we have $f_{\mathfrak{a}}(N, L) = f_{\mathfrak{a}}(\frac{N}{\Gamma_{\mathfrak{a}}(N)}, L)$. Now, the claim becomes clear by Theorem 2.1. □

Example 2.3. Let L, M and N be finitely generated R-modules such that $\mathrm{Supp}_R M = \mathrm{Supp}_R N$. In Theorem 2.1, we saw that $f_{\mathfrak{a}}(M, L) = f_{\mathfrak{a}}(N, L)$ for any ideal $\mathfrak{a}$ of R. One may ask whether the equality $f_{\mathfrak{a}}(L, M) = f_{\mathfrak{a}}(L, N)$ holds too. This would not be the case. To see this, let $(R, \mathfrak{m})$ be a local ring with $\mathrm{depth}\, R > 1$ and N' a 1-dimensional Cohen–Macaulay R-module. Set $L = M := R$ and $N := N' \oplus R$. Then $\mathrm{Supp}_R M = \mathrm{Supp}_R N$. Now, by [14, Remark 2.5], $H^1_{\mathfrak{m}}(N) \cong H^1_{\mathfrak{m}}(N')$ is not finitely generated. So $f_{\mathfrak{m}}(L, N) = f_{\mathfrak{m}}(N) = 1$, while $f_{\mathfrak{m}}(L, M) = f_{\mathfrak{m}}(R) > 1$.

If we fix the ideal $\mathfrak{a}$ and the R-module M, then we cannot say so much about $f_{\mathfrak{a}}(M, \cdot)$. However, we have the following result.

Proposition 2.4. *Let $\mathfrak{a}$ be an ideal of R and M, N two finitely generated R-modules such that $\mathrm{pd}\, N < \infty$. Then $f_{\mathfrak{a}}(M, N) \geq f_{\mathfrak{a}}(M, R) - \mathrm{pd}\, N$.*

Proof. We use induction on $n := \mathrm{pd}\, N$. If $n = 0$, then there is nothing to prove. Now, assume that $n > 0$ and that the assertion holds for $n - 1$. We can construct an exact sequence

$$0 \longrightarrow L \longrightarrow F \longrightarrow N \longrightarrow 0$$

of finitely generated R-modules such that F is free and $\mathrm{pd}\, L = n-1$. By the induction hypothesis, $f_{\mathfrak{a}}(M, L) \geq f_{\mathfrak{a}}(M, R)-n+1$. Let $i < f_{\mathfrak{a}}(M, R) - n$. Then, it follows from the exact sequence

$$H^i_{\mathfrak{a}}(M, F) \longrightarrow H^i_{\mathfrak{a}}(M, N) \longrightarrow H^{i+1}_{\mathfrak{a}}(M, L)$$

that $H^i_{\mathfrak{a}}(M, N)$ is finitely generated. Hence $f_{\mathfrak{a}}(M, N) \geq f_{\mathfrak{a}}(M, R) - n$, as required. □

Let $\mathfrak{a} \subseteq \mathfrak{b}$ be ideals of R. Recall that by Definition 5.3.6 [24] for a not necessary finitely generated R-module M, $\mathrm{grade}(\mathfrak{a}, M)$ is defined by

$$\mathrm{grade}(\mathfrak{a}, M) := \inf\{i \in \mathbb{N}_0 : \mathrm{Ext}^i_R(R/\mathfrak{a}, M) \neq 0\}.$$

Also, recall that $f^{\mathfrak{b}}_{\mathfrak{a}}(M)$, the $\mathfrak{b}$-finiteness dimension M relative to $\mathfrak{a}$, is defined by

$$f^{\mathfrak{b}}_{\mathfrak{a}}(M) := \inf\{i \in \mathbb{N}_0 : \mathfrak{b} \not\subseteq \mathrm{rad}(\mathrm{Ann}_R H^i_{\mathfrak{a}}(M))\}.$$

If M is finitely generated, then [6, Proposition 9.1.2] implies that $f^{\mathfrak{a}}_{\mathfrak{a}}(M) = f_{\mathfrak{a}}(M)$. Also, we remind the reader that for a finitely generated R-module M over a local ring $(R, \mathfrak{m})$, the filter depth of $\mathfrak{a}$ on M is defined as the length of any maximal M-filter regular sequence in $\mathfrak{a}$ and denoted by $f - \mathrm{depth}(\mathfrak{a}, M)$; see [19, Defintion 3.3]. By [20, Theorem 3.1], it is known that $f - \mathrm{depth}(\mathfrak{a}, M) = \inf\{i \in \mathbb{N}_0 : H^i_{\mathfrak{a}}(M)$ is not Artinian$\}$.

Theorem 2.5. *Let $\mathfrak{a}$ be an ideal of the local ring $(R, \mathfrak{m})$ and M an R-module. Then, the following holds:*

(i) *Assume that $r := f^{\mathfrak{a}}_{\mathfrak{a}}(M) < f^{\mathfrak{a}}_{\mathfrak{m}}(M)$. Then $H^r_{\mathfrak{a}}(M)$ is not Artinian. Moreover, if M is finitely generated, then $f - \mathrm{depth}(\mathfrak{a}, M) \leq f_{\mathfrak{a}}(M)$.*

(ii) *Assume that $r := \mathrm{grade}(\mathfrak{a}, M) < \mathrm{depth}\, M$. Then $H^r_{\mathfrak{a}}(M)$ is not Artinian. Moreover, if M is finitely generated, then $\mathrm{grade}(\mathfrak{a}, M) = f - \mathrm{depth}(\mathfrak{a}, M)$.*

Proof. (i) We argue by induction on r. Let $r = 0$. If $H^0_{\mathfrak{a}}(M)$ is Artinian, then

$$H^0_{\mathfrak{a}}(M) \cong H^0_{\mathfrak{m}}(H^0_{\mathfrak{a}}(M)) \cong H^0_{\mathfrak{m}}(M).$$

Since $0 < f^{\mathfrak{a}}_{\mathfrak{m}}(M)$, there is an integer n such that $\mathfrak{a}^n H^0_{\mathfrak{a}}(M) \cong \mathfrak{a}^n H^0_{\mathfrak{m}}(M) = 0$. So $0 < f^{\mathfrak{a}}_{\mathfrak{a}}(M)$ and we have arrived at a contradiction.

