Pseudo-Canonical Systems with Rational Weyl Functions: Explicit Formulas and Applications

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The paper extends earlier results of the authors for canonical systems with spectral functions of which the absolutely continuous part has a rational derivative to a class of differential systems with skew selfadjoint potentials. The corresponding direct and inverse spectral problems are solved explicitly, using state space methods from mathematical system theory. Applications to nonlinear integrable partial differential equations are included.

0. INTRODUCTION

We shall consider differential systems on the half axis $x \geq 0$ of the form

$$\frac{d}{dx} u(x, \lambda) = (\lambda j + V(x)) u(x, \lambda), \quad j = \begin{bmatrix} I_m & 0 \\ 0 & -I_m \end{bmatrix}, \quad V = \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix}. \quad (0.1)$$

Here $u$ and $V$ are $2m \times 2m$ matrix functions on the half axis, $v$ is $m \times m$ locally summable matrix function on $[0, \infty)$, while $I_m$ is $m \times m$ identity matrix and $\lambda$ is a spectral parameter ($\lambda \in \mathbb{C}$). Systems of the type (0.1) will be called pseudo-canonical.

When the entry $v^*$ in the left lower corner of $V$ is replaced by $-v^*$, then the system (0.1) is called a canonical system. Canonical systems with positive definite rational spectral densities were treated in [AG2], using state space methods from mathematical system theory. Analogous results for the discrete case were obtained in [AG1]. The next step in developing the state space approach for direct and inverse spectral problems was made...
in [GKS]. The results of the latter paper concern spectral functions of which the absolutely continuous part has a rational derivative and the singular part is a step function. As applications new explicit solutions were obtained in [GKS] for the well-known matrix nonlinear Schrödinger and modified Korteweg–de Vries equations associated with canonical systems.

In the present paper we study pseudo-canonical systems (0.1) with potentials \( v \) that can be presented in the form

\[
v(x) = -2i\theta_1^*(Pe^{-2ixA}|_{\text{Im}P})^{-1}\theta_2, \quad x \geq 0.
\]  

(0.2)

Here \( \theta_1, \theta_2 \) are \( n \times m \) matrices, \( A \) and \( P \) are square matrices of order \( 2n \) given by

\[
A = \begin{bmatrix}
\alpha^* & 0 \\
-\theta_1\theta_2^* & \alpha
\end{bmatrix}, \quad P = \begin{bmatrix}
I_n & -iI_n \\
0 & 0
\end{bmatrix},
\]  

(0.3)

\[
\alpha - \alpha^* = i(\theta_1\theta_1^* + \theta_2\theta_2^*),
\]  

(0.4)

where \( \alpha \) is an \( n \times n \) matrix. The symbol \( \text{Im}P \) in (0.2) denotes the range of the projector \( P \). Analogously to [GKS] we call a potential \( v \) described by (0.2)–(0.4) a pseudo-exponential potential determined by the triple \( \theta_1, \theta_2 \) and \( \alpha \), and we denote the class of these potentials for (0.1) by \( \text{PE}(2) \). (Notice that this class of potentials is different from the one in [GKS] which is denoted by \( \text{PE}(1) \).)

In this paper we introduce the Weyl function and we consider spectral problems for pseudo-canonical systems with potentials from the class \( \text{PE}(2) \). Since the operator corresponding to (0.1) has both selfadjoint and skew selfadjoint terms we cannot talk about the spectral function and the role of the spectral function is taken over by the Weyl function. Solutions to the direct and inverse problems are derived. Applications to matrix nonlinear Schrödinger and modified Korteweg–de Vries equations associated with pseudo-canonical systems are also given.

The paper consists of four sections. In the first section the fundamental solution of equation (0.1) with a pseudo-exponential potential \( v \) is derived. Weyl functions and spectral problems are studied in Section 2. The third section is dedicated to the problem of bispectrality. The applications to nonlinear partial differential equations appear in the fourth section.

We conclude this section with some terminology from mathematical system theory used in this paper. Consider a finite dimensional input-output system given by

\[
\begin{cases}
\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \\
y(t) = Cx(t) + Du(t).
\end{cases}
\]  

(0.5)
Here $A$, $B$, $C$ and $D$ are matrices of appropriate sizes, $A$ is a square matrix which is often referred to as the state matrix and $\dot{x}$ stands for $dx/dt$. If we assume the system $\Sigma$ to be at rest at time $t = 0$, that is, $x(0) = 0$, then the Laplace transform $\hat{y}$ of the output $y$ and the Laplace transform $\hat{u}$ of the output $u$ are related in the following way

$$\hat{y}(\lambda) = \{ D + C(\lambda I - A)^{-1} B \} \hat{u}(\lambda).$$

Here $I$ is an identity matrix of the same order as $A$. The matrix function

$$\Phi(\lambda) = D + C(\lambda I - A)^{-1} B$$

(0.6)
is called the transfer function of the system $\Sigma$. Notice that the transfer function $\Phi$ is a proper (i.e., analytic at infinity) rational matrix function. It is a basic fact from mathematical system theory (see [KFA]) that, conversely, any proper rational matrix function $\Phi$ is a transfer function of some system of the type (0.5), and hence can be represented in the form (0.6). In this case one refers to (0.6), or more precisely to the right hand side of (0.6), as a realization of $\Phi$. A realization (0.6) of $\Phi$ is called minimal if among all realizations of $\Phi$ the order of the state matrix $A$ is as small as possible. Minimal realizations are unique up to state space isomorphisms, that is, if (0.6) is a minimal realization of $\Phi$, then any other minimal realization of $\Phi$ is given by

$$\Phi(\lambda) = D + CS(\lambda I - S^{-1} AS)^{-1} S^{-1} B,$$

where $S$ is some non-singular matrix of the same order as $A$. The realization (0.6) is a minimal realization if and only if

$$\text{rank}[BA B A^{n-1} B] = n, \quad (0.7)$$

$$\text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n, \quad (0.8)$$

where $n$ is the order of the matrix $A$. If (0.7) is fulfilled, then the pair $(A, B)$ (or the system $\Sigma$) is said to be controllable. If (0.8) is satisfied, then the pair $(C, A)$ (or the system $\Sigma$) is called observable. Finally the pair $(C, A)$ is called detectable if there exists a matrix $R$ such that $A - RC$ has all its eigenvalues in open left half plane. The terms observable, controllable and detectable have an intrinsic system theoretic meaning; for this and related background material we refer to the books [K, KFA, Zh].
The fact that rational matrix functions may be viewed as transfer functions of input-output systems is used in this paper to obtain explicit solutions for direct and inverse problems of pseudocanonical systems.

1. FUNDAMENTAL SOLUTION

Our first aim is to prove an analogue of Proposition 1.1 in [GKS] for potentials \( v \) of the class PE(2). Let \( \theta_1, \theta_2 \) and \( \alpha \) be as above (in particular, (0.4) holds). We introduce matrix functions \( A \) and \( S \) in the following way

\[
A(x) = \begin{bmatrix} e^{-ix\theta_1} & e^{ix\theta_2} \end{bmatrix}, \quad (1.1)
\]

\[
S(0) = I_n, \quad S'(x) = A(x) jA(x)^* \left( S' = \frac{dS}{dx} \right), \quad (1.2)
\]

Notice that \( S(x) \) has size \( n \times n \).

