Limits for weighted $p$-variations and likewise functionals of fractional diffusions with drift

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Abstract

Let $X_t$ be the pathwise solution of a diffusion driven by a fractional Brownian motion $B^H_t$ with Hurst constant $H > 1/2$ and diffusion coefficient $\sigma(t,x)$. Consider the successive increments of this solution, $\Delta X_i = X_i/n - X_{(i-1)/n}$. Using a cylinder approximation for the solution $X_t$, our main result yields that if $1/2 < H < 3/4$ then, if $Z$ is a standard normal random variable which is independent of $B^H$, the process $\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} |\Delta X_i|^p - \sigma^p(i/n, X_i/n)\mathbb{E}(|Z|^p)$ converges weakly to $W(C_H, p \int_0^t \sigma^p(s, X_s)ds)$ as $n \to \infty$ where $W$ is a Wiener process which is independent of $B^H$ and $C_H, p$ is a constant which depends on $H$ and on $p$. In the place of $p$-variations we may consider functions that satisfy an almost multiplicative structure such as even polynomials or polynomials of absolute values. By considering second order increments of the discrete sample $X_i$ we obtain analogous results for the whole interval $1/2 < H < 1$. Finally, we show convergence is stable in the absence of drift and use this result to discuss weak convergence for weak solutions of the fractional diffusion equation.

Keywords: Fractional Brownian motion; $p$-variations; Fractional diffusions

1. Introduction

Let $B^H$ be a fractional Brownian motion (fBm), defined over a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with Hurst constant $0 < H < 1$. That is, a zero-mean, Gaussian process with $B^H_0 = 0$
and
\[ \text{Var}(B_t^H - B_s^H) = |t - s|^{2H} \text{Var}(B_1^H), \]
for all \( t, s \in [0, 1] \). In order to simplify notation assume \( \text{Var}(B_1^H) = 1 \). Let \( \mathcal{F}_t \) be the filtration generated by \( B_s^H, s \leq t \).

Consider the fractional one-dimensional diffusion equation with respect to fBm,
\[ X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s^H. \]

Following [23] if the Hurst coefficient \( H > 1/2 \), an integral with respect to fBm can be defined for \( \mathbb{P} \) almost all \( w \) as the path-wise limit of Riemann–Stieltjes sums. With this construction [20] show the existence of a unique strong solution \( X_t \) of Eq. (2) under certain conditions over the drift coefficient \( b \) and the diffusion coefficient \( \sigma \) which we state below. Although in the more general case, \( b \) and \( \sigma \) may be random functions, for our main result we shall assume they are deterministic.

Now assume we observe \( X_t \) at discrete time sample points \( s_i = i/n, i = 1, \ldots, n \) and consider the weighted normalized \( p \)-variation process
\[ z_n^p(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} f(X_{s_i}) \left[ \left| \frac{X_{s_{i+1}} - X_{s_i}}{n^{-H}} \right|^p - |\sigma(s_i, X_{s_i})|^p \mathbb{E}[Z]^p \right]. \]

More generally, in the place of \( p \)-variations we may consider any function \( G \) which satisfies an almost multiplicative condition (see B1 below). This includes polynomials of absolute values, for example. Define \( (G)_i(\sigma(t, X_t)) = \mathbb{E}_{\mathcal{F}_i} G(\sigma(t, X_t)Z) \), where \( \mathbb{E}_{\mathcal{F}_t} \) stands for the conditional expectation given \( B_s^H, s \leq t \) and \( Z \sim N(0, 1) \) independent of \( B_t^H \). With this notation set
\[ z_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} f(X_{s_i}) \left[ G \left( \frac{X_{s_{i+1}} - X_{s_i}}{n^{-H}} \right) - (G)_i(\sigma(s_i, X_{s_i})) \right]. \]

Our main result, see Theorem 2, is the weak convergence of \( z_n(t) \) toward a conditionally Gaussian variable which is independent of \( B_t^H \) if \( 1/2 < H < 3/4 \).

1.1. Related results

The problem we address in this article has been well studied in the case of diffusion processes, with a view towards estimating the diffusion coefficient (see [8] or [10], for example). In the fractional diffusion setting, Theorem 2 was obtained in [16] in the absence of drift. A first generalization of this result is based on results by [19], yielding a Girsanov formula for \( B_t^H + \mu t \) in terms of \( B_t^H \). This allowed us to extend our results to functionals of solutions defined by \( X_t = g(\mu t + \sigma B_t^H) \). The proofs are based on an extension of certain moment calculating techniques for centered stationary Gaussian process [21] and on the properties of the joint covariance structure of \( B_t^H \) and its increments (see proof Lemma 3, p. 21 in [16]). Actually, the underlying process need not be stationary and all that is important is the joint covariance structure. This observation allows a second generalization for problems with non vanishing drift.

Indeed, assume that we can write \( X_t = h(y_t + P_t), t \in [0, 1] \) with \( P_t = \int_0^t c(s)dB_s^H \) and that \( c(t) \) and \( y_t \) are deterministic functions. Assume moreover that there exist positive constants \( c, C \) such that \( c < c(t) < C \) for all \( t \in T \). Hence \( P_t \) is a Gaussian process with a covariance...
structure which satisfies the conditions of Lemma 3 in [16]. Following the proof of Theorem 2 in [16] weak convergence for \( z_n(t) \) defined in (3) is seen to hold in this case. This includes two well known examples.

- The fractional Black–Scholes model ([4], [15])
  \[
  X_t = x_0 e^{\int_0^t b(s) \, ds + \int_0^t \sigma(s) \, dB^H_t}
  \]
  which satisfies the diffusion equation
  \[
  dX_t = b(t) X_t \, dt + \sigma(t) X_t dB^H_t.
  \]
- The fractional O.U. model
  \[
  X_t = e^{\mu t} \sigma \int_0^t e^{-\mu s} \, dB^H_s + x_0
  \]
  which satisfies the diffusion equation
  \[
  dX_t = \mu X_t \, dt + \sigma \, dB^H_t.
  \]

In this article following the approach of [20], we take advantage of the pathwise construction approximating \( X_t \) by a process \( X^m_t \), for each \( w \in \Omega \), uniformly for all \( t \in [0, 1] \). This new process \( X^m_t \) depends only on a finite number of random variables \( B^H_{t_j} \), \( j = 1 \ldots m \) so it can be easily dealt with. The new extension considered in this article is interesting because it develops a method for studying functionals of \( X_t \) based solely on its path-wise properties. As a by product we obtain a numerical scheme for approximating \( X_t \) and study the properties of the approximate solution.

We remark that a weak solution approach is also possible. In this approach a Girsanov formula developed by Decreusefond and Ustünel [6] may be used and a weak solution \( X_t \) is obtained for Eq. (2). In this case weak convergence of \( z_n(t) \) may be obtained by showing a generalization of Theorem 2 which yields stable convergence in the case of vanishing drift. Related results were obtained by Gloter and Hoffmann [11] using a Schauder basis approach, although they do not actually consider weak convergence but rather \( L^2 \) bounds.

The article is organized as follows: in Section 2 we discuss strong solutions for fractional differential equations and we develop an approximation method for the true solutions. In Section 3 we state our main result. In Section 4 we consider the case \( 1/2 < H < 1 \). In Section 5 we discuss briefly the weak solution approach as outlined above. In Section 6 we give the proof of our main result.

2. Strong solutions for fractional diffusion equations

Let \( H > 1/2 \) be the Hurst coefficient of the fBm. As is well known \( B^H_t \) has a.s. finite \((H - \eta)\)-Hölder continuous trajectories for all \( \eta > 0 \).

In order to give more precise results, following [20], we state the next Lemma due to [9].

**Lemma 1.** Let \( p \geq 1 \) and \( \lambda > 1/p \). Then, there exists a constant \( C_{\lambda, p} > 0 \) such that for any continuous function \( f \) on \([0, T]\) for all \( s, t \in [0, T] \) the following holds true

\[
|f(s) - f(t)|^p \leq C_{\lambda, p} |s - t|^\lambda p - 1 \int_0^T \int_0^T \frac{|f(x) - f(y)|^p}{|x - y|^\lambda p + 1} \, dx \, dy
\]
An application of Lemma 1, [20], yields

**Lemma 2** (Lemma 7.4 [20]). Let $0 < H < 1$. For every $0 < \eta < H$ and $T > 0$, there exists a positive r.v. $\xi_{n,T}$ such that $E\xi_{n,T}^p < \infty$ for all $p \in [1, \infty)$ and for all $s, t < T$

$$|B_t^H - B_s^H| \leq \xi_{n,T}|t - s|^{H-\eta} \quad a.s.$$  

Actually, Lemma 2 can be shown without using Lemma 1 (using Theorem 1.2.3 of [7]), but we require the latter further on, so we have included it for the sake of completeness.

Set $C^{\alpha,\infty}([0, 1], \mathbb{R})$ to be the set of continuous functions $f : [0, 1] \to \mathbb{R}$ such that

$$\|f\|_{\alpha,\infty} := \sup_{t \in [0,1]} |f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{\alpha+1}}ds < \infty. \quad (8)$$

By Lemma 2, for $\mathbb{P}$—almost all $w \in \Omega$, $B^H_H(w) \in C^{\nu,\infty}([0, 1], \mathbb{R})$ as long as $\nu < H$.

Let $C^{\alpha}([0, 1], \mathbb{R})$ be the set of $\alpha$-Hölder continuous functions $f : [0, 1] \to \mathbb{R}$. Then clearly $C^{\alpha+\eta}([0, 1], \mathbb{R}) \subset C^{\alpha,\infty}([0, 1], \mathbb{R}) \subset C^{\alpha-\eta}([0, 1], \mathbb{R})$, for $\eta > 0$.

If $f \in C^{\alpha}$ for $1 - H < \alpha < 1/2$, then the stochastic integral with respect to fBm can be defined (23) by

$$\int_0^t f(s)dB^H_s = (-1)^\alpha \int_0^t D^\alpha_{0+}f_0+(s)D^1_{t-}B^H_t(s)ds + f(0)(B^H(t) - B^H_s(0+)), \quad (9)$$

where

$$D^\alpha_{0+}f(s) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(s)}{s^\alpha} + \alpha \int_0^s \frac{f(s) - f(u)}{(s-u)^{\alpha+1}}du \right) \mathbb{1}_{(0,1)}(s),$$

$$D^\alpha_{t-}f(s) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(s)}{(t-s)^\alpha} + \alpha \int_s^t \frac{f(s) - f(u)}{(u-s)^{\alpha+1}}du \right) \mathbb{1}_{(0,1)}(s),$$

$$f_{a+}(s) = (f(s) - f(a+))\mathbb{1}_{(0,1)}(s)$$

and

$$f_{b-}(s) = (f(s) - f(b-))\mathbb{1}_{(0,1)}(s)$$

with $f(a+) = \lim_{\eta \downarrow a} f(s + \eta)$, $f(b-) = \lim_{\eta \uparrow b} f(s - \eta)$, whenever the limits exist and are finite.