Now, assume that $r > 0$. Then there exists an integer $n \in \mathbb{N}$ such that $\mathfrak{a}^n H^0_{\mathfrak{a}}(M) = 0$. Hence the argument [5, Remark 1.3(ii)] yields that $f^{\mathfrak{a}}_{\mathfrak{m}}(M) = f^{\mathfrak{a}}_{\mathfrak{m}}(M/\Gamma_{\mathfrak{a}}(M))$ and $f^{\mathfrak{a}}_{\mathfrak{a}}(M) = f^{\mathfrak{a}}_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))$. Thus without loss of generality, we may and do assume that $H^0_{\mathfrak{a}}(M) = 0$. Now, we apply Melkersson's technic [21], so let E be an injective envelope of M and $N := E/M$. Then $\Gamma_{\mathfrak{a}}(E) = \Gamma_{\mathfrak{m}}(E) = 0$. From the exact sequence of local cohomology modules induced by

$$0 \longrightarrow M \longrightarrow E \longrightarrow N \longrightarrow 0,$$

we obtain that $H^i_{\mathfrak{a}}(N) \cong H^{i+1}_{\mathfrak{a}}(M)$ and $H^i_{\mathfrak{m}}(N) \cong H^{i+1}_{\mathfrak{m}}(M)$ for all $i \geq 0$. Hence $f^{\mathfrak{a}}_{\mathfrak{m}}(N) = f^{\mathfrak{a}}_{\mathfrak{m}}(M) - 1$ and $f^{\mathfrak{a}}_{\mathfrak{a}}(N) = f^{\mathfrak{a}}_{\mathfrak{a}}(M) - 1$ and the claim follows by the induction hypothesis.

(ii) The proof is similar to the proof of (i). Note that by [24, Proposition 5.3.15], $\operatorname{grade}(\mathfrak{a}, M) = \inf\{i \in \mathbb{N}_0 : H^i_{\mathfrak{a}}(M) \neq 0\}$. □

Example 2.6. (i) The assumption $f^{\mathfrak{a}}_{\mathfrak{a}}(M) < f^{\mathfrak{a}}_{\mathfrak{m}}(M)$ is crucial in Theorem 2.5(i). To realize this, let M be a Cohen–Macaulay R-module of positive dimension and $\mathfrak{a} = \mathfrak{m}$. Then $f^{\mathfrak{a}}_{\mathfrak{a}}(M) = \dim M$ and $H^{\dim M}_{\mathfrak{a}}(M)$ is Artinian.

(ii) Also, the assumption $\operatorname{grade}(\mathfrak{a}, M) < \operatorname{depth} M$ cannot be dropped in Theorem 2.5(ii). In fact, if M is a finitely generated R-module and $\mathfrak{a} = \mathfrak{m}$, then $\operatorname{grade}(\mathfrak{a}, M) = \operatorname{depth} M$ and all local cohomology modules $H^i_{\mathfrak{a}}(M)$ are Artinian.

Remark 2.7. (i) Let M be a finitely generated R-module such that $\operatorname{ht}_M \mathfrak{a} > 0$; then we have the inequality $f_{\mathfrak{a}}(M) \leq \operatorname{ht}_M \mathfrak{a}$. To see this, first note that we may assume that $\mathfrak{a}M \neq M$. Let $\mathfrak{p} \in \operatorname{Supp}_R(M/\mathfrak{a}M)$ be such that $\operatorname{ht}_M \mathfrak{a} = \operatorname{ht}_M \mathfrak{p}$, and set $t = \operatorname{ht}_M \mathfrak{a}$. Then, because of the natural isomorphisms

$$H^t_{\mathfrak{a}}(M)_{\mathfrak{p}} \cong H^t_{(\mathfrak{a}+\operatorname{Ann}_R M)R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \cong H^t_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})$$

and [14, Remark 2.5], it follows that $H^t_{\mathfrak{a}}(M)_{\mathfrak{p}}$ is not a finitely generated $R_{\mathfrak{p}}$-module. Hence the R-module $H^t_{\mathfrak{a}}(M)$ is not finitely generated, and consequently, $f_{\mathfrak{a}}(M) \leq \operatorname{ht}_M \mathfrak{a}$.

(ii) From the definition of $f_{\mathfrak{a}}(M)$, it becomes clear that $f_{\mathfrak{a}}(M) \geq \operatorname{grade}(\mathfrak{a}, M)$. There are some cases in which the equality holds. For example, let M be a Cohen–Macaulay R-module such that $\operatorname{ht}_M \mathfrak{a} > 0$; then by (i), $f_{\mathfrak{a}}(M) \leq \operatorname{ht}_M \mathfrak{a} = \operatorname{grade}(\mathfrak{a}, M)$, and so $f_{\mathfrak{a}}(M) = \operatorname{grade}(\mathfrak{a}, M)$.