**Proposition 1.1.** Let \( v \) be the pseudo-exponential potential determined by the matrices \( \theta_1, \theta_2 \) and \( \alpha \), and let \( S(x) \) be defined by (1.2). Then \( S(x) \) is invertible for each \( x \) and

\[
v(x) = -2\theta_1^* e^{i\alpha x} S(x)^{-1} e^{i\alpha x} \theta_2. \quad (1.3)
\]

**Proof.** First let us prove that \( S(x) \) is invertible. From (1.1) it follows that

\[
A'(x) = -iA(x) j. \quad (1.4)
\]

By virtue of (0.4), (1.1), (1.2) and (1.4) we get

\[
\alpha S(0) - S(0) \alpha^* = \alpha - \alpha^* = iA(0) A(0)^*, \quad \alpha S'(x) - S'(x) \alpha^* = i(A A^*)' (x).
\]

Hence the matrix function \( S(x) \) satisfies the operator identity

\[
\alpha S(x) - S(x) \alpha^* = iA(x) A(x)^*. \quad (1.5)
\]

According to (1.2) and (1.5) we have

\[
(e^{-ix} S(x) e^{i\alpha x})' = e^{-ix} (-i(\alpha S(x) - S(x) \alpha^*) + S'(x)) e^{i\alpha x} = e^{-ix} A(x)(I_{2n} + j) A(x)^* e^{i\alpha x},
\]

i.e., \( (e^{-ix} S(x) e^{i\alpha x})' \geq 0 \). As, by (1.2), the initial value \( S(0) = I_n \) so we see now that

\[
e^{-ix} S(x) e^{i\alpha x} \geq I_n. \quad (1.6)
\]
Hence the inequality
\[ \det S(x) \neq 0 \]  
(1.7) is true.

To prove (1.3) it remains to establish the equality
\[ Pe^{-2ixA} |_{\text{Im}P} = e^{-ixA} S(x) e^{-ixA^*}, \]  
(1.8)
from which (1.3) follows immediately. Since \( \text{Im}P \) may be identified with the space \( \mathbb{C}^n \), we may view the left-hand side as an \( n \times n \) matrix function, which we shall denote by \( Q_1(x) \). The right-hand side of (1.8) will be denoted by \( Q_2(x) \). According to (0.3) we have
\[ Q_1(x) = [I_n - il_0] e^{-2ixA} \begin{bmatrix} I_n \\ 0 \end{bmatrix}. \]

Therefore, taking into account (0.3), we obtain
\[ Q_1(x) = -i \left\{ [I_n - il_n] Ae^{-2ixA} \begin{bmatrix} I_n \\ 0 \end{bmatrix} + [I_n - il_0] e^{-2ixA} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \right\} \]
\[ = -i \left\{ [z^* + i\theta_1^* \theta_1^*] e^{-2ixA} \begin{bmatrix} I_n \\ 0 \end{bmatrix} + [I_n - il_n] e^{-2ixA} \begin{bmatrix} z^* \\ -\theta_1^* \theta_1^* \end{bmatrix} \right\}. \]  
(1.9)

From (0.4) and (1.9) it follows
\[ Q_1(x) = -i \left\{ [z - i\theta_2^* \theta_2^* - i\theta_1^*] e^{-2ixA} \begin{bmatrix} I_n \\ 0 \end{bmatrix} + [I_n - il_n] e^{-2ixA} \begin{bmatrix} z^* \\ -\theta_1^* \theta_1^* \end{bmatrix} \right\} \]
\[ = -i \left\{ [zQ_1(x) + Q_1(x) z^* - i[[\theta_2^* \theta_2^* \theta_1^*]] e^{-2ixA} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \\ + [I_n - il_n] e^{-2ixA} \begin{bmatrix} 0 \\ -\theta_1^* \theta_1^* \end{bmatrix} \right\}. \]

Therefore \( Q_1 \) satisfies the equation
\[ Q'(x) = -i(zQ(x) + Q(x) z^*) + e^{-2ixA} \theta_1^* \theta_1^* e^{-2ixA^*}. \]  
(1.10)

It can be easily seen from (1.1) and (1.2) that \( Q_2 \) also satisfies (1.10). As \( Q_1(0) = Q_2(0) = I_n \), so \( Q_1(x) = Q_2(x) \), and relation (1.3) is proved.
The fundamental solution \( u(x, \lambda) \) of the system (0.1) is by definition the unique solution of (0.1) satisfying the initial condition

\[
 u(0, \lambda) = I_{2m}. \tag{1.11}
\]

Theorem 1.2. Let \( v \) be the pseudo-exponential potential determined by \( \theta_1, \theta_2 \) and \( \alpha \). Then the fundamental solution \( u(x, \lambda) \) of the pseudo-canonical system (0.1) is given by

\[
 u(x, \lambda) = w_{\alpha, 0}(x, \lambda) e^{i\omega_j w_{\alpha, 0}(0, \lambda)^{-1}}, \tag{1.12}
\]

where

\[
 w_{\alpha, 0}(x, \lambda) = I_{2m} + iA(x)^* S(x)^{-1} (\lambda I_n - \alpha)^{-1} A(x) \tag{1.13}
\]

with \( A \) and \( S \) being given by (1.1) and (1.2).

Matrix-functions of the form (1.13) with property (1.5) were introduced by L. Sakhnovich in the context of his theory of \( S \)-nodes [SaL1] and used for the representation of the fundamental solution (see [SaL2] and the references therein).

It will be convenient to prove the following auxiliary result.

Proposition 1.3. For \( x \geq 0 \) let \( A(x) \) be an \( n \times 2m \) matrix function defined by (1.1), and let \( S(x) \) be the \( n \times n \) matrix function defined by (1.2). Let \( w_{\alpha, 0} \) be as in (1.13). Then we have

\[
 \frac{d}{dx} w_{\alpha, 0}(x, \lambda) = i(\lambda_j + V(x)) w_{\alpha, 0}(x, \lambda) - i\omega w_{\alpha, 0}(x, \lambda) j, \tag{1.14}
\]

where \( V \) is given by (0.1)–(0.4).