At this stage we also introduce the norm ([20])

$$\|f\|_{\alpha,\text{dif}} := \sup_{s < r \in [0,1]} \left( \frac{|f(t) - f(s)|}{(t-s)^\alpha} \right) + \sup_{t \in [0,1]} \int_0^t \frac{|f(t) - f(u)|}{(t-u)^{\alpha+1}}du < \infty \quad (10)$$

and the quantity

$$A_\alpha(g) := \frac{1}{\Gamma(\alpha)} \|g\|_{1-\alpha,\text{dif}}. \quad (11)$$

If $1 - H < \alpha < 1/2$, then by Lemma 2 for $\eta < \rho = \alpha - (1 - H)$,

$$\sup_{s \in [0,1]} |D^1_{t-}B^H_t(s)| \leq A_\alpha(B^H) \leq \frac{1}{\Gamma(\alpha)\rho} \xi_{n,1}. \quad (12)$$
Consider the fractional diffusion equation given in Eq. (2) and assume \( \sigma \) and \( b \) are \( \mathbb{P} \)-measurable functions that satisfy the following conditions:

**A1** Function \( \sigma (t, x) \in C(\mathbb{R}^+ \times \mathbb{R}) \) is differentiable in \( x \) and there exists a positive constant \( K \) such that

1. There exists \( 0 \leq \gamma \leq 1 \), such that
   \[ |\sigma (t, x)| \leq K (1 + |x|^\gamma). \]

2. \[ |\sigma (t, x) - \sigma (t, y)| \leq K |x - y|, \]
   for all \( t \in [0, 1] \).

3. There exists \( 1 - H < \beta < 1 \) such that
   \[ |\sigma (t, u) - \sigma (s, u)| + |\partial_x \sigma (t, u) - \partial_x \sigma (s, u)| \leq K |t - s|^{\beta}, \]
   for all \( t, s \in [0, 1] \) and \( u \in \mathbb{R} \).

4. For all \( M \) there exists a constant \( K_M \) such that
   \[ |\partial_x \sigma (t, u) - \partial_x \sigma (t, v)| \leq K_M |u - v|, \]
   for all \( t \in [0, 1] \).

**A2** Function \( b(t, x) \in C(\mathbb{R}^+ \times \mathbb{R}) \) is such that

1. There exists a positive constant \( C \) such that
   \[ |b(t, x)| \leq C (1 + |x|) \]

2. For all \( M \) there exists a constant \( C_M \) such that
   \[ |b(t, u) - b(t, v)| \leq C_M |u - v|, \]
   for all \( t \in [0, 1] \).

**A3** For all \( M \) there exists a constant \( L_M \) such that

\[ \frac{|b(t, x)|}{\sigma (t, x)} \leq L_M, \quad \forall |x| \leq M. \]

**Remark 1.** Assumption A3 is required in the proof of Theorems 2 and 3.

Assumptions A1 and A2 are essentially those given in [20], and constants \( K_M \) and \( C_M \) may be random. They yield the existence of a global and unique path-wise defined solution \( X_t \) which satisfies nice local regularity and integrability conditions, as given by norm \( \| \cdot \|_{\alpha, \infty} \) defined in (8). This is what is done by Nualart and Rascanu [20], of which we transcribe their main result.

**Theorem 1.** Assume A1 and A2 are satisfied and let \( \alpha \in (1 - H, \beta \lor 1/2) \). Then

1. There exists a unique stochastic process, solution of the fractional diffusion (2) and such that for \( \mathbb{P} \)-almost all \( w \in \Omega \)
   \[ X(w, \cdot) \in C^{1-\alpha} ([0, 1], \mathbb{R}), \]
   which in particular yields \( X(w, \cdot) \in C^{\alpha, \infty} ([0, 1], \mathbb{R}) \).

2. Moreover if \( \gamma = 0 \), then
   \[ \mathbb{E} \| X \|_{\alpha, \infty}^p < \infty, \quad \forall p \geq 1. \]
2.1. Approximating the solution

As is clear from the construction, the resulting process will not be Gaussian in general nor will it have independent increments. However, again by construction, it is tempting to believe that the joint behaviour of the solution process and its increments can be obtained based on the underlying Gaussian structure of the problem. This intuition turns out to be true if we take advantage of the pathwise properties of the solution discussed above and approximate $X_t$ by a cylinder process $X_t^m$ which we construct below.

Start by considering a fixed collection of time grid points $t_1, \ldots, t_m$ and construct the piecewise linear approximation of $B_t^H$ over this grid,

$$B_t^{H,m} = B_{t_j}^H + \left( t - t_j \right) \frac{B_{t_{j+1}}^H - B_{t_j}^H}{t_{j+1} - t_j}, \quad \text{if } t_j \leq t < t_{j+1}.$$

Note, $B_t^{H,m}$ is $\mathbb{P}$ a.s. a.e. differentiable over $[0, 1]$, so that

$$\int_0^t f(s) dB_{s,m}^H,$$

can be defined in the usual Riemann–Stieltjes sense for any continuous function $f$. Moreover, again by Lemma 2 we have, for $\rho$ defined as in Eq. (12),

$$\sup_{s \in [0,1]} |D_t^{1-\alpha} B_t^{H,m}(s)| \leq \Lambda_\alpha(B_t^{H,m}) \leq \frac{1}{\Gamma(\alpha)\rho} \zeta_{\eta,1}. \quad (13)$$

Thus, the integral can be defined by fractional derivatives as long as $f \in C^\alpha$ for $\alpha > 1 - H$. Note in Eq. (13) the bound is independent of index $m$.

This in turn implies we can introduce,

$$\int_0^t f(s) dB_{s,m}^{H,H} = \int_0^t f(s) dB_{s,m}^H - \int_0^t f(s) dB_s^H,$$

for any $f \in C^\alpha([0, 1], \mathbb{R})$ with $\alpha > 1 - H$.

Now define $X_t^m$ the solution of Eq. (2) with $B_t^{H,m}$ instead of $B_t^H$.

We have the following lemma

**Lemma 3.** 1. For $\mathbb{P}$–almost all $w \in \Omega$, $X_t^m \in C^{1-\alpha,\infty}([0, 1], \mathbb{R})$ for $1 - \alpha < H$. Thus, $\|X^m\|_{\alpha,\infty} < \infty$.

2. Moreover, let $A_M = \{ \|X^m\|_{\infty} < M, \|X\|_{\infty} < M \}$. Then, over $A_M$, $\|X^m - X\|_{1-\alpha,\infty} \to 0$ for $1 - H < \alpha < 1/2$.

**Proof.** The first part of the Lemma follows directly by Theorem 5.1 in [20] and the properties of $B_t^{H,m}$.

For the proof of the second part we will make use of certain inequalities due to [20]. We will begin by showing that

$$\|X^m - X\|_{\alpha,\infty} \to 0.$$ 

For this we must introduce an auxiliary norm, which is equivalent to $\| \cdot \|_{\alpha,\infty}$, defined by Nualart and Rascanu [20]

$$\| f \|_{\alpha,\lambda} := \sup_{t \in [0,1]} \exp(-\lambda t) \left( |f(t)| + \int_0^t |f(t) - f(s)| \frac{1}{(t-s)^{\alpha+1}} ds \right), \quad (14)$$

for all $f \in C^{\alpha,\infty}([0, 1], \mathbb{R})$ and $\lambda > 0$. 

Write
\[ \|X^m - X\|_{\alpha, \lambda} \leq \left\| \int_0^t (b(s, X^m_s) - b(s, X_s))ds \right\|_{\alpha, \lambda} \]
\[ + \left\| \int_0^t (\sigma(s, X^m_s) - \sigma(s, X_s))dB^H_m s \right\|_{\alpha, \lambda} \]
\[ + \left\| \int_0^t \sigma(s, X_s)dB^H s \right\|_{\alpha, \lambda} \]}
\[= I + II + III. \tag{15} \]

Let \( M = \max(\|X\|_\infty, \|X^m\|_\infty) \). By A2, and Proposition 4.4 in [20], there exists a (random) constant \( d_M \) which depends on \( \alpha \) and \( C_M \) such that for all \( \lambda \geq 1 \),
\[ I \leq \frac{d_M}{\lambda^{1-\alpha}} \|X^m - X\|_{\alpha, \lambda}. \]

Choosing a big enough (random) \( \lambda \) renders
\[ I \leq \|X^m - X\|_{\alpha, \lambda}/3. \tag{16} \]

On the other hand, Proposition 4.2 in [20] and A1 yield there exists a (random) constant \( c_M \) which depends on \( \alpha \) and \( K_M \) such that
\[ II \leq A_\alpha(B^{H, m} - B^H) \frac{c_M}{\lambda^{1-\alpha}} (1 + \|X\|_{\alpha, \infty} + \|X^m\|_{\alpha, \infty}) \|X^m - X\|_{\alpha, \lambda}. \]

Once again, for big enough (random) \( \lambda \)
\[ II \leq \|X^m - X\|_{\alpha, \lambda}/3. \tag{17} \]

Finally, consider III. By Proposition 4.2 and A1 in [20], there exists a (random) constant \( k_M \) which depends on \( \alpha \) and \( C_M \) such that for all \( \lambda > 0 \)
\[ III \leq A_\alpha(B^{H, m} - B^H) \frac{k_M}{\lambda^{1-\alpha}} (1 + \|X\|_{\alpha, \infty}). \]

It remains to bound \( A_\alpha(B^{H, m} - B^H) \). Consider \( \epsilon > 0 \). By definition, \( A_\alpha(B^{H, m} - B^H) \) is bounded by
\[ \frac{2}{\Gamma(\alpha)} \|B^{H, m} - B^H\|_\infty^\epsilon \sup_{s < t \leq [0, 1]} \left( \frac{|B^{H, m}_t - B^H_t - B^{H, m}_s - B^H_s|^{1-\epsilon}}{(t-s)^\alpha} \right) \]
\[ + \int_0^t \frac{|B^{H, m}_t - B^H_t - B^{H, m}_s - B^H_s|^{1-\epsilon}}{(t-s)^{\alpha+1}} ds. \]

If we choose \( \epsilon \) small enough, by Lemma 2, there exists a positive constant \( C \) and an a.s. bounded random variable \( \zeta \) which depends on \( \alpha, \epsilon \) and \( H \) but not on \( m \), such that
\[ A_\alpha(B^{H, m} - B^H) \leq C \zeta \|B^{H, m} - B^H\|_\infty^\epsilon. \tag{18} \]

To end the proof we have by (16)–(18) for big enough (random) \( \lambda \)
\[ \|X^m - X\|_{\alpha, \lambda} \leq 3C \zeta \|B^{H, m} - B^H\|_\infty^\epsilon, \]
or equivalently
\[\|X^m - X\|_{\alpha, \infty} \leq e^{-\lambda 3C \xi} \|B^{H,m} - B^H\|_{\infty}.\]

We end the proof by recalling that \(B^{H,m} \to B^H\) for almost all \(\mathbb{P} - w\).