(iii) If $(R, \mathfrak{m})$ is a regular local ring with $\dim R > 1$, then for any integer $0 < n < \dim R$, there exists an n-dimensional finitely generated R-module M such that $\operatorname{depth} M = 0$ and $f_{\mathfrak{m}}(M) = \dim M$. This holds, because by [6, Ex. 6.2.13], there is a finitely generated R-module M such that $H^i_{\mathfrak{m}}(M) \neq 0$ if and only if i is either 0 or n. So, $\operatorname{depth} M = 0$ and by [6, Exercise 6.1.6], $f_{\mathfrak{m}}(M) = n = \dim M$.

(iv) Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ a proper ideal of R and M, N finitely generated R-modules. Set $\mathfrak{a}_M := \operatorname{Ann}_R(\frac{M}{\mathfrak{a}M})$. By [2, Proposition 5.5], it follows that

$$\inf\{i \in \mathbb{N}_0 : H^i_{\mathfrak{a}}(M, N) \neq 0\} = \inf\{i \in \mathbb{N}_0 : H^i_{\mathfrak{a}_M}(N) \neq 0\} (= \operatorname{grade}(\mathfrak{a}_M, N)).$$

Also, [7, Theorem 2.2] yields that

$$\inf\{i \in \mathbb{N}_0 : H^i_{\mathfrak{a}}(M, N) \text{ is not Artinian}\} = \inf\{i \in \mathbb{N}_0 : H^i_{\mathfrak{a}_M}(N) \text{ is not Artinian}\}$$

$(= f - \operatorname{depth}(\mathfrak{a}_M, N))$. Having these facts in mind, one might ask whether $f_{\mathfrak{a}}(M, N) = f_{\mathfrak{a}_M}(N)$. This is not necessarily true. For instance, let $M := R/\mathfrak{m}$, $\mathfrak{a} := \mathfrak{m}$ and N be any non-Artinian finitely generated R-module. Then

$$H^i_{\mathfrak{a}}(M, N) = \varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N) \cong \operatorname{Ext}^i_R(R/\mathfrak{m}, N)$$

is finitely generated for all i, and so $f_{\mathfrak{a}}(M, N) = +\infty$, while $f_{\mathfrak{a}_M}(N) \leq \dim N < \infty$.

3. The Artinian dimension of modules

In this section, we focus on the invariant $q_{\mathfrak{a}}(M, N)$. In [9, Definition 2.1], the authors call an R-module N weakly Laskerian if any quotient of N has finitely many associated prime ideals. In what follows, for a not necessarily finitely generated R-module N, by dimension of N, we mean the dimension of $\operatorname{Supp}_R N$.

Theorem 3.1. *Let $\mathfrak{a}$ be an ideal of R, M a finitely generated R-module of finite projective dimension and N a weakly Laskerian R-module of finite dimension. Let $t > \operatorname{pd} M$ be an integer such that $H^j_{\mathfrak{a}}(M, N)$ is Artinian for all $j > t$. Then $H^t_{\mathfrak{a}}(M, N)/\mathfrak{a}H^t_{\mathfrak{a}}(M, N)$ is Artinian.*

Proof. We use induction on $n := \dim N$. By [8, Theorem 2.5], it follows that $H^i_{\mathfrak{a}}(M, L) = 0$ for all finitely generated R-modules L and all $i > \operatorname{pd} M + \dim L$. So, since the functor $H^i_{\mathfrak{a}}(M, \cdot)$ commutes with direct limits, it turns out that $H^i_{\mathfrak{a}}(M, N) = 0$ for all $i > \operatorname{pd} M + \dim N$. Thus the claim clearly holds for $n = 0$.

Now, assume that $n > 0$ and that the claim holds for $n - 1$. Since $t > \operatorname{pd} M$, in view of the long exact sequence of generalized local cohomology modules that is induced by the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(N) \longrightarrow N \longrightarrow N/\Gamma_{\mathfrak{a}}(N) \longrightarrow 0,$$

we may assume that N is $\mathfrak{a}$-torsion free. Note that since the functor $H^i_{\mathfrak{a}}(M, \cdot)$ is the ith right derived functor of the functor $\operatorname{Hom}_R(M, \Gamma_{\mathfrak{a}}(\cdot))$ and $\Gamma_{\mathfrak{a}}(N)$ possesses an injective resolution consisting of $\mathfrak{a}$-torsion injective R-modules, it follows that $H^i_{\mathfrak{a}}(M, \Gamma_{\mathfrak{a}}(N)) \cong \operatorname{Ext}^i_R(M, \Gamma_{\mathfrak{a}}(N))$ for all i. Take $x \in \mathfrak{a} \setminus \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R N} \mathfrak{p}$. Then N/xN is weakly Laskerian and $\dim N/xN \leq n - 1$. The exact sequence

$$0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0$$

implies the following long exact sequence of generalized local cohomology modules:

$$\cdots \longrightarrow H^j_{\mathfrak{a}}(M, N) \xrightarrow{x} H^j_{\mathfrak{a}}(M, N) \longrightarrow H^j_{\mathfrak{a}}(M, N/xN) \longrightarrow \cdots.$$

This yields that $H^j_{\mathfrak{a}}(M, N/xN)$ is Artinian for all $j > t$. Thus $\frac{H^t_{\mathfrak{a}}(M,N/xN)}{\mathfrak{a}H^t_{\mathfrak{a}}(M,N/xN)}$ is Artinian by the induction hypothesis.