Proof. By (1.2) and (1.4) we get

\[
 \frac{d}{dx} (iA(x)^* S(x)^{-1}) = -jA(x)^* S(x)^{-1} - iA(x)^* A(x) jA(x)^* S(x)^{-1}. \tag{1.15}
\]

From (1.5) and (1.14) it follows

\[
 \frac{d}{dx} (iA(x)^* S(x)^{-1}) = -jA(x)^* (S(x)^{-1} \alpha - iS(x)^{-1} A(x) A(x)^* S(x)^{-1})
\]

\[
 -iA(x)^* S(x)^{-1} A(x) jA(x)^* S(x)^{-1}. \tag{1.16}
\]
For
\[ \Gamma(x) = iA(x)^* S(x)^{-1} \]  
(1.17)
relation (1.16) yields the result
\[ \frac{d}{dx} \Gamma(x) = i\Gamma(x) + iH(x) \Gamma(x), \]  
(1.18)
where
\[ H(x) = i(A(x)^* S(x)^{-1} A(x)) j - f(A(x)^* S(x)^{-1} A(x)). \]  
(1.19)

By virtue of (1.4), (1.13), (1.17), and (1.18) we have
\[ \frac{d}{dx} w_{*,A}(x, \lambda) \]
\[ = i\Gamma(x)(\lambda - \lambda I_n + \lambda I_n)(\lambda I_n - x)^{-1} A(x) + iH(x)(w_{*,A}(x, \lambda) - I_{2m}) \]
\[ - i\Gamma(x)(\lambda I_n - x)^{-1} (\lambda - \lambda I_n + \lambda I_n) A(x) j \]
\[ = i(\lambda j + H(x))(w_{*,A}(x, \lambda) - I_{2m}) \]
\[ - i\Gamma(x) A(x) + iH(x) A(x) j - i\lambda(w_{*,A}(x, \lambda) - I_{2m}) j \]  
(1.20)
From (1.17), (1.19), and (1.20), the identity
\[ \frac{d}{dx} w_{*,A}(x, \lambda) = (i(\lambda j + H(x)) + i\lambda) w_{*,A}(x, \lambda) - \lambda w_{*,A}(x, \lambda) j \]  
(1.21)
easily follows. (Relation (1.21) was proved earlier in [SaA3].) Taking into account (1.1) and (1.3) we see that \( H \) given by (1.19) coincides with \( V \), hence (1.21) coincides with (1.14).

**Proof of Theorem 1.2.** According to (1.7) the matrix function \( S(x) \) is invertible. By (1.2) and (1.4) the conditions of Proposition 1.3 are fulfilled. Hence (1.14) is true for each \( x \geq 0 \). Relations (1.12) and (1.14) yield (0.1). The initial condition (1.11) also follows from (1.12). Therefore \( u \) of the form (1.12) is the fundamental solution of the pseudo-canonical system (0.1).

Let \( \lambda \notin \sigma(x) \), where \( \sigma(x) \) denotes the spectrum of \( x \) (i.e., the set of eigenvalues of \( x \)). According to (1.5) and (1.13) we get (see [SaL2])
\[ w_{*,A}(x, \lambda) \]
\[ = I^*_{2m} - i(\lambda - \lambda) A(x)^* (\lambda I_n - x)^{-1} S(x)^{-1} (\lambda I_n - x)^{-1} A(x). \]  
(1.22)
Hence, for $\lambda$ in the open lower half plane $\mathbb{C}_-$ we have
\[
i(\lambda - \tilde{\lambda}) A(x)^* (\lambda I_n - x^*)^{-1} S(x)^{-1} (\lambda I_n - x)^{-1} A(x) \leq I_{2m}. \tag{1.23}
\]
The inequality (1.23) yields the following proposition.

**Proposition 1.4.** Any matrix function $v \in PE(2)$ is bounded on the half axis $x > 0$.

**Proof.** Let $D$ be an open domain in $\mathbb{C}_-$. Then for any vector $f$ satisfying the identity $f^* (\lambda I_n - x)^{-1} \theta_k = 0$ for each $\lambda \in D$ it follows that $f^* \theta_k = 0$, i.e.,
\[
\text{span}_{\lambda \in D} (\lambda I_n - x)^{-1} \theta_k \equiv \theta_k \quad (k = 1, 2). \tag{1.24}
\]
By (1.1), (1.23) and (1.24) we have
\[
\sup_{x > 0} \|S(x)^{-1/2} e^{-ix\theta_1}\| < \infty, \quad \sup_{x > 0} \|S(x)^{-1/2} e^{ix\theta_2}\| < \infty. \tag{1.25}
\]
According to (1.3) and (1.25) there exists an $M > 0$ such that
\[
\sup_{x > 0} \|v(x)\| \leq M < \infty. \tag{1.26}
\]

2. WEYL FUNCTIONS AND SPECTRAL PROBLEMS

Recall at first that for all nonreal $\lambda$ there exists a square-integrable solution of the classical Sturm–Liouville equation [LS]. It is represented in the form $y(x, \lambda) = y_1(x, \lambda) + \phi(x) y_2(x, \lambda)$, where $y_k(x, \lambda)$ are solutions of the Sturm–Liouville equation with two different fixed boundary conditions. The function $\phi$ is called the Weyl–Titchmarsh or Weyl function and plays an essential role in the spectral theory. Following the spectral theory of the Sturm–Liouville equation a meromorphic $m \times m$ matrix function $\phi$ satisfying condition
\[
\left[ \begin{array}{cc}
\phi(\lambda) & I_m \\
0 & 0
\end{array} \right] u(x, \lambda) \left[ \begin{array}{cc}
\phi(\lambda) & I_m \\
0 & 0
\end{array} \right] ^* dx < \infty \quad (i(\lambda - \tilde{\lambda}) > \delta \geq 0), \tag{2.1}
\]
is called a Weyl function of the pseudo-canonical system (0.1) (compare with [SaA1]). Here $u$ is the fundamental solution of the system. (See Proposition 2.4 for the connections between the poles and zeros of $\phi$ with the eigenfunctions of the pseudo-canonical system.)

In this section we shall consider the spectral theory of pseudo-canonical systems in terms of the associated Weyl functions. The direct spectral
problem is, if given \( v \in \text{PE}(2) \), construct the Weyl function of the corresponding pseudo-canonical system.

**Theorem 2.1.** Assume that the potential \( v \in \text{PE}(2) \) of the pseudo-canonical system (0.1) is determined by the matrices \( \theta_1, \theta_2 \) and \( \alpha \). Then the system has a unique Weyl function \( \phi \), which satisfies (2.1) on all \( \mathbb{C} \), a finite number of poles excluded, and this function is given by the formula

\[
\phi(\lambda) = i \theta_1^\ast (\lambda I_n - \beta)^{-1} \theta_2,
\]

where

\[
\beta = \alpha - i \theta_2 \theta_2^\ast.
\]

**Proof.** Let \( w_{\alpha, \theta}(x, \lambda) \) be given by (1.13). Write \( w_{\alpha, \theta}(0, \lambda) \) as

\[
w_{\alpha, \theta}(0, \lambda) = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{bmatrix}.
\]

(2.4)

We first prove that

\[
b(\lambda) d(\lambda)^{-1} = i \theta_1^\ast (\lambda I_n - \beta)^{-1} \theta_2.
\]

(2.5)

Notice that the matrix functions \( b(\lambda) \) and \( d(\lambda) \) are given by

\[
b(\lambda) = i \theta_1^\ast (\lambda I_n - \alpha)^{-1} \theta_2,
\]

\[
d(\lambda) = I_m + i \theta_2^\ast (\lambda I_n - \alpha)^{-1} \theta_2.
\]

(2.6)

Taking into account (2.3), (2.6) we obtain

\[
d(\lambda)^{-1} = I_m - i \theta_2^\ast (\lambda I_n - \beta)^{-1} \theta_2.
\]

(2.7)

The equalities (2.3), (2.6), and (2.7) yield

\[
b(\lambda) d(\lambda)^{-1} = i \theta_1^\ast (\lambda I_n - \alpha)^{-1} \theta_2 + i \theta_2^\ast (\lambda I_n - \alpha)^{-1} (\beta - \alpha)(\lambda I_n - \beta)^{-1} \theta_2.
\]

(2.8)

From (2.8) formula (2.5) follows.