Next, as in Eq. (15), write
\[
|X_t^m - X_t - X_s^m + X_s| \leq \left| \int_s^t (b(s, X_s^m) - b(s, X_s)) \, ds \right| \\
+ \left| \int_s^t (\sigma(s, X_s^m) - \sigma(s, X_s)) \, dB_{s}^{H,m} \right| \\
+ \left| \int_s^t \sigma(s, X_s) d[B_{s}^{H,m} - B_s^H] \right| \\
= J + JJ + JJJ. \tag{19}
\]

By A2, there exists a (random) constant \(d_M\) which depends on \(\alpha\) and \(C_M\) such that
\[J \leq d_M \|X - X^m\|_{\alpha, \infty}|t - s|. \tag{20}\]

Next by proposition 4.1 in [20], there exists a random constant \(d_M'\) such that
\[JJ \leq c_\alpha \Delta \alpha(B^{H,m}) \|\sigma(\cdot, X^m) - \sigma(\cdot, X)\|_{\alpha, \infty}|t - s|^{1-\alpha} \leq d_M' \|X - X^m\|_{\alpha, \infty}|t - s|^{1-\alpha}. \tag{21}\]

Finally, again by proposition 4.1 in [20] there exists a random constant \(d_M''\) such that
\[JJJ \leq d_M'' \Delta \alpha(B^{H,m} - B^H)(1 + \|X\|_{\alpha, \infty})|t - s|^{1-\alpha}. \tag{22}\]

This ends the proof. \(\Box\)

3. Main result

Throughout the rest of the article we will assume \(\sigma\) and \(b\) are deterministic functions. Set \(\Delta X_k = X_{(k+1)\Delta} - X_k\Delta\) and \(\Delta B_k^H = B_{(k+1)\Delta}^H - B_k^H\). Define
\[\rho(k - i) = \mathbb{E} \Delta B_k^H / \Delta^H \Delta B_k^H / \Delta^H. \tag{23}\]

We have \(\rho(i) \sim H(2H - 1)i^{2H-2}\) for large values of \(i\).

Consider the broken line approximation of \(X_t\) at step \(\Delta = 1/n\),
\[X_n(t) = X_{k\Delta} + n(t - k\Delta)(X_{(k+1)\Delta} - X_k\Delta), \quad k\Delta \leq t < (k + 1)\Delta. \tag{24}\]

Note \(\hat{X}_n(t)\) is defined for each \(n\) except on a finite number of points. We will use this notation for the piece-wise derivative.

We are interested in the following functional of \(X_n(t)\)
\[
\hat{Z}_n(t) = \int_0^t f(X_n(s))G(\hat{X}_n(s)n^{H-1}) \, ds. \tag{25}\]

Set
\[
(G)_{(\sigma(t, X_t))} = \mathbb{E}_\mathcal{F}_t G(\sigma(t, X_t)Z), \tag{26}\]
for $Z \sim N(0, 1)$ and independent of $B^H_t$. Here $\mathbb{E}_{\mathcal{F}_t}$ stands for the conditional expectation given by $\mathcal{F}_t$.

Before we state our main result we must include some additional notation. Let $H_k = (-1)^k \exp(x^2)k!/(k-1) \exp(-x^2)$ stand for the $k$-th Hermite polynomial. For any function $g \in L^2(e^{-x^2}/2)$, let $c_k(g) = 1/k! \mathbb{E}g(Z)H_k(Z)$, with $Z \sim N(0, 1)$. As $\mathbb{E}H_k(Z)^2 = k!$, we have $\mathbb{E}g^2 = \sum_{k=0}^{\infty} c_k^2(g)k!$. If $g_1, g_2 \in L^2(e^{-x^2}/2)$, then we also have $\mathbb{E}g_1g_2 = \sum_{k=0}^{\infty} c_k(g_1)c_k(g_2)k!$.

We next define certain terms which will be useful in the sequel.

**Definition 1.**

1. For $H < 3/4$ and $k \geq 2$, define $\rho^k = 1 + 2 \sum_{i=1}^{\infty} \rho^k(i) < \infty$, where $\rho^k(i)$ was defined in Eq. (23).
2. Set $a_k(X_s, G, \sigma) = c_k(G(\sigma(s, X_s)Z))$. Define
   \[
   C_{H,G,\sigma}(X_s) = \sum_k a_k^2(X_s, G, \sigma)k!\rho^k.
   \] (27)

Note that since $\sigma$ is deterministic, $\sum_k a_k^2(X_s, G, \sigma)k!$ is just the conditional variance of $G(\sigma(t, X_s)Z)$ given $\mathcal{F}_t$.

Introduce
\[
Z(t) = \int_0^t f(X_s)G_s(\sigma(s, X_s))ds,
\] (28)
where $G_s$ was defined in Eq. (26) and define
\[
U_n(t) = \sqrt{n}(\hat{Z}_n(t) - Z(t)).
\] (29)

We also require the following definition.

**Definition 2.** $U(t)$ is the conditionally Gaussian centered process with independent increments, whose moment generating function is given by
\[
\mathbb{E}e^{s(U(t) - U(s))} = \mathbb{E}e^{s^2/2(\int_0^t f^2(X_s)C_{H,G,\sigma}(X_s)ds)}.
\]

The next assumption assures that variable $U(t)$ can be defined by its moments.

**C1** $f, \sigma, G$ and $X$ are such that there exists $\xi > 0$ such that, for all $t \in (0, 1)$,
\[
\mathbb{E}e^{s^2/2(\int_0^t f^2(X_s)C_{H,G,\sigma}(X_s)ds)} < \infty.
\]

Note that under **C1** $U(t)$ can be interpreted as $W(\int_0^t f^2(X_s)C_{H,G,\sigma}(X_s)ds)$ with $W$ a Wiener process, independent of $X$. Using this interpretation, we can write $U(t) = \int_0^t f(X_s)\sqrt{C_{H,G,\sigma}}(X_s)dW_s$.

Given a function $h$, we will say it has exponential growth if there exists $a > 0$, such that $|f(x)| \leq e^{a|x|}$. In this case $\mathbb{E}f^p(Z) < \infty$, for all $p$, and $Z \sim N(0, 1)$. Given a function $f$ we will say it has sub-exponential growth if $f \circ h$ has exponential growth for all $h$ with exponential growth. Note that if $f$ has sub-exponential growth, $|f(x)| \leq C(1 + |x|^p)$ for some positive constants $C, p$. We introduce the following conditions. The first condition is related to an almost multiplicative structure for function $G$. This assumption is true for polynomials and polynomials of the absolute value, for example.
B1 Function $G$ defined in Eq. (25) is such that there exist functions $g_j : \mathbb{R}^+ \to \mathbb{R}$, $j = 0, \ldots, r$ and $T_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \ldots, r$ in such a way that for strictly positive $a$

$$G(xa) = \sum_{j=1}^r g_j(a)T_j(x) + g_0(a).$$

In this case we have the representation $G(x) = \sum_{j=1}^r g_j(1)T_j(x) + g_0(1)$. Assume
1. The functions $g_j(x)$, $j = 1, \ldots, r$ are twice continuously differentiable for $x > 0$ and have sub-exponential growth as well as their derivatives.
2. The functions $T_j(x)$, $j = 1, \ldots, r$ are even.
3. The functions $T_j(x)$, $j = 1, \ldots, r$ are continuous and twice continuously differentiable for $x \neq 0$.
4. There exist positive functions $R_1$ and $R_2$ with sub-exponential growth such that $|T_j'(x + y)| \leq R_1(x) + R_2(y)$.
5. There exist positive functions $S_1$ and $S_2$ with sub-exponential growth such that $|T_j''(x + y)| \leq S_1(x) + S_2(y)$.

B2 Function $f$ defined in Eq. (25) is continuously differentiable. Both $f$ and $f'$ have sub-exponential growth.

Our main result is then,

**Theorem 2.** Assume that function $G$ satisfies condition B1. Assume that function $f$ satisfies B2. Assume $\sigma$ and $b$ are deterministic functions that satisfy A1–A3. Assume condition C1 holds. Then, for $1/2 < H < 3/4$

$$U_n(t) \xrightarrow{\text{weakly}} U(t).$$

(30)

4. The case $3/4 \leq H < 1$

Theorem 2 cannot be extended to consider the case $H \geq 3/4$ as becomes apparent by analyzing the variance, which depends on $\zeta(2k(1 - H))$, for $k \geq 2$ and $\zeta(x)$ the Zeta function. One solution is of course to consider functions $T_j$, $j = 1, \ldots, r$ with higher order vanishing moments, i.e. such that $c_k(T_j) = 0$ for $k > 1/(2 - 2H)$, but this is not satisfactory if we only know that $H < 1$. Another approach is the following. Consider the process $\Delta^2B_t^H = B^H((i + 1)/n) - 2B^H(i/n) + B^H((i - 1)/n)/\Delta^H$. Then $\Delta^2B_t^H$ is a centered, stationary Gaussian process with covariance function $\eta(i)$ such that $\eta(0) = 2(2 - 2^H)/2$ and $\eta(i) \sim H(2H - 1)(2H - 2)(2H - 3)i^{2H-4}$ for large values of $i$. Note that for $H = 1$, the variance is zero.

On the other hand, by direct calculation $\text{Cov}(\Delta^2B_t^H, B_s^H) = O((i^{2H-2} - [(ns) - i]^{2H-2})/([ns]^{H-1})) = O((ns)^{-H}) = O(\Delta^H)$ for fixed $0 < s \leq 1$.

Now set $\eta^k = 1 + 2\sum_{i=1}^\infty (\eta(i)/\eta(0))^k < \infty$, for all $k \geq 1$, and let

$$D_{H,G,\sigma}(X_s) = \sum_k a_k^2(X_s, G, \sigma)k!\eta^k.$$ 

Define the following functional closely related to $Z_n(t)$,

$$\tilde{Z}_n(t) = \frac{1}{n} \sum_{i=1}^{|nt|} f(X(i/n))G(\Delta^2X(i/n)/(|\eta(0)\Delta^H|), (31)$$
where \( \Delta^2 X(i/n) = X((i + 1)/n) - 2X(i/n) + X((i - 1)/n) \). Also define,
\[
\tilde{U}_n(t) = \sqrt{n}(\tilde{Z}_n - Z(t)).
\]

Let \( \tilde{U}(t) \), be a centered conditionally Gaussian process with independent increments and whose moment generating function is given by
\[
E e^{\xi (\tilde{U}(t) - \tilde{U}(s))} = E e^{\xi^2 / 2(\int_s^t f^2(X_u)D_{H,G,\sigma}(X_u)du)}.
\]

Assume

**C1’** \( f, \sigma, G \) and \( X \) are such that there exists \( \xi > 0 \) such that, for all \( t \in (0, 1) \),
\[
E e^{\xi^2 / 2(\int_0^t f^2(X_s)D_{H,G,\sigma}(X_s)ds)} < \infty.
\]

We have,

**Theorem 3.** Assume that function \( G \) satisfies condition **B1**. Assume that function \( f \) satisfies **B2**. Assume \( \sigma \) and \( b \) are deterministic functions that satisfy **A1–A3**. Assume condition **C1’** holds. Then, for \( 1/2 < H < 1 \)
\[
\tilde{U}_n(t) \xrightarrow{\text{weakly}} \tilde{U}(t).
\]

**Proof.** It follows exactly as for the proof of Theorem 2 from the following lemma. \( \square \)

**Lemma 4.** Define \( S_n^{G} = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} G(\Delta^2 B_i^H) \). Let \( G \) be such that \( E G(Z) = 0 \) and \( E G'(Z) < \infty \) for some \( r \geq 3p, p > 2 \) and \( Z \sim N(0, 1) \). Then, there exists a constant \( C_p \), which depends on \( p, G, H \) but not on \( n \) such that
\[
E \| S_n^{G} \|_{\text{var}(p)} \leq C_p.
\]

The proof of the Lemma follows exactly that of Lemma 6. Note that function \( G \) is not required to be even since \( \sum_i \eta(i) < \infty \).