Now, consider the exact sequence

$$H^t_{\mathfrak{a}}(M, N) \xrightarrow{x} H^t_{\mathfrak{a}}(M, N) \xrightarrow{f} H^t_{\mathfrak{a}}(M, N/xN) \xrightarrow{g} H^{t+1}_{\mathfrak{a}}(M, N),$$

which induces the following two exact sequences:

$$0 \longrightarrow \operatorname{im} f \longrightarrow H^t_{\mathfrak{a}}(M, N/xN) \longrightarrow \operatorname{im} g \longrightarrow 0$$

and

$$H^t_{\mathfrak{a}}(M, N) \xrightarrow{x} H^t_{\mathfrak{a}}(M, N) \longrightarrow \operatorname{im} f \longrightarrow 0.$$

Therefore, we can obtain the following two exact sequences:

$$Tor^R_1(R/\mathfrak{a}, \operatorname{im} g) \longrightarrow \operatorname{im} f/\mathfrak{a} \operatorname{im} f \longrightarrow \frac{H^t_{\mathfrak{a}}(M, N/xN)}{\mathfrak{a}H^t_{\mathfrak{a}}(M, N/xN)} \longrightarrow \operatorname{im} g/\mathfrak{a} \operatorname{im} g \longrightarrow 0(*)$$

and

$$\frac{H^t_{\mathfrak{a}}(M, N)}{\mathfrak{a}H^t_{\mathfrak{a}}(M, N)} \xrightarrow{x} \frac{H^t_{\mathfrak{a}}(M, N)}{\mathfrak{a}H^t_{\mathfrak{a}}(M, N)} \longrightarrow \operatorname{im} f/\mathfrak{a} \operatorname{im} f \longrightarrow 0.$$

Since $x \in \mathfrak{a}$, from the latter exact sequence, we deduce that $\frac{H^t_{\mathfrak{a}}(M,N)}{\mathfrak{a}H^t_{\mathfrak{a}}(M,N)} \cong \operatorname{im} f/\mathfrak{a} \operatorname{im} f$. Now, since $Tor^R_1(R/\mathfrak{a}, \operatorname{im} g)$ and $\frac{H^t_{\mathfrak{a}}(M,N/xN)}{\mathfrak{a}H^t_{\mathfrak{a}}(M,N/xN)}$ are Artinian, the claim follows by $(*)$. □

The following corollary improves [10, Theorem 4.7].

Corollary 3.2. *Let $\mathfrak{a}$ be an ideal of R and N a weakly Laskerian R-module of finite dimension. Let t be a positive integer. If $H^i_{\mathfrak{a}}(N)$ is Artinian for all $i > t$, then $H^t_{\mathfrak{a}}(N)/\mathfrak{a}H^t_{\mathfrak{a}}(N)$ is Artinian.*

Corollary 3.3. *Let $\mathfrak{a}$ be an ideal of R, M a finitely generated R-module of finite projective dimension and N a weakly Laskerian R-module of finite dimension. If $q := q_{\mathfrak{a}}(M, N) > \operatorname{pd} M$, then $H^q_{\mathfrak{a}}(M, N)$ is not finitely generated, and so $f_{\mathfrak{a}}(M, N) \leq q_{\mathfrak{a}}(M, N)$. In particular, if $r := q_{\mathfrak{a}}(N) > 0$, then $H^r_{\mathfrak{a}}(N)$ is not finitely generated and so $f_{\mathfrak{a}}(N) \leq q_{\mathfrak{a}}(N)$.*

Proof. To the contrary, assume that $H^q_{\mathfrak{a}}(M, N)$ is finitely generated. Then, there exists an integer n such that $\mathfrak{a}^n H^q_{\mathfrak{a}}(M, N) = 0$. Since $\mathfrak{a}^n$ and $\mathfrak{a}$ have the same radical, it turns out that $H^i_{\mathfrak{a}^n}(M, N) \cong H^i_{\mathfrak{a}}(M, N)$ for all i. Thus, Theorem 3.1 yields that $H^q_{\mathfrak{a}}(M, N) \cong \frac{H^q_{\mathfrak{a}^n}(M,N)}{\mathfrak{a}^n H^q_{\mathfrak{a}^n}(M,N)}$ is Artinian, and so we have arrived at a contradiction. □

In what follows, we use the notion of cohomological dimension. Recall that for an R-module N, the cohomological dimension of N with respect to $\mathfrak{a}$ is defined by $\operatorname{cd}(\mathfrak{a}, N) := \sup\{i \in \mathbb{N}_0 : H^i_{\mathfrak{a}}(N) \neq 0\}$. Also, recall that the arithmetic rank $\operatorname{ara}(\mathfrak{a})$ of the ideal $\mathfrak{a}$ is the least number of elements of R required to generate an ideal which has the same radical as $\mathfrak{a}$. By [6, Corollary 3.3.3 and Theorem 6.1.2], it turns out that $\operatorname{cd}(\mathfrak{a}, N) \leq \min\{\operatorname{ara}(\mathfrak{a}), \dim N\}$.

Example 3.4. (i) In Corollary 3.2, the positivity assumption on t is really necessary. To see this, let $(R, \mathfrak{m})$ be a local ring and consider the weakly Laskerian R-module $M := \bigoplus_{\mathbb{N}} R/\mathfrak{m}$. We have $H^i_{\mathfrak{m}}(M) = 0$ for all $i > 0$, but $\frac{H^0_{\mathfrak{m}}(M)}{\mathfrak{m}H^0_{\mathfrak{m}}(M)} = \bigoplus_{\mathbb{N}} R/\mathfrak{m}$ is not Artinian.