Let \( \phi \) be defined by (2.2), and thus by virtue of (2.5) we have

\[
\phi(\lambda) = b(\lambda) d(\lambda)^{-1}.
\]

(2.9)

By (2.4), (2.9) and the representation (1.12) of the fundamental solution we get

\[
u(x, \lambda) \begin{bmatrix} \phi(\lambda) \\ I_m \end{bmatrix} = w_{\alpha, \theta}(x, \lambda) \begin{bmatrix} 0 \\ e^{-i\lambda d(\lambda)^{-1}} \end{bmatrix}.
\]

(2.10)
As the second term in the right-hand side of (1.22) is nonpositive, formula (1.22) yields
\[ w_{a,d}(x, \lambda) \leq I_{2m} \quad (\lambda \in \mathbb{C}_-). \tag{2.11} \]
Taking into account (2.10) and (2.11) we obtain (2.1), i.e., \( \phi \) is a Weyl function.

It remains only to prove the uniqueness of the Weyl function. Suppose now that \( \tilde{\phi} \) is also a Weyl function of (0.1) and that for some fixed \( \lambda_0 \in \mathbb{C}_- \) we have \( \tilde{\phi}(\lambda_0) \neq \phi(\lambda_0) \). Put
\[ L_1 = \text{Im} \left[ \begin{array}{c} \phi(\lambda_0) \\ I_m \end{array} \right] + \text{Im} \left[ \begin{array}{c} \tilde{\phi}(\lambda_0) \\ I_m \end{array} \right]. \]
By definition of the Weyl function we have
\[ \int_0^\infty f^* u(x, \lambda_0)^* u(x, \lambda_0) f \, dx < \infty \quad (f \in L_1). \tag{2.12} \]
Consider now
\[ L_2 = \text{Im} \left[ \begin{array}{c} f_m \\ 0 \end{array} \right]. \]
Without loss of generality we can demand \( i(\lambda - \tilde{\lambda}) \geq 2M \), where \( M \) is the upper bound of the \( \|v\| \) as in the inequality (1.26). Then according to (0.1) and (1.26) we get
\[ \frac{d}{dx} \left( u(x, \lambda)^* j u(x, \lambda) \right) = u(x, \lambda)^* \left( (i(\lambda - \tilde{\lambda}) I_{2m} + 2jV(x)) u(x, \lambda) \right) \geq 0. \tag{2.13} \]
In particular we have
\[ f^* u(x, \lambda)^* u(x, \lambda) \geq f^* u(x, \tilde{\lambda})^* j u(x, \tilde{\lambda}) f \geq f^* f \quad (f \in L_2). \tag{2.14} \]
From (2.14) it follows
\[ \int_0^\infty f^* u(x, \lambda)^* u(x, \lambda) f \, dx = \infty \quad (f \in L_2, f \neq 0). \tag{2.15} \]
As \( \dim L_1 > m \) and \( \dim L_2 = m \), there is a non-zero vector \( f \) such that \( f \in (L_1 \cap L_2) \), which contradicts (2.12) and (2.15). \[ \square \]

The inverse spectral problem is, if given a strictly proper rational function \( \phi \) (here \textit{strictly proper} means that \( \phi \) is proper and \( \phi(\infty) = 0 \)), to construct a potential \( v \in PE(2) \) such that \( \phi \) proves to be the Weyl function of the
corresponding pseudo-canonical system. To solve the inverse problem we need the following result from the theory of algebraic Riccati equations.

**Proposition 2.2.** If $D \geq 0$, $C \geq 0$ and the pairs $(D, \gamma)$ is detectable and $(\gamma, C)$ is controllable, then there exists a positive solution $X > 0$ of the algebraic Riccati equation

$$\gamma X - X\gamma^* = iXD - C.$$  \hspace{1cm} (2.16)

Proposition 2.2 goes essentially back to [Kal] (see also [LR2], pp. 358 and 369). Let us consider now a minimal realization of the strictly proper rational function $\phi$: \hspace{1cm} (2.17)

Putting $C = \mathcal{A}_2 \mathcal{F}_2$ and $D = \mathcal{A}_1 \mathcal{F}_1$, we see that the conditions of Proposition 2.2 are fulfilled. Hence we can introduce matrices $\theta_1$, $\theta_2$, and $\beta$ by the equalities

$$\theta_1 = X^{1/2} \mathcal{A}_1, \quad \theta_2 = X^{-1/2} \mathcal{A}_2, \quad \beta = X^{-1/2} \mathcal{A}^{1/2}.$$  \hspace{1cm} (2.18)

By (2.16)–(2.18) we have

$$\beta - \beta^* = i(\theta_1 \mathcal{F}_1 - \theta_2 \mathcal{F}_2)$$  \hspace{1cm} (2.19)

and hence $\phi$ admits the realization (2.2).

**Theorem 2.3.** Let $\phi$ be a strictly proper rational $m \times m$ matrix function, given by the minimal realization (2.17). Define matrices $\theta_1$, $\theta_2$, and $\alpha = \beta + i\theta_2 \mathcal{F}_2$, by (2.18). Then $\theta_1$, $\theta_2$, and $\alpha$ determine a unique pseudo-exponential solution $v$ of the inverse spectral problem, i.e., a unique system (0.1) with a pseudo-exponential potential and the given matrix function $\phi$ as a Weyl function.

**Proof.** According to (2.19) the matrix $\alpha$ satisfies (0.4). From Theorem 2.1 and formulas (2.17), (2.18) it follows now that the potential $v$ determined by $\theta_1$, $\theta_2$ and $\alpha = \beta + i\theta_2 \mathcal{F}_2$ is indeed a solution of the inverse problem.

Consider two pseudo-canonical systems with potentials $v_1 \in \text{PE}(2)$ and $v_2 \in \text{PE}(2)$, respectively, and assume that these systems have the same Weyl function $\phi$. The fundamental solutions of these systems will be denoted by $u_1$ and $u_2$, respectively. Put