**Remark 2.** The set of conditions **B1** can be relaxed by not requiring the functions \( T_j, j = 0, \ldots, r \) to be even since \( \sum_i \eta(i) < \infty \).

### 5. Weak solutions

Another approach results from considering weak solutions for Eq. (2). This possibility is based on a Girsanov theorem proven by Decreusefond and Ustunel [6] for fractional Brownian motion (also see [5] and [11]).

For this we will assume

**A1’** \( \sigma : \mathbb{R} \to \mathbb{R} \) is a \( C^1 \) function such that
\[
\sigma(x) \geq c > 0,
\]
\[
\sigma(x) \leq M
\]
and
\[
|\sigma(x) - \sigma(y)| \leq K|x - y|.
\]
A2' \ b : \mathbb{R} \times [0, 1] \to \mathbb{R} \text{ is a } C^1 \text{ function such that }
\begin{align*}
|b(x, t)| & \leq M, \\
|b(x, t) - b(x, s)| & \leq K|t - s|
\end{align*}

and

\[|b(x, t) - b(y, t)| \leq K|x - y|.\]

Set \( X_t = h(B^H_t) \) where \( h \) satisfies the O.D.E

\[
\begin{cases}
    dh(t) = \sigma(h(t))dt \\
h(0) = x_0.
\end{cases}
\]  

Now consider \( B^{H,1}_t = B^H_t - y(t) \) where \( y(t) \) solves the random O.D.E.

\[
\begin{cases}
    dy(t) = \frac{b(h(B^H_t), t)}{\sigma(h(B^H_t))}dt \\
y(0) = 0.
\end{cases}
\]

With this notation \( X_t = h(B^{H,1}_t + y(t)) \) and solves the stochastic differential equation

\[X_t = x_0 + \int_0^T b(s, X_s)ds + \int_0^T \sigma(X_s)dB^{H,1}_s.\]  

The above stochastic integral is well defined as \( B^{H,1} \) has the same pathwise properties as \( B^H \). By Theorem 4.1 in [5] and Theorem 6.1 in [6] there exists a probability measure \( P^1 \) which is absolutely continuous with respect to \( P \) and such that the law of \( B^{H,1} \) under \( P^1 \) is the same as the law of \( B^H \) under \( P \), as long as we show that for any \( C > 0 \), (Lemma 9 in [11])

\[\mathbb{E} \|D^{H,1/2}_0y\|_2^2 < \infty.\]  

This we do below. Hence, with the change of measure \( X_t \) solves Eq. (2) under \( P^1 \). On the other hand, \( X_t = h(B_t) \), so that under \( P \), \( \sqrt{n}(Z_n(t) - Z(t)) \) is asymptotically conditionally Gaussian. In order to deal with the change of measure we require a modified version of Theorem 2 for the vanishing drift case. Before stating the theorem we introduce some necessary notation (the notation is as in [13] and [14]).

Let \( Y_n \) be a sequence of r.v. with values in a Polish space \( E \), defined over a probability space \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}) \) and let \( \mathcal{G} \) be a sub-\( \sigma \)-field of \( \mathcal{F} \). Let \( Y \) be an \( E \) valued r.v. defined over an extension \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}) \) of the original probability space. Then, \( Y_n \) converges \( \mathcal{G} \)-stably to \( Y \) if

\[\lim_n \mathbb{E}(Uh(Y_n)) = \tilde{\mathbb{E}}(Uh(Y)),\]  

for any bounded continuous \( h : E \to \mathbb{R} \) and all bounded \( \mathcal{G} \) measurable r.v. \( U \). In particular, if \( Y_n \) converges \( \mathcal{G} \)-stably to \( Y \), and if \( Q \ll P \), then under \( Q \), \( Y_n \) converges to \( Y \) in distribution. We will apply this definition to the sequence \( \tilde{Z}_n(t) \), defined by

\[\tilde{Z}_n(t) = \sqrt{n}(\hat{Z}_n(t) - Z(t)).\]  

We assume \( E = C[0, 1] \) endowed with the uniform topology. Let \( \mathcal{F}_t \) be the \( \sigma \)-field generated by \( B^H_s \), \( 0 \leq s \leq t \). Set \( \mathcal{F} = \bigvee \mathcal{F}_t \), \( t \leq 1 \). Now consider an extension of the original filtered space \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{P}) \), such that there exists a Wiener process, \( W_t \) with respect to a filtration \( \mathcal{G}_t \) (sub
\( \sigma \)-field of \( \tilde{F}_t \), which is independent of the original process \( B^H \) (this can be done by defining \( W \) as the canonical process on the canonical space \( (\Omega_1, \mathcal{G}, \mathcal{G}_1, P_1) \) and setting \( \tilde{\Omega}, \tilde{F}, \tilde{F}_t \) to be the product of \( (\Omega, \mathcal{F}, \mathcal{F}_t) \) by \( (\Omega_1, \mathcal{G}, \mathcal{G}_1) \), \( P(dw, dw_1) = P(dw)P_1(dw_1) \)). We have,

**Theorem 4.** Assume that function \( G \) satisfies condition \( B1 \). Assume that function \( f \) satisfies \( B2 \). Assume \( X_t = h(B_t^H) \) where \( h \) is a solution of (33). Assume \( \sigma \) is a deterministic function that satisfies \( A1 \). Assume condition \( C1 \) holds. Then, for \( 1/2 < H < 3/4 \)

\[
\zeta_n(t) = \sqrt{n}(\hat{Z}_n(t) - Z(t)) \xrightarrow{\text{stably}} U(t).
\]

In turn this result yields

\[
\zeta_n(t) \xrightarrow{\text{weakly}} U(t),
\]

if (36) holds.

In what follows we will show that (36) holds if assumptions \( A1' \), \( A2' \) are satisfied. To begin with, note that

\[
D_{0+}^\alpha f(u) = \frac{1}{1 - [\alpha]} \frac{d^{[\alpha]+1}}{du^{[\alpha]+1}} \int_0^u f(s)(u-s)^{-[\alpha]} ds,
\]

for \( 1 < \alpha \). Here \([\alpha]\) and \([\alpha]\) denote the integer and fractional part of \( \alpha \) respectively.

A bit of manipulation shows that if \( f \in C^{1+\alpha+\epsilon} \) for \( \epsilon > 0 \), then

\[
D_{0+}^\alpha f(u) = \frac{1}{1 - [\alpha]} \left( f'(u) u^{[\alpha]} + \left[\alpha\right] \int_0^u f'(u) - f'(s) (u-s)^{1+[\alpha]} \right).
\]

Now consider \( y(t) \) the solution of (34). Hence, for \( \alpha = H + 1/2 \),

\[
\left\| D_{0+}^{H+1/2} y \right\|_2 \leq C \int_0^1 \left( \frac{y'(u)}{u^{H-1/2}} \right)^2 + C \int_0^1 \left( \int_0^u \frac{y'(u) - y'(s)}{(u-s)^{H+1/2}} \right)^2 + C \int_0^1 \left( \int_{|s-u|<\delta} \frac{y'(u) - y'(s)}{(u-s)^{H+1/2}} \right)^2.
\]

Under \( A1' \) and \( A2' \) for any fixed \( \delta \) the first two terms are bounded. In order to prove (36) it is necessary to show that there exists \( \delta \) such that

\[
\mathbb{E} e^{C \int_0^1 \left( \int_{|s-u|<\delta} \frac{y'(u) - y'(s)}{(u-s)^{H+1/2}} \right)^2} < \infty.
\]

Recall under \( A1' \), \( A2' \) function \( |y'| \) is uniformly bounded by a certain constant \( c \). Set \( Z(u, s) = \frac{|B^H - B^H_{|u-s|}|}{|u-s|^H} \). Applying Jensen’s inequality three times and setting \( C' = 4Cc^2 \),

\[
\mathbb{E} e^{C \int_0^1 \left( \int_{|u-s|<\delta} \frac{y'(u) - y'(s)}{(u-s)^{H+1/2}} \right)^2} \leq \mathbb{E} e^{\frac{1}{4\delta} \int_0^1 \int_0^\delta \int_0^\delta \exp\{C'\delta^2Z(u, s)\} \exp\{C'\delta^2Z(u, s)\}\left[\frac{1}{|u-s|^{1/2}|u-t|^{1/2}} \right]^2 dt ds du} \leq \frac{1}{4\delta} \int_0^1 \int_0^\delta \int_0^\delta \mathbb{E} \left[ \left| \frac{1}{|u-s|^{1/2}|u-t|^{1/2}} \right|^2 \right] ds du \leq \frac{1}{4\delta} \int_0^1 \left( \int_0^\delta \mathbb{E}^{1/2} \left| \frac{1}{|u-s|^{1/2}} \right| ds \right)^2 du.
\]

The last expression is bounded if \( \delta \) is small enough. This ends the proof.
6. Proofs

Proof of Theorem 2: Before giving the proof we introduce some notation and some technical lemmas. Let \( \| f \|_{\text{var}(p)} = \sup_{\Pi} \sum_{r_1 \in \Pi} | f(r_1) - f(r_{1-1}) |^p / p \) be the \( p \)-variation semi-norm of a function \( f \). We will say that \( f \) has finite \( p \)-variation if \( \| f \|_{\text{var}(p)} < \infty \). \( p \)-variations are of course related to the local pathwise continuity of the processes we are considering.

Based on this notion of \( p \)-variation we have the following inequality due to [22] (as cited in [17]).