(ii) In Corollary 3.2, if $t := \mathrm{cd}(\mathfrak{a}, N)$, then it can be seen easily that $\frac{H^t_{\mathfrak{a}}(N)}{\mathfrak{a}H^t_{\mathfrak{a}}(N)} \cong H^t_{\mathfrak{a}}(N/\mathfrak{a}N) = 0$. But, in general $\frac{H^t_{\mathfrak{a}}(N)}{\mathfrak{a}H^t_{\mathfrak{a}}(N)}$ might not even be finitely generated. To see this, let $R := k[[X_1, X_2, X_3, X_4]]$, where k is a field. Let $\mathfrak{p}_1 := (X_1, X_2)$, $\mathfrak{p}_2 := (X_3, X_4)$ and $\mathfrak{a} := \mathfrak{p}_1 \cap \mathfrak{p}_2$. By [12, Example 3], one has that $q_{\mathfrak{a}}(R) = 2$ and $H^2_{\mathfrak{a}}(R) \cong H^2_{\mathfrak{p}_1}(R) \oplus H^2_{\mathfrak{p}_2}(R)$. Now consider the following isomorphisms:

$$\frac{H^2_{\mathfrak{a}}(R)}{\mathfrak{a}H^2_{\mathfrak{a}}(R)} \cong \frac{H^2_{\mathfrak{p}_1}(R)}{\mathfrak{a}H^2_{\mathfrak{p}_1}(R)} \bigoplus \frac{H^2_{\mathfrak{p}_2}(R)}{\mathfrak{a}H^2_{\mathfrak{p}_2}(R)} \cong H^2_{\mathfrak{p}_1}(R/\mathfrak{a}) \bigoplus H^2_{\mathfrak{p}_2}(R/\mathfrak{a}).$$

By the Hartshorne–Lichtenbaum Vanishing Theorem, $H^2_{\mathfrak{p}_1}(R/\mathfrak{a}) \neq 0$. Therefore $\mathrm{cd}(\mathfrak{p}_1, R/\mathfrak{a}) = 2$, and so by [14, Remark 2.5], $H^2_{\mathfrak{p}_1}(R/\mathfrak{a})$ is not finitely generated. Consequently, $\frac{H^2_{\mathfrak{a}}(R)}{\mathfrak{a}H^2_{\mathfrak{a}}(R)}$ is not finitely generated.

Recall that for an R-module M, $\mathrm{Coass}_R M$, the set of coassociated prime ideals of M, is defined to be the set of all prime ideals $\mathfrak{p}$ of R such that $\mathfrak{p} = \mathrm{Ann}_R L$ for some Artinian quotient L of M. In the case that $(R, \mathfrak{m})$ is local, it is known that $\mathrm{Coass}_R M = \mathrm{Ass}_R(\mathrm{Hom}_R(M, E(R/\mathfrak{m})))$ (see e.g. [26, Theorem 1.7]).

Example 3.5. Let $\mathfrak{a}$ be an ideal of R and M a finitely generated R-module. Let t be a natural integer. Assume that $H^i_{\mathfrak{a}}(M)$ is finitely generated for all $i < t$. Then by [1, Theorem 1.2], $\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ is finitely generated. (This can be viewed as a dual of Corollary 3.2.) Since $H^t_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-torsion, it follows that $\mathrm{Ass}_R(H^t_{\mathfrak{a}}(M)) = \mathrm{Ass}_R(\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)))$, and so $\mathrm{Ass}_R(H^t_{\mathfrak{a}}(M))$ is finite. Now, assume that $H^i_{\mathfrak{a}}(M)$ is Artinian for all $i > t$. Then, it is rather natural to expect that the set $\mathrm{Coass}_R(H^t_{\mathfrak{a}}(M))$ is finite. But, this is not the case. To see this, let $(R, \mathfrak{m})$ be an equicharacteristic local ring with $\dim R > 2$. Let $\mathfrak{p}$ be a prime ideal of R of height 2 and take $x \in \mathfrak{m} - \mathfrak{p}$. Then by [13, Corollary 2.2.2], $\mathrm{Coass}_R(H^1_{(x)}(R)) = \mathrm{Spec}\, R \setminus \mathrm{V}((x))$. Since $\mathrm{ht}\,\mathfrak{p} = 2$, there are infinitely many prime ideals of R which are contained in $\mathfrak{p}$, and so $\mathrm{Coass}_R(H^1_{(x)}(R))$ is infinite.

We apply the following lemma in the proof of the next theorem.

Lemma 3.6. *Let A be an Artinian R-module and S a multiplicatively closed subset of R. Then as an R-module, $S^{-1}A$ is isomorphic to a submodule of A, and so $S^{-1}A$ is Artinian both as an R-module and as an $S^{-1}R$-module.*

Proof. Since A is Artinian, it is supported in finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ of R. It is known that there is a natural isomorphism $A \cong \oplus_{i=1}^t \Gamma_{\mathfrak{m}_i}(A)$. We can assume that there is an integer $1 \leq \ell \leq t$ such that S does not intersect $\mathfrak{m}_1, \ldots, \mathfrak{m}_\ell$, while $S \cap \mathfrak{m}_i \neq \emptyset$ for the remaining i's. For any maximal ideal $\mathfrak{m}$, one can check that $S^{-1}(\Gamma_{\mathfrak{m}}(A)) = 0$ if $S \cap \mathfrak{m} \neq \emptyset$ and that the natural map $\Gamma_{\mathfrak{m}}(A) \longrightarrow S^{-1}(\Gamma_{\mathfrak{m}}(A))$ is an isomorphism otherwise. Hence, we have a natural R-isomorphism $S^{-1}A \cong \oplus_{i=1}^{\ell} \Gamma_{\mathfrak{m}_i}(A)$. This shows that as an R-module, $S^{-1}A$ is isomorphic to a submodule of A. In particular, $S^{-1}A$ is an Artinian R-module, and so it is also Artinian as an $S^{-1}R$-module. □

Part (iv) of the next result has been proved by Chu and Tang [7, Theorem 2.6] under the extra assumption that R is local.