$$U_k(x, \lambda) = u_k(x, \lambda) \begin{bmatrix} I_m & \phi(\lambda) \\ 0 & I_m \end{bmatrix} e^{-\kappa^* \mathcal{F}} (k = 1, 2).$$  \hspace{1cm} (2.20)
Take an $\bar{\tilde{M}}>0$ such that the matrix functions $w_{\omega,j}(0, \lambda)^{-1}$, $d(\lambda)^{-1}$, and $\varphi(\lambda)$ have no poles in the domain $D = \{\tilde{\lambda} - \bar{\lambda} \geq \bar{\tilde{M}}\}$. By virtue of (1.12), (2.10), and (2.11) we obtain that
\[
\sup_{\lambda \in D} \| U_{\lambda}(x, \lambda) \| < \infty \quad (k = 1, 2). \tag{2.21}
\]
Taking into account (1.12), (2.2) and the definitions (1.13) and (2.20) we get
\[
\int_{-\infty}^{\infty} \left(U_{\lambda}\left(x, \frac{\xi - i}{2} \bar{\tilde{M}}\right) - I_{2m}\right)^* \left(U_{\lambda}\left(x, \frac{\xi - i}{2} \bar{\tilde{M}}\right) - I_{2m}\right) d\xi < \infty \quad (k = 1, 2), \tag{2.22}
\]
\[
\lim_{\xi \to \infty} U_{\lambda}\left(x, \frac{\xi - i}{2} \bar{\tilde{M}}\right) = I_{2m}. \tag{2.23}
\]
Notice also that by (0.1) and (1.11) the matrix $u(x, \lambda)$ is invertible as
\[
u(x, \tilde{\lambda})^* u(x, \lambda) = I_{2m}. \tag{2.24}
\]
(Identity (2.24) is proved by the direct differentiation of its two parts.) Using the Paley–Wiener theorem it follows from (2.20)–(2.22) and (2.24) that
\[
\sup_{\lambda \in D} \| U_{\lambda}(x, \lambda)^{-1} \| < \infty \quad (k = 1, 2). \tag{2.25}
\]
Formulas (2.20), (2.21) and (2.25) yield
\[
\sup_{\lambda \in D} \| U_{\lambda}(x, \lambda) U_{\mu}(x, \lambda)^{-1} \| = \sup_{\lambda \in D} \| u_{\lambda}(x, \lambda) u_{\mu}(x, \lambda)^{-1} \| < \infty \quad (k, p = 1, 2). \tag{2.26}
\]
From (2.24) and (2.26) we have
\[
\sup_{\lambda \in D_1} \| u_{\lambda}(x, \lambda) u_{\mu}(x, \lambda)^{-1} \| < \infty, \tag{2.27}
\]
\[
D_1 = \{\lambda : \| \tilde{\lambda} - \bar{\lambda} \| \geq \bar{\tilde{M}}\} \cup \{\lambda : - \| \tilde{\lambda} - \bar{\lambda} \| \geq \bar{\tilde{M}}\}.
\]
If (2.27) is true in $D_1$, it is also true in the whole plane $\mathbb{C}$ and hence (2.27) yields $u_{\lambda}(x, \lambda) u_{\mu}(x, \lambda)^{-1} \equiv \text{const}$. Moreover, taking into account (2.23) we see $u_{\lambda}(x, \lambda) u_{\mu}(x, \lambda)^{-1} \equiv I_{2m},$ i.e., $u_{\lambda}(x, \lambda) \equiv u_{\mu}(x, \lambda)$ and hence $v_{\lambda}(x) \equiv v_{\lambda}(x)$.\]
In the proof of the uniqueness of the solution of the inverse problem the asymptotics of the solution

\[ u(x, \lambda) = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \phi(\lambda) \]

of (0.1) was used. (See also [SaA2].) Other interesting generalizations of the notion of the Weyl function connected with the asymptotics of the solution of the differential equation one can find, for instance, in [L, BC, BDT, Y, DZ, BDZ, and FI].

Certain connections exist between the Weyl function \( \phi \) and the spectrum of the operators \( H_k \) \((k = 1, 2)\), defined by the differential expression

\[ Hf = \left( -\partial d \frac{d}{dx} - jV(x) \right) f \]

on the absolutely continuous functions \( f (f \in L^2_{ac}(0, \infty), f' \in L^2_{ac}(0, \infty)) \) satisfying the initial conditions

\[ [0 \quad I_m] f(0) = 0 \quad \text{in case } k = 1, \quad [I_m \quad 0] f(0) = 0 \quad \text{in case } k = 2. \]

(2.29)

**Proposition 2.4.** Let \( \phi \) be the Weyl function of the system (0.1) with the potential \( v \in PE(2) \). Then the poles of \( \phi \) in \( \mathbb{C}_- \) are the eigenvalues of \( H_1 \) and the zeros of \( \phi \) in \( \mathbb{C}_- \) are the eigenvalues of \( H_2 \).

**Proof.** Let \( \phi \) admit the representation

\[ \phi(\lambda) = \frac{v}{(\lambda - \mu)^r} + O((\lambda - \mu)^{1-\epsilon}) \quad (\lambda \to \mu \in \mathbb{C}_-). \]

By virtue of (2.10) and (2.11) it follows that the columns of

\[ f(x) = u(x, \mu) \begin{bmatrix} v \\ 0 \end{bmatrix} \]

belong to \( L^2_{ac}(0, \infty) \). According to (0.1) and (2.28) we have also \( Hf = \mu f \). Hence it can be easily seen that the columns of \( f \) are the eigenvectors of \( H_1 \) corresponding to the eigenvalue \( \mu \).

Let \( \phi \) satisfy the equality

\[ \phi(\lambda) h = O(\lambda - \mu) \quad (\lambda \to \mu \in \mathbb{C}_-, h \in \mathbb{C}^m). \]
By virtue of (2.10) and (2.11) it follows now that

\[ f(x) = u(x, \mu) \begin{bmatrix} 0 \\ h \end{bmatrix} \]

belongs to \( L_2^+(0, \infty) \). According to (0.1) and (2.28) we have again \( Hf = \mu f \)
and the initial condition for \( H_2 \) is satisfied. Hence \( f \) is the eigenvector of \( H_2 \)
corresponding to the eigenvalue \( \mu \). □

3. PARAMETRIZATION OF PSEUDO-EXPONENTIAL POTENTIALS AND BISPECTRALITY

3.1. Parametrization

We say that the triples \( \theta_1, \theta_2 \) and \( \pi \) and \( \tilde{\theta}_1, \tilde{\theta}_2 \) and \( \tilde{\pi} \) are unitarily equivalent if there exists a unitary matrix \( q \) such that \( \tilde{\theta}_1 = q \theta_1, \tilde{\theta}_2 = q \theta_2 \) and \( \tilde{\pi} = q \pi q^* \). It is easily seen from (1.1)–(1.3) that unitarily equivalent triples
determine the same function \( v \in \text{PE}(2) \).

**Proposition 3.1.** Every pseudo-exponential potential is determined by
some triple \( \theta_1, \theta_2 \) and \( \pi \), for which the pair \( \beta \) and \( \theta_2 \), where \( \beta = \pi - i \theta_1 \theta_2^* \),
is controllable, the pair \( \theta_2^* \) and \( \beta \) is observable and the additional property
(0.4) holds. This correspondence is unique up to the unitary equivalence of the
triples.

**Proof.** According to Theorems 2.1 and 2.3 we can obtain such a triple
\( \theta_1, \theta_2 \) and \( \pi \) by formula (2.18) from the minimal realization of the Weyl
function. Moreover any other triple \( \tilde{\theta}_1, \tilde{\theta}_2 \) and \( \tilde{\pi} \) with the same properties
determining the same potential gives a minimal realization of the same
Weyl function. Hence we have

\[
\tilde{\theta}_1 = (q^*)^{-1} \theta_1, \quad \tilde{\theta}_2 = q \theta_2, \quad \tilde{\beta} = q \beta q^{-1}
\]

for some invertible matrix \( q \). Taking into account property (0.4) for both
triples we see that the matrices \( X = I_m \) and \( X = q^{-1}(q^*)^{-1} \) satisfy the
algebraic Riccati equation

\[
\beta X - X \beta^* = i(X \theta_1 \theta_1^* X - \theta_2 \theta_2^* X).
\]

As this equation has a unique nonnegative solution [LR1, Section 2],
so \( q^{-1}(q^*)^{-1} = I_m \) and the matrix \( q \) is unitary, i.e., the triples are unitarily
equivalent. □
3.2. Bispectrality

As in [GKS] a certain subclass of the potentials considered in the present paper gives rise to a phenomenon of modified bispectrality. To consider this phenomenon we have to make some preparations.