**Lemma 5.** Assume \( g \) has finite \( p \)-variation and \( f \) has finite \( p' \)-variation with \( 1/p + 1/p' > 1 \). Then,

\[
\left| \sum_{r_1 \in \Pi} f(r_1)(g(r_1) - g(r_{1-1})) \right| \\
\leq | f(t_0)(g(t_N) - g(t_0)) | + \zeta(1/p + 1/p') \| g \|_{\text{var}(p)} \| f \|_{\text{var}(p')}.
\]

Set \( S^G_n(t) = 1 / \sqrt{n} \sum_{i=1}^{[nt]} G(\Delta B^H_t^i / \Delta H) \). The following two lemmas bound the \( p \)-variations of \( S^G_n(t) \). In both cases the bound depends on the quantity \( \sum_{u \leq n} \rho^m(u) \), where \( m \) is the Hermite rank of function \( G \) and \( \rho(u) \) was defined in (23), although the methods of proof differ. The fact that if \( m \geq 2 \), since \( H < 3/4 \), then \( \lim_{n \to \infty} \sum_{u \leq n} \rho^m(u) < \infty \), yields Lemma 6. However if function \( G \) has Hermite rank one, there is no hope of bounding the \( p \)-variation by a constant. **Lemma 7** yields the divergence rate in this case.

**Lemma 6.** Let \( G \) be an even function such that \( \mathbb{E}G(Z) = 0 \) and \( \mathbb{E}G^r(Z) < \infty \) for some \( r \geq 3p \), \( p > 2 \) and \( Z \sim N(0, 1) \). Then, there exists a constant \( C_p \), which depends on \( p, G \) and \( H \) but not on \( n \) such that

\[
\mathbb{E}\| S^G_n \|_{\text{var}(p)} \leq C_p.
\]

**Proof.** The proof follows exactly as that of Lemma 2, using Lemma 1. Let \((S^G_n)^c(t) \) be the linear interpolation of \( S^G_n(t) \). For \( \epsilon > 0 \) we have

\[
|(S^G_n)^c(t) - (S^G_n)^c(s)| \leq |t - s|^{1/2 - \epsilon} \xi,
\]

with \( \mathbb{E} |\xi|^p < \infty \) for all \( p \geq 1 \). Indeed, set \( \lambda = (3/2 - \epsilon)/q \). As \((S^G_n)^c(t) \) is continuous by Lemma 1

\[
|(S^G_n)^c(t) - (S^G_n)^c(s)| \leq C_{\lambda, q} |s - t|^{1+ q-1} \int_0^1 \int_0^1 \frac{|(S^G_n)^c(u) - (S^G_n)^c(v)|^q}{|u - v|^2 q+1} \, du \, dv.
\]

Set

\[
\xi = \int_0^1 \int_0^1 \frac{|(S^G_n)^c(u) - (S^G_n)^c(v)|^q}{|u - v|^2 q+1} \, du \, dv
\]

it remains to see that \( \mathbb{E} |\xi|^p < \infty \). Without losing generality assume \( p \) is an integer.

For this we study \( |(S^G_n)^c(u) - (S^G_n)^c(v)| \) for \( |u - v| \leq 2/n \) and \( |u - v| > 2/n \). Set \( \Delta G(u) = [G(\Delta B^H_{[nu]+1} / \Delta H) - G(\Delta B^H_{nu} / \Delta H)] \). If \([nu] = [nv] \),

\[
|(S^G_n)^c(u) - (S^G_n)^c(v)| = \frac{\Delta G(u)}{\Delta^{1/2}} |u - v|.
\]
Hence
\[
\frac{|(S^G_n)^c(u) - (S^G_n)^c(v)|^q}{|u - v|^{2q+1}} \leq 2^q [G^q(\Delta B^H_{[nu]+1}/\Delta H) + G^q(\Delta B^H_{[nv]}/\Delta H)] \frac{\Delta^{q/2-3/2}}{|u - v|^{1-\epsilon}}.
\]
If \([nu] - [nv]| = 1,
\[
| (S^G_n)^c(u) - (S^G_n)^c(v) | \leq \frac{\max_{w=u,v} |\Delta G(w)|}{\Delta^{1/2}} |u - v|.
\]
So that,
\[
\frac{|(S^G_n)^c(u) - (S^G_n)^c(v)|^q}{|u - v|^{2q+1}} \leq 2^q \max_{u=v} [G^q(\Delta B^H_{[nu]+1}/\Delta H) + G^q(\Delta B^H_{[nv]}/\Delta H)] \frac{\Delta^{q/2-3/2}}{|u - v|^{1-\epsilon}}.
\]
If \(|u - v| > 2/n\), (assume \(u > v\))
\[
\frac{|(S^G_n)^c(u) - (S^G_n)^c(v)|^q}{|u - v|^{2q+1}} \leq C_q \max_{u=v} [G^q(\Delta B^H_{[nu]+1}/\Delta H) + G^q(\Delta B^H_{[nv]}/\Delta H)] \frac{\Delta^{q/2-3/2}}{|u - v|^{1-\epsilon}} + \frac{|(S^G_n)^c([nu]/n) - (S^G_n)^c(([nv] + 1)/n)|^q}{|u - v|^{5/2-\epsilon}}.
\]
Integrating \(p\) times, by Minkowski’s inequality
\[
\mathbb{E} \left( \int_0^1 \int_0^1 \frac{|(S^G_n)^c(u) - (S^G_n)^c(v)|^q}{|u - v|^{2q+1}} \, du \, dv \right)^p = \int_0^1 \cdots \int_0^1 \mathbb{E} \prod_{i=1}^p \frac{|(S^G_n)^c(u_i) - (S^G_n)^c(v_i)|^q}{|u_i - v_i|^{2q+1}} \, du_1 \cdots du_p \, dv_p
\]
\[
\leq C_{q,p} \left( \int_0^1 \int_0^1 \mathbb{E}^{1/p} \left[ \max_{u,v} G^q(\Delta B^H_{[nu]+1}/\Delta H) \right]^p \right)^{1/p} \frac{\Delta^{q/2-3/2}}{|u - v|^{1-\epsilon}} \, du \, dv
\]
\[
+ C_{q'} \int_{|u-v| > 2/n} \mathbb{E}^{1/p} \frac{|(S^G_n)^c([nu]/n) - (S^G_n)^c(([nv] + 1)/n)|^q}{|u - v|^{5/2-\epsilon}} \, du \, dv
\]
\[
\leq C_{1,q} + C_{2,q} \left( \int_0^1 \int_0^1 \mathbb{E}^{1/p} \frac{1}{|u - v|^{1-\epsilon}} \, du \, dv \right)^p.
\]
The last inequality comes from \(\mathbb{E}^{1/p} |(S^G_n)^c([nu]/n) - (S^G_n)^c(([nv] + 1)/n)|^q \leq C_{H,q} |[nu] - [nv]| - 1)^{q/2} n^{q/2}\) by Lemma 4.5 in [21], under the conditions over \(G\), and by taking \(q = 3\).
To end the proof, \((S_n^G)^c(t)\) and \(S_n^G(t)\) coincide over the grid \(s_i, i = 1, \ldots, n\). Thus,
\[
\mathbb{E}\|S_n^G\|_{\text{var}(p)} \leq \mathbb{E}\|(S_n^G)^c\|_{\text{var}(p)} \leq C_p'. \quad \square
\]

**Lemma 7.** Let \(G\) be such that \(\mathbb{E}G(Z) = 0\) and \(\mathbb{E}G^{p'}(Z) < \infty\) for \(Z \sim N(0, 1)\). Then, there exists a constant \(C_{p'}\), which depends on \(p'\), \(G\) and \(H\) but not on \(n\) such that
\[
\mathbb{E}\|S_n^G\|_{\text{var}(p')} \leq C_{p'} n^{H - 1/2}.
\]

**Proof.** The arguments are slightly different in this case, as we are only interested in the divergence rate. By definition, if \(\Pi\) is any partition of \([0, 1]\),
\[
\|S_n^G\|_{\text{var}(p')} = \sup_{\Pi} \sum_{i} |S_n^G(r_i) - S_n^G(r_i)|^{p'/2}.
\]
Since \(S_n^G\) is a step function these differences will only vary along the points \(s_i\), which are multiples of \(n\). Thus we may bound the last expression by
\[
\max_{1 < k_1 < k_2 < n} \frac{1}{(k_2 - k_1)^{p'/2}} \left| \sum_{i=k_1}^{k_2} G(\Delta B_i^H / \Delta H) \right|^{p'/2} \sup_{\Pi} \sum_{i} |r_{i+1} - r_i|^{p'/2}
\]
\[
\leq C \max_{1 < k_1 < k_2 < n} \frac{1}{(k_2 - k_1)^{p'/2}} \left| \sum_{i=k_1}^{k_2} G(\Delta B_i^H / \Delta H) \right|^{p'/2}
\]
\[
\leq C \max_{1 < k < n} \frac{1}{(k)^{p'/2}} \left| \sum_{i=1}^{k} G(\Delta B_i^H / \Delta H) \right|^{p'/2},
\]
the last equality in distribution because fBm has stationary increments. By Lemma 4.5 in [21], if \(G\) has a vanishing expectation, for any \(1 \leq k \leq n,
\[
\mathbb{E}\left( \frac{1}{(k)^{p'/2}} \left| \sum_{i=1}^{k} G(\Delta B_i^H / \Delta H) \right|^{p'/2} \right) \leq C \left( \sum_{u=-n}^{n} |\rho(u)| \right)^{p'/2},
\]
for \(\rho(u) = \text{Cov}(\Delta B_{i+1}^H / \Delta H, \Delta B_i^H / \Delta H) \sim H(2H - 1)|u|^{2(H - 1)}\). Set \(q(n) = \sum_{u=-n}^{n} |u|^{2(H - 1)}\). Function \(q\) satisfies the conditions of Theorem 3.1 in [18] with \(Q = 2^{2(1-H)} < 2\). Hence
\[
\mathbb{E}\left( \max_{1 \leq k < n} \frac{1}{(k)^{p'/2}} \left| \sum_{i=1}^{k} G(\Delta B_i^H / \Delta H) \right|^{p'/2} \right) \leq A(n^{2H-1})^{p'/2}
\]
and
\[
\mathbb{E}\|S_n^G\|_{\text{var}(p')} \leq C n^{H - 1/2}.
\]

The next lemmas give the basic technical tools for convergence in distribution. \(\square\)

**Lemma 8.** Assume \(Y = \{Y_i\}_{i=1,\ldots,\infty}\) is a collection of centered Gaussian variables such that \((Y_i, \ldots, Y_n)\) has correlation matrix \(C\) with \(C(i_1, i_2) = O(|i_1 - i_2|^{-\beta})\), \(\beta > 1/2\). Assume \(\nu^k := 1 + \lim_{n \to \infty} 2/n \sum_{1 \leq i < j \leq n} C(i, j)^k\) exists for \(k \geq 2\). Assume \(T\) is an even function such that \(\mathbb{E}T(Z) = 0\) and \(\text{Var}(T(Z)) < \infty\) for \(Z \sim N(0, 1)\). Let \(c_k(T)\) be the \(k\)-th Hermite coefficient.
of $T$ and define $C_{v,T} = \sum_k c_k^2(T)k!v^k$. Let $a(s)$ be a continuous square integrable function and set $a_i = a(i/n)$. Let $b_{i,n}$ and $\sigma_{i,n}$ be triangular arrays such that

1. $\sup_i b_{i,n} = O(\Delta^\gamma)$ with $\gamma > 1/4$.
2. $\sup_i (1 - \sigma_{i,n}^2) = O(\Delta^{2\gamma})$ with $\gamma > 1/4$.