Theorem 3.7. *Let L, M and N be finitely generated R-modules and $\mathfrak{a} \subseteq \mathfrak{b}$ ideals of R. Then the following holds:*

(i) $q_{\mathfrak{b}}(M, N) \leq q_{\mathfrak{a}}(M, N) + \mathrm{ara}(\mathfrak{b}/\mathfrak{a})$.
(ii) *If $(R, \mathfrak{m})$ is local and* $\mathrm{ara}(\mathfrak{m}/\mathfrak{a}) \leq 1$, *then* $q_{\mathfrak{a}}(M, N) = \sup\{i : \mathrm{Supp}_R(H^i_{\mathfrak{a}}(M, N)) \nsubseteq \{\mathfrak{m}\}\}$.
(iii) *If S is a multiplicatively closed subset of R, then $q_{S^{-1}\mathfrak{a}}(S^{-1}M, S^{-1}N) \leq q_{\mathfrak{a}}(M, N)$.*
(iv) *If $\mathrm{pd}\, M < \infty$ and $\mathrm{Supp}_R N \subseteq \mathrm{Supp}_R L$, then $q_{\mathfrak{a}}(M, N) \leq q_{\mathfrak{a}}(M, L)$. In particular, if $\mathrm{Supp}_R N = \mathrm{Supp}_R L$, then $q_{\mathfrak{a}}(M, N) = q_{\mathfrak{a}}(M, L)$.*
(v) *If $\mathrm{pd}\, M < \infty$, then $q_{\mathfrak{a}}(M, N) = \sup\{q_{\mathfrak{a}}(M, R/\mathfrak{p}) : \mathfrak{p} \in \mathrm{Supp}_R N\}$.*

Proof. (i) Since the generalized local cohomology functors with respect to ideals with the same radicals are equivalent, we may assume that there are $x_1, x_2, \ldots x_n \in R$ such that $\mathfrak{b} = \mathfrak{a} + (x_1, x_2, \ldots x_n)$. By induction on n, it is enough to show the claim for the case $n = 1$. So, let $\mathfrak{b} = \mathfrak{a} + (x)$ for some $x \in R$. By [8, Lemma 3.1], there is the following long exact sequence of generalized local cohomology modules:

$$\cdots \longrightarrow H^{i-1}_{\mathfrak{a}R_x}(M_x, N_x) \longrightarrow H^i_{\mathfrak{b}}(M, N) \longrightarrow H^i_{\mathfrak{a}}(M, N) \longrightarrow \cdots.$$

Let $i > q_{\mathfrak{a}}(M, N) + 1$. Then $H^i_{\mathfrak{a}}(M, N)$ and $H^{i-1}_{\mathfrak{a}}(M, N)$ are Artinian. Also, by Lemma 3.6, it turns out that $H^{i-1}_{\mathfrak{a}R_x}(M_x, N_x) \cong H^{i-1}_{\mathfrak{a}}(M, N)_x$ is Artinian as an R-module. Thus $H^i_{\mathfrak{b}}(M, N)$ is Artinian, and so $q_{\mathfrak{b}}(M, N) \leq q_{\mathfrak{a}}(M, N) + 1$.

(ii) First of all note that $H^i_{\mathfrak{m}}(M, N)$ is Artinian for all i. Similar to (i), we may assume that $\mathfrak{m} = \mathfrak{a} + (x)$ for some $x \in R$. Clearly,

$$t := \sup\{i : \mathrm{Supp}_R(H^i_{\mathfrak{a}}(M, N)) \nsubseteq \{\mathfrak{m}\}\} \leq q_{\mathfrak{a}}(M, N).$$

Let $j > t$ be an integer. Then $H^j_{\mathfrak{a}}(M, N)$ is $\mathfrak{m}$-torsion, and so

$$H^j_{\mathfrak{a}R_x}(M_x, N_x) \cong H^j_{\mathfrak{a}}(M, N)_x = 0.$$

Hence, from the exact sequence

$$\cdots \longrightarrow H^j_{\mathfrak{m}}(M, N) \longrightarrow H^j_{\mathfrak{a}}(M, N) \longrightarrow H^j_{\mathfrak{a}R_x}(M_x, N_x) \longrightarrow \cdots,$$

we conclude that $H^j_{\mathfrak{a}}(M, N)$ is Artinian. This implies that $q_{\mathfrak{a}}(M, N) \le t$.

(iii) This is clear by Lemma 3.6.

(iv) For any finitely generate R-module C, one has $H^i_{\mathfrak{a}}(M, C) = 0$ for all $i > \operatorname{pd} M + \operatorname{ara}(\mathfrak{a})$; see e.g. [8, Theorem 2.5]. Hence by using decreasing induction on $q_{\mathfrak{a}}(M, L) \le i \le \operatorname{pd} M + \operatorname{ara}(\mathfrak{a}) + 1$, one can prove the claim by slight modification of the proof of Theorem 2.1.