First we notice that in a triple \( \theta_1, \theta_2 \) and \( x \) the matrix \( x \) can be taken to be lower triangular. Indeed by a proper choice of \( q \) we can always obtain a lower triangular matrix \( \tilde{x} = g q x^q \). So all the functions \( v \in \text{PE}(2) \) are determined by triples \( \theta_1, \theta_2 \) and \( x \) with a lower triangular matrix \( x \).

We shall suppose \( x \) to be lower triangular in this subsection. By virtue of (1.1) and (0.4) we have

\[
\alpha - \alpha^* = iA(0) A(0)^* \quad \text{and} \quad A(0) = [\theta_1, \theta_2].
\]

Therefore the lower triangular matrix \( \alpha \) is except for its main diagonal uniquely determined by \( \theta_1 \) and \( \theta_2 \). If, additionally, the rows in \( A(0) \) have the same norm, we see from (3.1) that we can choose the matrix \( x \) to be of the form

\[
\alpha = i \hbar \nu + \sigma_0, \quad (3.2)
\]

where the Euclidean norms of the rows of \( A(0) \) equal \( \sqrt{2\hbar} \) and \( \sigma_0 \) is a nilpotent matrix.

For the case of canonical systems we could choose the matrix \( x \) to be nilpotent, in which case the corresponding potential is rational. Here \( x \) may be nilpotent (or equivalently the main diagonal in the right-hand side of (3.1) may be equal zero) only for the trivial potential \( v = 0 \). Therefore rational potentials cannot be obtained in this way.

Still, if the representation (3.2) holds, the transfer matrix function \( w_{\alpha}(x, \lambda) \) proves to be polynomial in \((\lambda - i\hbar)^{-1}\). Suppose \( \sigma_0 = 0 \). Then we have

\[
(\lambda I_n - x)^{-1} = ((\lambda - i\hbar) I_n - \sigma_0)^{-1} = (\lambda - i\hbar)^{-1} \sum_{p=0}^{r-1} \left( \frac{\sigma_0}{\lambda - i\hbar} \right)^p.
\]

Hence, from the definition (1.13) we obtain

\[
w_{\alpha, A}(x, \lambda) = I_{2m} + (\lambda - i\hbar)^{-1} A(x)^* S(x)^{-1} \left( \sum_{p=0}^{r-1} \left( \frac{\sigma_0}{\lambda - i\hbar} \right)^p \right) A(x). \quad (3.3)
\]

By virtue of relation (3.3) the pseudo-canonical systems with pseudo-exponential potentials considered above have an interesting property of bispectrality. This notion was introduced by Duistermaat and Grünbaum [DG] for the Schrödinger operator \( L = -\partial_x^2 + v(x) \). The operator \( L \) is said to have the bispectral property if there is a differential equation
\[ B(\lambda, \partial \lambda) \psi(x, \lambda) = \alpha(x) \psi(x, \lambda) \quad (B(\lambda, \partial \lambda) = \sum_{n=0}^{k} B_n(\lambda)(\partial/\partial \lambda)^n) \] which has nonzero solutions in two variables \( x \) and \( \lambda \) in common with \( L \psi = \lambda^2 \psi. \n\] In this case \( \psi \) is called a bispectral eigenfunction. In [Zu] an analogous definition of bispectrality was introduced for the canonical systems and bispectrality for several cases of rational potentials was proved. The basic fact in the proof (see [Zu, p. 79]) was connected with a representation of the fundamental solution \( u(x, \lambda) \) that could also be obtained from Theorem 4.2 [GKS] in the case of a nilpotent \( x \). The notion of the modified bispectral property from [GKS] may be easily transferred to the case of the pseudo-canonical system (0.1). We say that the pseudo-canonical system (0.1) with the potential \( v \) (or the potential \( v \) itself) has the modified bispectral property if there is a set of complex numbers \( c_p, 0 \leq p \leq k \), such that the solution \( u(x, \lambda) \) of the pseudo-canonical system satisfies for each \( x \) a differential equation in \( \lambda \):

\[ \sum_{p=1}^{k} c_p(\lambda - c_p)^p \frac{\partial^p}{\partial \lambda^p} (u(x, \lambda) e^{-i\alpha x}) = 0 \quad \left( \sum_{p=1}^{k} |c_p| \neq 0 \right). \]

**Proposition 3.2.** Let \( v \in PE(2) \) and \( \sigma(x) = i\hbar \). Then the pseudo-canonical system (0.1) has the modified bispectral property.

**Proof.** By Theorem 1.2 the matrix function \( u(x, \lambda) = w_{x,A}(x, \lambda) e^{i\alpha x} \) satisfies (0.1). Therefore if \( \sigma(x) = i\hbar \) (i.e., \( \alpha_* = 0 \)), then \( u(x, \lambda) e^{-i\alpha x} \) can be represented in the form (3.3). Put now \( c_0 = i\hbar \) and the proposition follows.

Let us calculate potentials with the modified bispectral property in case \( \sigma(x) = i\hbar \) \((h > 0)\). There exists a unique solution \( s \) of the equation

\[ zs - sz^* = i\partial_1 \theta^*_1. \quad (3.4) \]

Notice that by (3.1) and (3.4) we have

\[ z(I - s) - (I - s) z^* = i\partial_2 \theta^*_2. \quad (3.5) \]

Hence the matrix function \( Q(x) = e^{-2i\alpha x} + (I - s) e^{-2i\alpha x^*} \) satisfies equation (1.10), i.e.,

\[ Q'(x) = -i(\sigma Q(x) + Q(x) z^*) + e^{-2i\alpha x} \partial_1 \theta^*_1 - \partial_2 \theta^*_2 e^{-2i\alpha x^*}, \]

and \( Q \) has initial value \( Q(0) = I_\nu \). Thus \( Q(x) \) satisfies the same differential equation as \( e^{-i\alpha x^*} S(x) e^{-i\alpha x^*} \) with the same initial condition (see the proof of Proposition 1.1), i.e., we have \( e^{-i\alpha x^*} S(x) e^{-i\alpha x^*} = e^{-2i\alpha x} + (I - s) e^{-2i\alpha x^*} \). Representation (1.3) of the potential \( v \) yields now
4. NONLINEAR EQUATIONS

The representation (0.2) of a PE(2) function proves to be very useful in the theory of the integrable nonlinear equations.