Set $T_{i,n}(x) = T(\sigma_{i,n}x + b_{i,n})$. Then,

$$
\mathbb{E} \left( \frac{1}{n} \sum_{i=1}^{[nt]} a_i T_{i,n}(Y_i) \right)^p = p!! \left[ C_{v,T} \int_0^t a^2(s)ds \right]^{p/2} + o(1)
$$

for even $p$ and

$$
\mathbb{E} \left( \frac{1}{n} \sum_{i=1}^{[nt]} a_i T_{i,n}(Y_i) \right)^p = o(1)
$$

for odd $p$.

**Proof.** We divide the proof in two steps. Step 1:

1. $|\mathbb{E}T_{i,n}(Z)| = O(b_{i,n}^2 + 1 - \sigma_{i,n}^2)$ and
2. $|\mathbb{E}ZT_{i,n}(Z)| = O(b_{i,n}^2 + 1 - \sigma_{i,n}^2)$.

In order to simplify, we drop sub-indexes $i$, $n$ and will write $\sigma$ and $b$. Recall $\mathbb{E}T(Z) = 0$. Thus, for big enough $n$

$$
|\mathbb{E}T_{i,n}(Z)| = \left| \int_{-\infty}^{\infty} \frac{T(u)}{\sqrt{2\pi}} e^{-u^2/2} \left( e^{-u^2/(2\sigma^2)} - \frac{e^{-(u-b)^2/(2\sigma^2)}}{\sqrt{\sigma^2}} \right) du \right|
$$

$$
= \left| \int_{-\infty}^{\infty} \frac{T(u)}{\sqrt{2\pi}} e^{-u^2/2} \left( 1 - \frac{e^{-(1-\sigma^2)u^2/(2\sigma^2) e^{ub/\sigma^2} e^{b^2/(2\sigma^2)}}}{\sqrt{\sigma^2}} \right) du \right|
$$

$$
= 2 \left| \int_0^{\infty} \frac{T(u)}{\sqrt{2\pi}} e^{-u^2/2} \left( 1 - \frac{e^{-(1-\sigma^2)u^2/(2\sigma^2) e^{ub/\sigma^2} e^{b^2/(2\sigma^2)}}}{\sqrt{\sigma^2}} \right) du \right|
$$

$$
\leq \int_0^{\infty} \frac{T(u)}{\sqrt{2\pi}} e^{-u^2/2} \left( 1 - \frac{e^{-(1-\sigma^2)u^2/(2\sigma^2) e^{ub^2/(2\sigma^2) e^{b^2/(2\sigma^2)}}}}{\sqrt{\sigma^2}} \right) du
$$

$$
\leq 2\sqrt{2} \int_0^{\infty} \frac{T(u)}{\sqrt{2\pi}} \left( \frac{(bu)^2}{2\sigma^3} + \frac{b^2}{2\sigma^3} + \frac{(1-\sigma^2)u^2}{2\sigma^3} + 1 - \sigma^2 \right) e^{-u^2/2+(1-\sigma^2)u^2/(2\sigma^2)+a^2/(2\sigma^2) + b^2/(2\sigma^2)} du
$$

$$
\leq K \left( \int_0^{\infty} \left( \frac{(bu)^2}{2\sigma^3} + \frac{(1-\sigma^2)u^2}{2\sigma^3} + 1 - \sigma^2 \right)^{1/2} e^{-u^2/4} du \right)
$$

which yields the desired result. Equality 3 follows because $T(u)$ is even. Inequality 4 follows by remarking that $|e^x - 1| \leq |e^{|x|} - 1|$, expanding the cosh function and using the bound $(2k)! \geq 2^k k!$. Inequality 5 follows because if $0 < a < 1/2$,

$$
\left| \frac{e^x}{\sqrt{1-a}} - 1 \right| \leq \frac{|x| e^x}{\sqrt{1-a}} + \sqrt{2a}.
$$
In order to show the second assertion we proceed likewise, recalling that $E Z T(Z) = 0$ and the bound we just obtained.

**Step 2:** Set $T_{i,n}^1(x) = T_{i,n}(x) - E T_{i,n}(Z) - x E Z T_{i,n}(Z)$ for $Z \sim N(0, 1)$. Thus $T_{i,n}^1$ has Hermite rank 2. Write

$$
\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} a_{i,n} T_{i,n}(Y_i) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} a_{i,n} [T_{i,n}^1(Y_i) + E T_{i,n}(Z) + Y_i E Z T_{i,n}(Z)]
$$

$$
= \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} a_{i,n} T_{i,n}^1(Y_i) + r_n.
$$

Remark, $r_n$ is a Gaussian r.v. with $E(r_n)^2 = O(n^{1-4\gamma})$ and $\text{Var}(r_n) = O(n^{-2\gamma+1-\beta})$, so that $E(r_n^q) \to 0$ for all $q \geq 1$. By the binomial theorem

$$
E \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} a_i T_{i,n}(Y_i) \right)^p
$$

$$
= E \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} a_i T_{i,n}^1(Y_i) \right)^p + E \sum_{j=0}^{p-1} \binom{p}{j} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} a_i T_{i,n}^1(Y_i) \right)^j r_n^{p-j}.
$$

Hence, using Holder’s inequality for each term in the sum

$$
E \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} a_i T_{i,n}(Y_i) \right)^p = E \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} a_i T_{i,n}^1(Y_i) \right)^p + o(1),
$$

if we show that for all $q \geq 1$,

$$
E \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} a_i T_{i,n}^1(Y_i) \right)^q < \infty.
$$

The proof follows by expanding each $T_{i,n}^1$ in Hermite polynomials and by applying standard moment calculating results for products of Hermite polynomials as in [21] (similar techniques are also developed in [2] or [1]). Indeed, for example an application of Lemmas 4.3, 4.4 and 4.4 in [21] (much as in the proof of Proposition 4.2 in [21]) leads in the case of odd $p$ to

$$
E \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} a_i T_{i,n}^1(Y_i) \right)^p = o(1)
$$

and for even $p$ to

$$
E \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} a_i T_{i,n}^1(Y_i) \right)^p = p!! \left[ \frac{1}{n} \right]^{p/2} \sum_{i_1=1}^{[nt]} \cdots \sum_{i_p=1}^{[nt]} \sum_{k_1=1}^{k_p/2} \cdots \sum_{k_p=1}^{k_p/2} a_{i_1} \cdots a_{i_p}
$$

$$
\times k_1! \cdots k_p! c_{k_1}(T_{i_1}) c_{k_2}(T_{i_2}) \cdots c_{k_p/2}(T_{i_{p-1}}) c_{k_p/2}(T_{i_p})
$$

$$
\times C(i_1, i_2)^{k_1} \cdots C(i_{p-1}, i_p)^{k_p/2} + o(1).
$$

Set, for each $k$, dropping subscript $j$ from the notation,

$$
d_{k,n} = \left| \frac{1}{n} \sum_{i_1, i_2} a_{i_1} a_{i_2} c_k(T_{i_1}) c_k(T_{i_2}) C(i_1, i_2)^k \right|
$$

Remark, since in each of the above sums \( k_j \geq 2 \), there exists a constant \( A \), independent of \( k \) and \( n \), such that
\[
d_{k,n} \leq \frac{2A}{n} \sum_{i_1} a_{i_1}^2 c_k^2(T_{i_1}) := e_{k,n}.
\]
On the other hand, since \( a(s) \) is a continuous function it may be approximated by piece-wise constant functions. And if \( a(s) \) is a piece-wise constant function it can be seen that for each \( x \)
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i_1 \leq n} a_{i_1}^2 T_{i_1}^2(x) = T^2(x) \int_0^1 a^2(t) dt.
\]
By Fubini and the dominated convergence theorem
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i_1 \leq n} a_{i_1}^2 c_k^2(T_{i_1}) = \lim_{n \to \infty} \int \frac{1}{n} \sum_{i_1 \leq n} a_{i_1}^2 T_{i_1}(x) H_k(x)e^{-x^2} dx = c_k^2(T) \int_0^1 a^2(t) dt,
\]
since the integrand is uniformly bounded by an integrable function for all \( n \). Now since,
\[
\sum_k k! \frac{1}{n} \sum_{i_1 \leq n} a_{i_1}^2 c_k^2(T_{i_1}) = \int \frac{1}{n} \sum_{i_1 \leq n} a_{i_1}^2 T_{i_1}(x)e^{-x^2} dx,
\]
again by Fubini and the dominated convergence theorem
\[
\lim_{n \to \infty} \sum_k k! e_{k,n} = \int \int_0^1 a^2(t) T^2(x)e^{-x^2} dxdt
\]
\[
= \int_0^1 a^2(t) dt \sum_k k! c_k^2(T) = \sum_k k! \lim_{n \to \infty} e_{k,n}.
\]
Hence, for each \( \varepsilon > 0 \) there exist \( M \) and \( n_0 \) such that if \( n > n_0 \),
\[
\sum_{k > M} k! d_{k,n} < \varepsilon
\]
which in turn yields
\[
\lim_{n \to \infty} \sum_k k! \frac{1}{n} \sum_{i_1,i_2} a_{i_1,i_2} c_k(T_{i_1}) c_k(T_{i_2}) C(i_1,i_2)^k
\]
\[
= \sum_k k! \lim_{n \to \infty} \frac{1}{n} \sum_{i_1,i_2} a_{i_1,i_2} c_k(T_{i_1}) c_k(T_{i_2}) C(i_1,i_2)^k. \tag{39}
\]
As above, for \( a(s) \) a continuous function, it can be seen that
\[
d_{k,n} \to c_k(T)^2 v^k \int_0^1 a^2(s) ds.
\]
Thus, we have shown
\[
\mathbb{E} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} a_i T_{i,n}(Y_i) \right)^p \to p!! \left( \sum_k c_k(T)^2 k! v^k \int_0^1 a^2(s) ds \right)^{p/2}.
\]
In what follows we will be interested in applying Lemma 8 when \( \nu^k = \rho^k \), introduced in Definition 1. Since \( \rho^k = \rho^k(H) \) we will introduce the notation \( C_{H,T} = \sum_{k=0}^{\infty} c_k^2(T) k! \rho^k \) to stress this dependence.

**Lemma 9.** Assume \( T \) is an even function with vanishing expectation and assume \( f \) has sub-exponential growth. Let \( X^m \) be defined as in Section 2. Then,

\[
\mathbb{E} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} f(X^m_{s_j}) T \left( \frac{\Delta B^H_i}{\Delta H} \right) \right)^p = p!! \left[ C_{H,T} \mathbb{E} \int_0^t f^2(X^m_s) ds \right]^{p/2} + o(1)
\]

for even \( p \) and

\[
\mathbb{E} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} f(X^m_{s_j}) T \left( \frac{\Delta B^H_i}{\Delta H} \right) \right)^p = o(1)
\]

for odd \( p \).