(v) This can be deduced by an argument similar to that used in the proof of Corollary 2.2(iii). □

Remark 3.8. It is known that if $\mathfrak{a}$ is an ideal of a regular local ring $(R, \mathfrak{m})$, then (under some mild assumptions on R), we have $q_{\mathfrak{a}}(R) = \sup\{i \in \mathbb{N}_0 : \operatorname{Supp}_R(H^i_{\mathfrak{a}}(R)) \not\subseteq \{\mathfrak{m}\}\}$. See [17, Corollary 2.4] for the case that the characteristic of R is prime and [22, Theorem 2.7] for the characteristic 0 case. Part (ii) of the above theorem might be considered as a generalization of the Hartshorne–Speiser and Ogus's results. But, the reader should be aware that the assumption $\operatorname{ara}(\mathfrak{m}/\mathfrak{a}) \le 1$ is necessary. To see this, let $R = k[[U, V, X, Y]]/(UX + VY)$; k a field, $\mathfrak{m} := (U, V, X, Y)R$ and $\mathfrak{a} = (U, V)R$. Then $\operatorname{Supp}_R(H^i_{\mathfrak{a}}(R)) \subseteq \{\mathfrak{m}\}$ for all $i \ge 2$ and $H^2_{\mathfrak{a}}(R)$ is not Artinian; see [15, Theorem 1.1]. Note that $\operatorname{ara}(\mathfrak{m}/\mathfrak{a}) = 2$ and $(R, \mathfrak{m})$ is a complete intersection ring which is not regular.

In what follows, we need the following definition from [8].

Definition 3.9. Let M and N be finitely generated R-modules. We define *projective dimension of M relative to N* by

$$\operatorname{pd}_N M := \sup\{\operatorname{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} : \mathfrak{p} \in \operatorname{Supp}_R M \cap \operatorname{Supp}_R N\}.$$

Also, we define *Gorenstein projective dimension of M relative to N* by

$$\operatorname{Gpd}_N M := \sup\{i \in \mathbb{N}_0 : \operatorname{Ext}^i_R(M, N) \neq 0\}.$$

Lemma 3.10. *Let M and N be finitely generated R-modules.*

(i) $\operatorname{grade}(\operatorname{Ann}_R M, N) \le \operatorname{Gpd}_N M$.
(ii) *If $\operatorname{pd}_N M$ is finite, then $\operatorname{Gpd}_N M = \operatorname{pd}_N M$. In particular, if R is local and $\operatorname{pd} M$ is finite, then $\operatorname{Gpd}_N M = \operatorname{pd} M$.*
(iii) $\operatorname{Gpd}_N M \le \min\{\operatorname{pd} M, \operatorname{id} N\}$. *In particular, if either $\operatorname{pd} M$ or $\operatorname{id} N$ is finite, then $\operatorname{Gpd}_N M$ is finite.*
(iv) *If $\operatorname{pd}_N M$ is finite, then $\bigcup_{i \in \mathbb{N}_0} \operatorname{Supp}_R(\operatorname{Ext}^i_R(M, N)) = \operatorname{Supp}_R M \cap \operatorname{Supp}_R N$, and so $\dim(M \otimes_R N) = \sup\{\dim(\operatorname{Ext}^i_R(M, N)) : i \in \mathbb{N}_0\}$.*
(v) *If $\operatorname{pd}_N M$ is finite, then $q_{\mathfrak{a}}(L, M \otimes_R N) = \max\{q_{\mathfrak{a}}(L, \operatorname{Ext}^i_R(M, N)) : i \in \mathbb{N}_0\}$ for all finitely generated R-modules L of finite projective dimension.*
(vi) *If R is local and $\operatorname{id} N < \infty$, then $\operatorname{Gpd}_N M = \operatorname{depth} R - \operatorname{depth} M$, and if in addition M is maximal Cohen–Macaulay, then $\operatorname{Gpd}_N M = 0$.*

Proof. We only prove (v). The other parts are proved in [8, Lemma 2.2].

By (iv) and Theorem 3.7(v), we have

$$\begin{aligned} q_{\mathfrak{a}}(L, M \otimes_R N) &= \sup\{q_{\mathfrak{a}}(L, R/\mathfrak{p}) : \mathfrak{p} \in \operatorname{Supp}_R(M \otimes_R N)\} \\ &= \sup\{q_{\mathfrak{a}}(L, R/\mathfrak{p}) : \mathfrak{p} \in \bigcup_{i \in \mathbb{N}_0} \operatorname{Supp}_R(\operatorname{Ext}^i_R(M, N))\} \\ &= \max_{i \in \mathbb{N}_0}(\sup\{q_{\mathfrak{a}}(L, R/\mathfrak{p}) : \mathfrak{p} \in \operatorname{Supp}_R(\operatorname{Ext}^i_R(M, N))\}) \\ &= \max\{q_{\mathfrak{a}}(L, \operatorname{Ext}^i_R(M, N)) : i \in \mathbb{N}_0\}. \quad \square \end{aligned}$$

Theorem 3.11. *Let M and N be finitely generated R-modules. For any ideal $\mathfrak{a}$ of R, the inequality $q_{\mathfrak{a}}(M, N) \le \operatorname{Gpd}_N M + q_{\mathfrak{a}}(M \otimes_R N)$ holds.*

Proof. We may assume that $\operatorname{Gpd}_N M < \infty$. Let $F(\cdot) := \Gamma_{\mathfrak{a}}(\cdot)$ and $G(\cdot) := \operatorname{Hom}_R(M, \cdot)$. It is straightforward to see that $H^i_{\mathfrak{a}}(\operatorname{Hom}_R(M, E)) = 0$ for any injective R-module E and all $i \ge 1$. So, since $(GF)(\cdot) = \operatorname{Hom}_R(M, \Gamma_{\mathfrak{a}}(\cdot))$, by [23, Theorem 11.38], one has the following Grothendieck's spectral sequence:

$$H^i_{\mathfrak{a}}(\operatorname{Ext}^j_R(M, N)) \underset{i}{\Longrightarrow} H^{i+j}_{\mathfrak{a}}(M, N).$$