**Theorem 4.1.** Let \( \theta_1, \theta_2 \) and \( \alpha \) be a matrix triple with the additional property

\[
\alpha - \alpha^* = i(\theta_1 \theta_1^* + \theta_2 \theta_2^*). \tag{4.1}
\]

Then the matrix function

\[
v(x, t) = -2i\theta_1^* \left( [I_n, -iA_n] e^{-2ixA + i\varepsilon^t} \right)^{-1} \theta_2, \tag{4.2}
\]

where

\[
A = \begin{bmatrix}
\alpha^* & 0 \\
-\theta_1 \theta_1^* & \alpha
\end{bmatrix}, \tag{4.3}
\]

belongs to PE(2) in \( x \) for each \( t \) in \( 0 \leq t < \varepsilon \) for some \( \varepsilon > 0 \). Moreover for \( k = 2 \) the function \( v \) is a solution of the matrix nonlinear Schrödinger equation (matrix NSE)

\[
2\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial x^2} + 2ivv^*v = 0, \tag{4.4}
\]

and for \( k = 3 \) of the matrix modified Korteweg–de Vries equation (matrix MKdVE)

\[
4\frac{\partial v}{\partial t} + \frac{\partial^3 v}{\partial x^3} + 3 \left( \frac{\partial v}{\partial x} v^*v + vv^* \frac{\partial v}{\partial x} \right) = 0. \tag{4.5}
\]
Proof. Step 1. Let us consider the matrix function
\[ Q(x, t) = \begin{bmatrix} I_n & -iL_n \end{bmatrix} e^{-2i(xA + tA'^*)} \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \] (4.6)
whose inverse appears in the right-hand side of (4.2). At first we shall obtain expressions for the derivatives \( Q_x, Q_t \) and prove that \( Q(x, t) \) is invertible in some half strip \( D = \{ (x, t) \colon x \geq 0, 0 \leq t < t' \} \) and so \( \nu(x, t) \) is well defined in the same half strip \( D \). Taking into account (4.1) we get, analogously to (1.10), the formula
\[ \frac{\partial Q(x, t)}{\partial x} = -i(\alpha Q(x, t) + Q(x, t) \alpha^*) + e^{-2i(xA + tA'^*)}(\theta_1 \theta_1^* - \theta_2 \theta_2^*) e^{-2i(xA + tA'^*)}(\alpha^*)^{k-h}. \] (4.7)
From (4.1) it follows also that
\[ \alpha^k - (\alpha^*)^k = \sum_{h=1}^{k} \alpha^{h-1}(\alpha - \alpha^*)^{(\alpha^*)^{k-h}} = i \sum_{h=1}^{k} \alpha^{h-1}(\theta_1 \theta_1^* + \theta_2 \theta_2^*)^{(\alpha^*)^{k-h}}. \] (4.8)
The equality
\[ A^k = \begin{bmatrix} (\alpha^*)^k \\ -\sum_{h=1}^{k} \alpha^{h-1} \theta_1 \theta_1^* (\alpha^*)^{k-h} \alpha^k \end{bmatrix}, \] (4.9)
can be easily proved by induction. According to (4.8) and (4.9) we have
\[ \frac{\partial Q(x, t)}{\partial t} = -i \left\{ \begin{bmatrix} I_n & -iL_n \end{bmatrix} A^k e^{-2i(xA + tA'^*)} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \right\} + \begin{bmatrix} I_n & -iL_n \end{bmatrix} e^{-2i(xA + tA'^*)} A^k \begin{bmatrix} I_n \\ 0 \end{bmatrix} = -i(\alpha^k Q(x, t) + Q(x, t) (\alpha^*)^k) \]
\[ + \sum_{h=1}^{k} \alpha^{h-1}(e^{-2i(xA + tA'^*)}(\theta_1 \theta_1^* - \theta_2 \theta_2^*) e^{-2i(xA + tA'^*)}(\alpha^*)^{k-h}). \] (4.10)
We shall introduce now matrix functions $S(x, t)$ and $A(x, t)$ by the formulas

$$
A(x, t) = [A_1(x, t) \ A_2(x, t)] = [e^{-i(x+td)\theta_1} \ e^{i(x+td)\theta_2}],
$$

$$
S(x, t) = e^{i(x+td)Q(x, t)} e^{i(x+td)P}. \tag{4.11}
$$

Taking into account (4.7), (4.10), and (4.11) we obtain

$$
S(0, 0) = I, \quad \frac{\partial S(x, t)}{\partial x} = A(x, t) j A^*(x, t); \tag{4.12}
$$

$$
\frac{\partial S(x, t)}{\partial t} = \sum_{k=1}^{K} A(x, t) j A^*(x, t)(\pi^*)^{k-1}; \tag{4.13}
$$

$$
\frac{\partial A(x, t)}{\partial t} = -ix A(x, t) j, \quad \frac{\partial A(x, t)}{\partial t} = -ix A(x, t) j. \tag{4.14}
$$

According to (4.1) and (4.12)–(4.14) we get

$$
\pi S(0, 0) - S(0, 0) \pi^* = iA(0, 0) A(0, 0)^*,
$$

$$
\pi \frac{\partial S}{\partial x} - \frac{\partial S}{\partial x} \pi^* = i \frac{\partial A A^*}{\partial x},
$$

and hence

$$
\pi S(x, t) - S(x, t) \pi^* = iA(x, t) A(x, t)^*. \tag{4.15}
$$

By virtue of (4.12) and (4.15) we have

$$
\frac{\partial}{\partial x} \left( e^{-i\pi x S(x, t)} e^{i\pi x^*} \right) = e^{-i\pi x} \left( -i(\pi S(x, t) - S(x, t) \pi^*) + \frac{\partial}{\partial x} S(x, t) \right) e^{i\pi x^*}
$$

$$
= e^{-i\pi x} A(x, t)(I_{2m} + j) A(x, t)^* e^{i\pi x^*}.
$$

In particular, $(\partial/\partial x)(e^{-i\pi x S(x, t)} e^{i\pi x^*}) \geq 0$. As, by (4.12), $S(0, 0) = I$, we see now that for sufficiently small $\pi > 0$ the matrix function $S(0, t) > 0$ for $t$ in $0 \leq t < \varepsilon$ and hence

$$
S(x, t) = S(x, t)^* > 0 \quad (0 \leq t < \varepsilon). \tag{4.16}
$$

Invertibility of $Q$ follows from (4.11) and (4.16), in particular, and so the right-hand side of (4.2) is well defined.
Step 2. To prove that \( v(x, t) \in \text{PE}(2) \) we shall introduce new matrix functions \( \hat{\theta}_1, \hat{\theta}_2, \hat{\alpha}, \hat{S}, \hat{A} \) by setting

\[
\hat{S}(x, t) = S(0, t)^{-1/2} S(x, t) S(0, t)^{-1/2},
\]
\[
\hat{A}(x, t) = \begin{bmatrix} \hat{A}_1(x, t) & \hat{A}_2(x, t) \end{bmatrix} = S(0, t)^{-1/2} A(x, t),
\]
\[
[\hat{\theta}_1(t) \quad \hat{\theta}_2(t)] = \hat{A}(0, t),
\]
\[
\hat{\alpha}(t) = S(0, t)^{-1/2} z S(0, t)^{1/2}.
\]  

(4.17)