**Proof.** Recall \( t_1, \ldots, t_m \) are a fixed collection of grid points and that by definition \( f(X^m_i) \) is a function of \( B^H_1, \ldots, B^H_m \).

We have

\[
|\text{Cov}(B^H_{ij}, \Delta B^H_i / \Delta H)| = \left| \frac{s_i^{2H} - s_{i-1}^{2H} - |t_j - s_i|^{2H} - |t_j - s_{i-1}|^{2H}}{\Delta H} \right| \leq \frac{s_i^{2H-1} \Delta}{\Delta H} + \frac{t_j^{2H-1} \Delta}{\Delta H} = O(\Delta^{1-H}),
\]

and

\[
|\text{Cov}(\Delta B^H_k / \Delta H, \Delta B^H_i / \Delta H)| = O(|k - i|^{2H-2}),
\]

with \( 2 - 2H > 1/2 \) as \( H < 3/4 \). Define \( B^{H,m} = (B^H_1, \ldots, B^H_m) \) and \( \Delta B^H = (\Delta B^H_1 / \Delta H, \ldots, \Delta B^H_{n-1} / \Delta H) \). Set \( A = \mathbb{E} B^{H,m} B^{H,m \Gamma}, \Sigma = \mathbb{E} \Delta B^H \Delta B^{H\Gamma} \) and \( D = \mathbb{E} B^{H,m} \Delta B^H \). Then, \( \Delta B^H | B^{H,m} \sim N(-D^\Gamma A^{-1} B^{H,m}, \Sigma - D^\Gamma A^{-1} D) \). Thus, conditionally to \( B^{H,m}, \Delta B^H \) satisfies the conditions of Lemma 8, with \( b_{i,n} = -(D^\Gamma A^{-1} B^{H,m})_i = O(\Delta^{1-H}), \sigma_{i,n} = (\Sigma - D^\Gamma A^{-1} D)_{i,i} = 1 - O(\Delta^{2(1-H)}) \) and \( C \) the conditional correlation matrix of \( \Delta B^H \).

In this case \( \nu^k \), defined in Lemma 8, is equal to \( \rho^k \). The result follows by calculating conditional expectations. □

We are now ready to return to the proof of Theorem 2.

**Proof of Theorem 2.** We divide the proof into a series of steps.

**Step 1:** For any given \( h \) define \( h_N \) to be the truncated function at level \( N \). Set

\[
\hat{Z}_{N,m,n}(t) = \frac{1}{n} \sum_{i=1}^{[nt]} f_N(X^m_{s_i}) G \left( \sigma_N(X^m) \frac{\Delta B^H_i}{\Delta H} \right),
\]

\[
Z_{N,m}(t) = \int_0^t f_N(X^m_s) G_s(\sigma_N(X^m_s)) ds,
\]
for $G_s$ defined in Eq. (26), and
$$z_{N,m,n}(t) = \sqrt{n}(\hat{Z}_{N,m,n}(t) - Z_{N,m}(t)).$$

Finally define
$$U_{N,m}(t) = W \left( \int_0^t f_N^2(X_s^m)C_{H,G,\sigma_N}(X_s^m)ds \right).$$

Here $W$ is a Wiener process which is independent of $B^H$. Then, by Lemma 9, the finite dimensional distributions of $z_{N,m,n}$ converge weakly, as $n \to \infty$, towards $U_{N,m}(t)$.

Tension follows by checking, as in the proof of Lemma 9 that there exists a positive constant $C$ such that
$$\mathbb{E}(z_{N,m,n}(t) - z_{N,m,n}(s))^4 \leq C \mathbb{E} \left( \int_s^t f_N^2(X_u^m)C_{H,G,\sigma_N}(X_u^m)du \right)^2.$$

Step 2: By Step 1,
$$\mathbb{E}e^{i\gamma(z_{N,m,n}(t) - z_{N,m,n}(s))} \to \mathbb{E}e^{-\gamma^2/2 \int_s^t f_N^2(X_u^m)C_{H,G,\sigma_N}(X_u^m)du}.$$

Thus using the dominated convergence Theorem twice ($|f_N| \leq f$ and $\sigma_N \leq N$)
$$\mathbb{E}e^{i\gamma(z_{N,m,n}(t) - z_{N,m,n}(s))} \to \mathbb{E}e^{-\gamma^2/2 \int_s^t f^2(X_u)C_{H,G,\sigma}(X_u)du}.$$

Hence $z_{N,m,n}(t)$ converges weakly to $W(\int_0^t f^2(X_s)C_{H,G,\sigma}(X_s)ds)$, where $W$ is a Wiener process which is independent of $B^H$.

Step 3: Set $z_n(t) = \sqrt{n}(\hat{Z}_n(t) - Z(t))$. Consider now the quantity $\delta_{N,m,n} = z_n(t) - z_{N,m,n}(t)$. In what follows we will show the latter converges to zero in probability as $m, N \to \infty$, uniformly for all $n$. That is, for all $\varepsilon$,
$$\lim_{m,N} \limsup_n \mathbb{P}(\delta_{N,m,n} > \varepsilon) \to 0. \quad (40)$$

By Theorem 4.1 in [3], this shows the required weak convergence.

Start by defining
$$\hat{Z}_{n,1}(t) = \frac{1}{n} \sum_{i=1}^{[nt]} f(X_{s_i})G \left( \frac{\Delta X_i}{\Delta^H} \right),$$
$$\hat{Z}_{n,2}(t) = \frac{1}{n} \sum_{i=1}^{[nt]} f(X_{s_i})G \left( \sigma(X_{s_i}) \left( \frac{\Delta B^H_i}{\Delta^H} + \Delta^{1-H} \frac{b}{\sigma}(X_{s_i}) \right) \right),$$
$$\hat{Z}_{n,3}(t) = \sum_{j=1}^{r} \frac{1}{n} \sum_{i=1}^{[nt]} f(X_{s_i})g_j(\sigma(X_{s_i}))T_j \left( \frac{\Delta B^H_i}{\Delta^H} \right),$$
$$\hat{Z}_{n,4}(t) = \sum_{j=1}^{r} \frac{1}{n} \sum_{i=1}^{[nt]} f_N(X_{s_i})g_j(\sigma_N(X_{s_i}))T_j \left( \frac{\Delta B^H_i}{\Delta^H} \right),$$
$$\hat{Z}_{n,5}(t) = \sum_{j=1}^{r} \frac{1}{n} \sum_{i=1}^{[nt]} f_N(X_{s_i})g_j(\sigma_N(X_{s_i}))\mathbb{E}T_j(Z).$$
\[ \hat{Z}_{n,6}(t) = \sum_{j=1}^{r} \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} f(X_{s_i}) \sigma(X_{s_i}) \hat{E}_j(Z) \]

\[ \hat{Z}_{n,7}(t) = \sum_{j=1}^{r} \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} f(X_{s_i}^m) \sigma(X_{s_i}^m) \hat{E}_j(Z). \]

We have

\[ \delta_{N,m,n} = \sqrt{n} (\hat{Z}_n(t) - \hat{Z}_{n,1}(t) + \hat{Z}_{n,2}(t)) + \hat{Z}_{n,2}(t) - \hat{Z}_{n,3}(t) + \hat{Z}_{n,3}(t) - \hat{Z}_{n,4}(t) + \sqrt{n} (\hat{Z}_{n,4}(t) - \hat{Z}_{n,5}(t)) - z_{N,m,n}(t) + \sqrt{n} (\hat{Z}_{n,5}(t) - \hat{Z}_{n,6}(t)) + \hat{Z}_{n,6}(t) - Z(t). \quad (41) \]

Set \( A_N := \{ w : \|X\|_\infty \leq N, \sup_n \|X^n\|_\infty \leq N \} \). Since \( \|X\|_{\alpha,\infty} < \infty \) and \( \|X^n\|_{\alpha,\infty} \to \|X\|_{\alpha,\infty} \) \( \mathbb{P} \)-a.s., \( \lim_{n,N} \limsup_n P(A_N^c) = 0 \). Also, for \( p' > 2 \) set \( B_N = \{ w : \sup_j \|S_{n}^{T_j-\hat{E}_j}\|_{\text{var}(p')} < N \} \). Because of Lemma 6 \( \lim_n \limsup_n P(B_N^c) = 0 \).

We consider each of the terms in (41).

1. Firstly,

\[ \limsup_n \mathbb{E}[\sqrt{n} (\hat{Z}_n(t) - \hat{Z}_{n,1}(t))]^2 1_{A_N} = 0 \]

and

\[ \limsup_n \mathbb{E}[\sqrt{n} (Z(t) - \hat{Z}_{n,6}(t))]^2 1_{A_N} = 0 \]

as \( f \in C^1 \) and \( X_s \in C^{1-\alpha} \) for \( 1/2 < 1 - \alpha < H \). The proof follows by Chebyshev’s inequality.

2. Next, set \( a_i(s) = [\sigma(s, X_s) - \sigma(s_i, X_{s_i})] \) and \( d_i(s) = [b(s, X_s) - b(s_i, X_{s_i})] \) for \( s_i \leq s < s_{i+1} \). Recalling the pathwise definition of the fractional integral given in (9) and inequality (12) for \( 1 - H < \alpha < 1/2 \) over \( A_N \)

\[ \left| \int_{s_i}^{s_{i+1}} a_i(s) dB_s^H \right| \leq K \|X\|_{\alpha,\infty} A_\alpha (B^H) \int_{s_i}^{s_{i+1}} \left[ |s - s_i|^{1-2\alpha} + \int_{s_i}^{s} |s - u|^{-2\alpha} du \right] ds \]

\[ \leq 2K N A_\alpha (B^H) \Delta^{2(1-\alpha)}. \quad (42) \]

On the other hand,

\[ \left| \int_{s_i}^{s_{i+1}} d_i(s) ds \right| \leq K \|X\|_{\alpha,\infty} \int_{s_i}^{s_{i+1}} \left[ |s - s_i|^{1-2\alpha} + \int_{s_i}^{s} |s - u|^{-2\alpha} du \right] ds \]

\[ \leq 2K N \Delta^{2-\alpha}. \quad (43) \]

Let \( \eta > 0 \) be such that \( H - 2\eta > 1/2 \) and set \( \alpha = 1 - H + \eta \). By (42) and (43) over \( A_N \), there exists a constant \( C(N) \) such that

\[ |R| := \left| \frac{\Delta X_i}{\Delta^H} - \sigma(s_i, X_{s_i}) \frac{\Delta B_i^H}{\Delta^H} + b(s_i, X_{s_i}) \Delta^{1-H} \right| \leq C(N) A_\alpha (B^H) \Delta^{H-2\eta}. \]