Hence, for each $n \in \mathbb{N}_0$, there exists a chain

$$(*)\ 0 = H^{-1} \subseteq H^0 \subseteq \cdots \subseteq H^n := H^n_{\mathfrak{a}}(M, N)$$

of submodules of $H^n_{\mathfrak{a}}(M, N)$ such that $H^i/H^{i-1} \cong E^{i,n-i}_\infty$ for all $i = 0, \ldots, n$. Since $\operatorname{Ext}^j_R(M, N)$ is supported in $\operatorname{Supp}_R M \cap \operatorname{Supp}_R N$, it follows by [11, Theorem 3.2] that $H^i_{\mathfrak{a}}(\operatorname{Ext}^j_R(M, N))$ is Artinian for all $i > q_{\mathfrak{a}}(M \otimes_R N)$. If $n > \operatorname{Gpd}_N M + q_{\mathfrak{a}}(M \otimes_R N)$, then either $i > q_{\mathfrak{a}}(M \otimes_R N)$ or $n - i > \operatorname{Gpd}_N M$. In each case, it turns out that $E^{i,n-i}_2$ is Artinian. Therefore, by splitting the chain $(*)$ into short exact sequences, we deduce that $H^n_{\mathfrak{a}}(M, N)$ is Artinian for all $n > \operatorname{Gpd}_N M + q_{\mathfrak{a}}(M \otimes_R N)$. Note that $E^{i,n-i}_\infty$ is a subquotient of $E^{i,n-i}_2$ for all i. □

In what follows, we consider some cases in which the equality holds in Theorem 3.11.

Corollary 3.12. *Let $\mathfrak{a}$ be an ideal of R and M, N two finitely generated R-modules.*

(i) *If $p := \operatorname{Gpd}_N M = \operatorname{grade}(\operatorname{Ann}_R M, N)$, then $H^i_{\mathfrak{a}}(M, N) = H^{i-p}_{\mathfrak{a}}(\operatorname{Ext}^p_R(M, N))$ for all i, and consequently $q_{\mathfrak{a}}(M, N) = q_{\mathfrak{a}}(\operatorname{Ext}^p_R(M, N)) + p$. Also, $f_{\mathfrak{a}}(M, N) = f_{\mathfrak{a}}(\operatorname{Ext}^p_R(M, N)) + p$.*
(ii) *If $p := \operatorname{pd}_N M = \operatorname{grade}(\operatorname{Ann}_R M, N)$, then $q_{\mathfrak{a}}(M, N) = q_{\mathfrak{a}}(M \otimes_R N) + p$.*
(iii) *Let $\mathfrak{a}$ be an ideal of the local ring $(R, \mathfrak{m})$ and M, N two finitely generated R-modules such that M is faithful and maximal Cohen–Macaulay and $\operatorname{id} N < \infty$, then $q_{\mathfrak{a}}(M, N) = q_{\mathfrak{a}}(N)$ and $f_{\mathfrak{a}}(M, N) = f_{\mathfrak{a}}(\operatorname{Hom}_R(M, N))$.*

Proof. (i) The spectral sequence

$$H^i_{\mathfrak{a}}(\operatorname{Ext}^j_R(M, N)) \underset{i}{\Longrightarrow} H^{i+j}_{\mathfrak{a}}(M, N)$$

collapses at $j = p$, and so $H^n_{\mathfrak{a}}(M, N) \cong H^{n-p}_{\mathfrak{a}}(\operatorname{Ext}^p_R(M, N))$ for all n. Hence

$$\begin{aligned} q_{\mathfrak{a}}(M, N) &= \sup\{n \in \mathbb{N}_0 : H^{n-p}_{\mathfrak{a}}(\operatorname{Ext}^p_R(M, N)) \text{ is not Artinian}\} \\ &= \sup\{j + p : j \in \mathbb{N}_0 \text{ and } H^j_{\mathfrak{a}}(\operatorname{Ext}^p_R(M, N)) \text{ is not Artinian}\} \\ &= q_{\mathfrak{a}}(\operatorname{Ext}^p_R(M, N)) + p. \end{aligned}$$

A similar argument shows that $f_{\mathfrak{a}}(M, N) = f_{\mathfrak{a}}(\operatorname{Ext}^p_R(M, N)) + p$.

(ii) This follows by (i), Lemma 3.10(ii) and Lemma 3.10(v).

(iii) By Lemma 3.10(vi), $\operatorname{Gpd}_N M = 0$, and so in the light of the proof of (i), it turns out that $H^n_{\mathfrak{a}}(M, N) = H^n_{\mathfrak{a}}(\operatorname{Hom}_R(M, N))$ for all n. Hence $f_{\mathfrak{a}}(M, N) = f_{\mathfrak{a}}(\operatorname{Hom}_R(M, N))$. On the other hand, by [3, Exersice 1.2.27],

$$\operatorname{Ass}_R(\operatorname{Hom}_R(M, N)) = \operatorname{Supp}_R M \cap \operatorname{Ass}_R N = \operatorname{Ass}_R N,$$

and so $\operatorname{Supp}_R(\operatorname{Hom}_R(M, N)) = \operatorname{Supp}_R N$. By Theorem 3.7(iv), this shows that

$$q_{\mathfrak{a}}(M, N) = q_{\mathfrak{a}}(\operatorname{Hom}_R(M, N)) = q_{\mathfrak{a}}(N). \quad \square$$

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