By virtue of (4.11) and (4.17) formula (4.2) can be rewritten in the form

\[
v(x, t) = -2i \hat{A}_1(x, t)^* S(x, t)^{-1} A_2(x, t) = -2i \hat{A}_1(x, t)^* \hat{S}(x, t)^{-1} \hat{A}_2(x, t).
\]  

(4.18)

From (4.11) and (4.17) we obtain

\[
\hat{A}(x, t) = \begin{bmatrix} e^{-ix(t)} \hat{\theta}_1(t) & e^{ix(t)} \hat{\theta}_2(t) \end{bmatrix},
\]
\[
\hat{S}(0, t) = I_n, \quad \frac{\partial \hat{S}(x, t)}{\partial x} = \hat{A}(x, t) j \hat{A}(x, t)^*.
\]  

(4.19)  

(4.20)

From (4.15) and (4.17) we get also

\[
\hat{\alpha}(t) - \hat{\alpha}(t)^* = i \hat{A}(0, t) \hat{A}(0, t)^*.
\]  

(4.21)

Introducing the operator \( \hat{A}(t) \) by the equality

\[
\hat{A}(t) = \begin{bmatrix} \hat{\alpha}(t)^* & 0 \\ -\hat{\theta}_1(t) \hat{\theta}_2(t)^* & \hat{\alpha}(t) \end{bmatrix}
\]  

(4.22)

and taking into account (4.19)–(4.22) we can repeat the considerations of Section 1 and get

\[
P e^{-2ix(\hat{A}(t))|_{\text{Im}}} = e^{-ix(t)} S(x, t) e^{-ix(t)^*}.
\]  

(4.23)

By virtue of (4.18), (4.19), and (4.23) the representation

\[
v(x, t) = -2i \hat{\theta}_1^*(t)(Pe^{-2ix(\hat{A}(t))}|_{\text{Im}})^{-1} \hat{\theta}_2(t)
\]  

(4.24)

is true, i.e., \( v \in \text{PE}(2) \).
Step 3. We shall prove now that \( v \) satisfies nonlinear equations. The important part of the proof is the computation of \((\partial^2 Q/\partial x^2) - (\partial Q/\partial x) Q^{-1}(\partial Q/\partial x)\). From (4.11) and (4.15) we get

\[
\sigma Q(x, t) - Q(x, t) x^* = i(e^{-(2i(\alpha x + \gamma t))\theta_0} + \theta_2 \theta_2^* e^{-2i(\alpha x^* + \gamma(\alpha x^)*)}).
\]

Formulas (4.7) and (4.25) yield

\[
\frac{\partial Q(x, t)}{\partial x} = -2i\alpha Q(x, t) - 2\theta_2 \theta_2^* e^{-2i(\alpha x^* + \gamma(\alpha x^)*)},
\]

\[
\frac{\partial Q(x, t)}{\partial x} = -2i Q(x, t) x^* + 2e^{-2i(\alpha x + \gamma t)} \theta_0 \theta_0^*.
\]

Let us differentiate both parts of (4.26) with respect to \( x \) and substitute the term \( \partial Q/\partial x \) in the right-hand side of the resulting identity by the right-hand side of (4.27). We get

\[
\frac{\partial^2 Q(x, t)}{\partial x^2} = -4 \left\{ \sigma Q(x, t) x^* + i \left( e^{-(2i(\alpha x + \gamma t))\theta_0} + \theta_2 \theta_2^* e^{-2i(\alpha x^* + \gamma(\alpha x^)*)} \right) \right\}.
\]

Taking into account (4.11) we obtain also

\[
Q(x, t)^{-1} = e^{2i(\alpha x + \gamma t)} (Q(x, t))^*^{-1} e^{2i(\alpha x + \gamma t)}.
\]

Step 4. Let us consider the case \( k = 2 \). By virtue of (4.6) it is easily seen that

\[-2i(\partial Q/\partial t) = (\partial^2 Q/\partial x^2) \]

and therefore we get
\[
2 \frac{\partial (Q(x,t)^{-1})}{\partial t} + i \frac{\partial^2 (Q(x,t)^{-1})}{\partial x^2}
= -2iQ(x,t)^{-1} \left( \frac{\partial^2 Q(x,t)}{\partial x^2} - \frac{\partial Q(x,t)}{\partial x} Q(x,t)^{-1} \frac{\partial Q(x,t)}{\partial x} \right) Q(x,t)^{-1}.
\]

(4.32)

From (4.2), (4.31), and (4.32) we derive
\[
2 \frac{\partial (Q(x,t)^{-1})}{\partial t} + i \frac{\partial^2 (Q(x,t)^{-1})}{\partial x^2} = -4Q(x,t)^{-1} \theta_2 \psi(x,t)* \theta_1^* Q(x,t)^{-1}.
\]

(4.33)

Finally according to (4.2) and (4.33) we obtain (4.4).

Let us consider now the case \( k = 3 \). In this case we have
\[
-4 \frac{\partial Q}{\partial t} = \frac{\partial^3 Q}{\partial x^3},
\]
\[
\frac{\partial^3 (Q(x,t)^{-1})}{\partial x^3} = -Q^{-1} \left( \frac{\partial^3 Q}{\partial x^3} + 3 \left( \frac{\partial^2 Q}{\partial x^2} - \frac{\partial Q}{\partial x} Q^{-1} \frac{\partial Q}{\partial x} \right) Q^{-1} \frac{\partial Q}{\partial x} \right) Q^{-1} \frac{\partial Q}{\partial x} - 3 \frac{\partial Q}{\partial x} Q^{-1} \left( \frac{\partial^2 Q}{\partial x^2} - \frac{\partial Q}{\partial x} Q^{-1} \frac{\partial Q}{\partial x} \right) Q^{-1} \frac{\partial Q}{\partial x} - Q^{-1} \left( \frac{\partial^2 Q}{\partial x^2} - \frac{\partial Q}{\partial x} Q^{-1} \frac{\partial Q}{\partial x} \right) Q^{-1} \frac{\partial Q}{\partial x}.
\]

Therefore, taking into account (4.2) and (4.31) we obtain
\[
4 \frac{\partial (Q(x,t)^{-1})}{\partial t} + \frac{\partial^3 (Q(x,t)^{-1})}{\partial x^3}
= 6iQ(x,t)^{-1} \theta_2 \psi(x,t)* \theta_1^* \frac{\partial (Q(x,t)^{-1})}{\partial x}
+ \frac{\partial (Q(x,t)^{-1})}{\partial x} \theta_2 \psi(x,t)* \theta_1^* Q(x,t)^{-1}.
\]

(4.34)

According to (4.2) and (4.34) the function \( \psi \) satisfies (4.5). 

The famous N-soliton solutions (see [M, ASe] and references therein), the well-known rational solutions ([ASa] and [EKK]) and the so-called rational-exponential solutions [Bez] can be presented in the form (4.2) by choosing \( \theta_1, \theta_2 \) and \( x \) in appropriate way. In [SaA3] these solutions were expressed through matrix functions \( A \) and \( S \) satisfying (4.12)-(4.14). The inverse problems on the half axis are essential also in the initial-boundary value problems for the integrable nonlinear equations (see [SaL2, SaA1, F1] and references therein).
REFERENCES