Thus we have
Lemma 2

The next step is bounding. Note that because the

Set \( \delta \) difference is zero over \( E \) first order terms we use

For this we consider a second order expansion of

Hence, under assumptions \( (\sigma (s, X_s)) \) and by Lemma 2, \( \mathbb{E} \vert T_j (\zeta) \vert < \infty \) and

and by Chebyshev’s inequality,

\[ \lim_{N, m} \lim_{n} P (|\sqrt{n}(\hat{Z}_{n,1}(t) - \hat{Z}_{n,2}(t))| > \epsilon) A_N = 0. \]

3. Set \( \delta_i = b(X_{s_i})/\sigma(X_s) \Delta^{1-H} \). The next step requires bounding for each \( j = 1, \ldots, r \)

For this we consider a second order expansion of \( T_j \). Since \( \delta^2 = o(n^{1/2}) \) over \( A_N \), under assumption \( A_3 \), second order terms will be negligible. In order to control first order terms we use Lemma 5. Note that because the \( T_j \) are even functions with integrable derivatives \( \mathbb{E} T_j'(Z) = 0 \) for \( Z \sim N(0,1) \). Hence setting \( h(s) = g_j(\sigma(s, X_s))f(X_s)b(s, \sigma(s, X_s))/\sigma(s, X_s) \) we can write

\[
\frac{1}{\sqrt{n}} \left| \sum_{i=1}^{[nt]} f(X_s_i) g_j(\sigma(s, X_s)) \delta_i T_j'(\Delta B_i^H / \Delta H) \right|
\]

\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} h(s_i) \Delta S_{\nu j}^t j(s_i) \Delta^{1-H} T_j'(\Delta B_i^H / \Delta H) \]

\[
\leq \Delta^{1-H} \| h \|_{\varphi(p)} \| S_{\nu j}^t j - ET_j' \|_{\varphi(p')},
\]

for \( p = 1/(H - \eta) \) and \( p' > 2 \) such that \( 1/p + 1/p' > 1 \). Over \( A_N \), there exists a constant \( C_1(N) \) such that \( \| h \|_{\varphi(p)} \leq C_1(N) \| X \|_{\alpha, \infty} \leq C_1(N) N \) with \( \alpha = 1 - H + \eta \). Moreover, by Lemma 7, for each \( j = 1, \ldots, r \), \( \mathbb{E} \| S_{\nu j}^t j - ET_j' \|_{\varphi(p')} \leq C_p H^{1/2} \). To complete the argument use Chebyshev and note that \( \Delta^{1-H} \| n^H H^{1/2} = \Delta^{3/2 - 2H} \) which tends to zero when \( H < 3/4 \).

4. The next step is bounding \( |\hat{Z}_{n,3}(t) - \hat{Z}_{n,4}(t)| \) and \( |\hat{Z}_{n,5}(t) - \hat{Z}_{n,6}(t)| \). This is trivial since the difference is zero over \( A_N \).
5. The core of the proof remains to bound \(|\sqrt{n}(\hat{Z}_{n,t} - Z_{n,t}) - z_{n,m,n}|\). First observe this expression is bounded by

\[
\left| \sum_{j=1}^{\lfloor n \rfloor} \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor n \rfloor} \left[ f_N(X_{s_i})g_j(\sigma_N(X_{s_i})) - f_N(X_{s_i}^m)g_j(\sigma_N(X_{s_i}^m)) \right] \left( T_j - ET_j \right) \left( \frac{\Delta B_j^H}{\Delta H} \right) \right| \\
+ |\sqrt{n}(Z_{n,\gamma} - Z_{n,m}(t))| = I + II.
\]

The second term can be bounded as in step 1. For the first term we resort to Lemma 5. We will show that over \(B_N \cap A_N\), we can bound \(I \leq C(N)\|X - X^m\|_{1-\alpha, \infty}\). The result then follows by Lemma 3. Consider, to simplify the notation and the proof, that \(j = 1\). The general case is similar but more cumbersome. Set \(L_{n,m} = f_N(X_{s_i})g(\sigma_N(X_{s_i})) - f_N(X_{s_i}^m)g(\sigma_N(X_{s_i}^m))\). So that

\[
|I| = \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor n \rfloor} L_{n,m}(s_i)(T - ET) \left( \frac{\Delta B_i^H}{\Delta H} \right) \right| \leq C\|L_{n,m}\|_{\text{var}(p')}\|S_n\|_{\text{var}(p)},
\]

with \(p > 2\). Here \(p'\) is such that \(1/p + 1/p' > 1\) and \(p'(1 - \alpha) > 1\) for some \(1 - H < \alpha < 1/2\). To compute \(\|L_{n,m}\|_{\text{var}(p')}\) write

\[
L_{n,m}(t_i) - L_{n,m}(t_{i-1}) = f_N(X_{t_i})[g(\sigma_N(t_i, X_{t_i})) - g(\sigma_N(t_i, X_{t_i}^m))]
+ g(\sigma_N(t_i, X_{t_i}^m))[f_N(X_{t_i}) - f_N(X_{t_i}^m)]
- f_N(X_{t_{i-1}})[g(\sigma_N(t_{i-1}, X_{t_{i-1}})) - g(\sigma_N(t_{i-1}, X_{t_{i-1}}^m))]
- g(\sigma_N(t_{i-1}, X_{t_{i-1}}^m))[f_N(X_{t_{i-1}}) - f_N(X_{t_{i-1}}^m)].
\]

This in turn can be written as

\[
[f_N(X_{t_i}) - f_N(X_{t_{i-1}})][g(\sigma_N(t_i, X_{t_i})) - g(\sigma_N(t_i, X_{t_i}^m))]
+ f_N(X_{t_{i-1}})[g(\sigma_N(t_i, X_{t_i})) - g(\sigma_N(t_i, X_{t_i}^m))]
- (g(\sigma_N(t_{i-1}, X_{t_{i-1}})) - g(\sigma_N(t_{i-1}, X_{t_{i-1}}^m))]
+ g(\sigma_N(t_{i-1}, X_{t_{i-1}}^m))[f_N(X_{t_i}) - f_N(X_{t_{i-1}}) - f_N(X_{t_{i-1}}^m)]
+ g(\sigma(t_i, X_{t_i}^m)) - g(\sigma_N(t_{i-1}, X_{t_{i-1}}^m)][f_N(X_{t_i}) - f_N(X_{t_i}^m)].
\]

By using this equality we get

\[
\|L_{n,m}\|_{\text{var}(p')} \leq C(N) \left( \|X\|_{\text{var}(p')}\|X - X^m\|_{\infty}^{p'} \right.
+ \|f_N\|_{\infty}^{p'} \sup_{\Pi} \sum_{t_i \in \Pi} |g(\sigma_N(t_i, X_{t_i})) - g(\sigma_N(t_i, X_{t_i}^m))|
- (g(\sigma_N(t_{i-1}, X_{t_{i-1}})) - g(\sigma_N(t_{i-1}, X_{t_{i-1}}^m))|^{p'}
+ \|g \circ \sigma_N\|_{\infty}^{p'} \sup_{\Pi} \sum_{t_i \in \Pi} |f_N(X_{t_i}) - f_N(X_{t_i})|
- (f_N(X_{t_{i-1}}^m) - f_N(X_{t_{i-1}}^m))|^{p'}
+ \|X^m\|_{\text{var}(p')}\|X - X^m\|_{\infty}^{p'} \right).
\]
On the other hand,
\[
\sup_{\Pi} \sum_{t_i \in \Pi} |g(\sigma_N(t_i, X_{t_i})) - g(\sigma_N(t_i, X_{t_i}^m))
\]
\[
- (g(\sigma_N(t_{i-1}, X_{t_{i-1}})) - g(\sigma_N(t_{i-1}, X_{t_{i-1}}^m)))|^{p'}
\]
\[
\leq 2^p \left( \| (g \circ \sigma)^{'''} \|_\infty' \sup_{\Pi} \sum_{t_i \in \Pi} |X_{t_i} - X_{t_i}^m|^{p'}
\]
\[
+ \| (g \circ \sigma)^{'} \|_\infty' \sup_{\Pi} \sum_{t_i \in \Pi} |X_{t_i} - X_{t_i}^m - X_{t_{i-1}} - X_{t_{i-1}}^m|^{p'} \right),
\]
and finally from the definition of the norm \( \| \cdot \|_{1-\alpha, \infty} \), we obtain by an application of Lemma 1
\[
L_{N,m}^{p'} \| \text{var}(p') \leq C(N)\| X - X^m \|_{1-\alpha, \infty}. \quad \Box
\]

**Proof of Theorem 4.** By Theorem 3.1 in [12] stable convergence follows from Theorem 2 if for all bounded, \( \mathcal{F}_t \) measurable r.v. \( U \) and for all \( \eta \) we have
\[
\lim_{n \to \infty} \mathbb{E}(Ue^{i\eta Z_n(t)}) = \mathbb{E}(Ue^{iU(t)}).
\]
Recall we are dealing with the vanishing drift case. That is, \( X_s = h(B^H_s) \). Hence, for given \( \eta \), \( \mathbb{E}(e^{i\eta Z(t)}|\mathcal{F}_t) = e^{-\eta^2/2} \int_0^t f^2(x_s)C_{H,G,\sigma}(x_s)dx \). On the other hand, let \( U \) be a bounded, \( \mathcal{F}_t \) measurable r.v. Construct a bounded cylinder sequence \( U_m = U_m(x_1, \ldots, x_n) \), such that \( U_m \to U \), almost everywhere. As in the proof of Theorem 2 it can be checked directly that,
\[
\mathbb{E}(U_m e^{i\eta Z_n(t)}) \to \mathbb{E}U_m \left( \int_0^t f^2(x_s)C_{H,G,\sigma}(x_s)dx \right)^{p'/2}
\]
if \( p \) is even and to zero if \( p \) is odd.

Thus,
\[
\mathbb{E}(U_m e^{i\eta Z_n(t)}) \to \mathbb{E}U_m e^{-\eta^2/2} \int_0^t f^2(x_s)C_{H,G,\sigma}(x_s)dx.
\]  \( (44) \)

Now since
\[
|\mathbb{E}[U e^{i\eta Z_n(t)} - U e^{iU(t)}]| \leq |\mathbb{E}[U e^{i\eta Z_n(t)} - U_m e^{i\eta Z_n(t)}]| + |\mathbb{E}[U e^{iU(t)} - U_m e^{iU(t)}]|
\]
\[
+ |\mathbb{E}[U_m e^{i\eta Z_n(t)} - U_m e^{iU(t)}]| \leq 2|U_m - U| + |\mathbb{E}[U_m e^{i\eta Z_n(t)} - U_m e^{iU(t)}]|,
\]
by the dominated convergence Theorem and Eq. \( (44) \), \( \mathbb{E}(U e^{i\eta Z_n(t)}) \to \mathbb{E}(U e^{iU(t)}). \quad \Box
\]

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**References**


