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The Linear Algebra of the Generalized Pascal Matrix

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ABSTRACT

This paper discusses three kinds of generalized Pascal matrix, and generalizes the results of R. Brawer and M. Pirovino. © *Elsevier Science Inc., 1997*

Let x be any nonzero real number. The generalized Pascal matrix of the first kind, $P_n[x]$, is defined as (see [1])

$$P_n(x;i,j) = x^{i-j}\binom{i}{j}, \qquad i,j = 0,\dots,n,$$

with

$$\binom{i}{j} = 0 \quad \text{if} \quad j > i.$$

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Further we define the $(n+1)\times(n+1)$ matrices I_n, $S_n[x]$, and $D_n[x]$ by

$$I_n = \operatorname{diag}(1, 1, \ldots, 1),$$

$$S_n(x; i, j) = \begin{cases} x^{i-j} & \text{if } j \leqslant i, \\ 0 & \text{if } j > i, \end{cases}$$

$$D_n(x; i, i) = 1 \qquad \text{for } i = 0, \ldots, n,$$

$$D_n(x; i+1, i) = -x \qquad \text{for } i = 0, \ldots, n-1,$$

$$D_n(x; i, j) = 0 \qquad \text{if } j > i \text{ or } j < i - 1.$$

It is easy to see that

LEMMA 1.

$$S_n[x] = D_n^{-1}[x],$$

$$P_n^{-1}[x] = P_n[-x].$$

EXAMPLE.

$$S_2[x]D_2[x] = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -x & 1 & 0 \\ 0 & -x & 1 \end{bmatrix} = I_2,$$

$$P_3[x]P_3[-x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 & 2x & 1 & 0 \\ x^3 & 3x^2 & 3x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -x & 1 & 0 & 0 \\ x^2 & -2x & 1 & 0 \\ -x^3 & 3x^2 & -3x & 1 \end{bmatrix} = I_3.$$

Furthermore we need the matrices

$$\bar{P}_k[x] = \begin{bmatrix} 1 & 0^T \\ 0 & P_k[x] \end{bmatrix} \in R^{(k+2)\times(k+2)}, \qquad k \geqslant 0,$$

$$G_k[x] = \begin{bmatrix} I_{n-k-1} & 0 \\ 0 & S_k[x] \end{bmatrix} \in R^{(n+1)\times(n+1)}, \qquad k = 1, \ldots, n-1,$$

and $G_n[x] = S_n[x]$.

LEMMA 2.

$$S_k[x]\overline{P}_{k-1}[x] = P_k[x] \qquad \textit{for} \quad k \geqslant 1.$$

Proof. The (i, j) element of $\overline{P}_{k-1}[x]$ is

$$\binom{i-1}{j-1} x^{i-j} \qquad (i, j = 1, 2, \dots, k),$$

or 1 $(i = 0,\ j = 0)$, or 0 $(i \neq 0,\ j = 0)$ or $(i = 0,\ j \neq 0)$.

Let $S_k[x]\overline{P}_{k-1}[x] = (C_k(x; i, j))$. Obviously, $C_k(x; i, 0) = x^{i-0}$ $(i = 0, 1, 2, \dots, n)$ and $C_k(x; i, j) = 0$ $(i < j)$. When $i > j$, we have

$$\begin{aligned} C_k(x; i, j) &= \sum_{h=0}^{k} x^{i-h} \binom{h-1}{j-1} x^{h-j} \\ &= \left[\sum_{h=0}^{i} \binom{h-1}{j-1} \right] x^{h-j} = \binom{i}{j} x^{i-j} \end{aligned}$$

Thus, $S_k[x]\overline{P}_{k-1}[x] = P_k[x]$. ■

EXAMPLE.

$$\begin{aligned} S_3[x]\overline{P}_2[x] &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 & x & 1 & 0 \\ x^3 & x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & x^2 & 2x & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 & 2x & 1 & 0 \\ x^3 & 3x^2 & 3x & 1 \end{bmatrix}. \end{aligned}$$

An immediate consequence of Lemma 2 and the definition of the $G_k[x]$'s is

THEOREM 1. *The generalized Pascal matrix of first kind, $P_n[x]$, can be factorized by the summation matrices $G_k[x]$:*

$$P_n[x] = G_n[x] G_{n-1}[x] \cdots G_1[x]. \tag{1}$$

EXAMPLE.

$$P_3[x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 & 2x & 1 & 0 \\ x^3 & 3x^2 & 3x & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 & x & 1 & 0 \\ x^3 & x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & x^2 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & x & 1 \end{bmatrix}.$$

For the inverse of the generalized Pascal matrix of the first kind, $P_n[x]$, we get

$$P_n^{-1}[x] = G_1^{-1}[x]G_2^{-1}[x] \cdots G_n^{-1}[x]$$
$$= F_1[x]F_2[x] \cdots F_n[x]$$

with

$$F_k[x] = G_k^{-1}[x] = \begin{bmatrix} I_{n-k-1} & 0 \\ 0 & D_k[x] \end{bmatrix}, \qquad k = 1, \dots, n-1,$$

and

$$F_n[x] = G_n^{-1}[x] = D_n[x].$$

Using Lemma 1, we have

THEOREM 2.

$$P_n^{-1}[x] = P_n[-x] = F_1[x]F_2[x] \cdots F_n[x]. \tag{2}$$

In particular,

$$P_n^{-1}[x] = P_n[-x] = J_n P_n[x] J_n, \tag{3}$$

where

$$J_n = \operatorname{diag}\left(1, -1, 1, \dots, (-1)^n\right) \in R^{(n+1)\times(n+1)}.$$

Equation (3) represents the well-known inverse relation

$$x^{n-k}\delta_{n,k} = \sum_{j=k}^{n} (-1)^{j+k} x^{n-j}\binom{n}{j} x^{j-k}\binom{j}{k},$$

that is,

$$\delta_{n,k} = \sum_{j=k}^{n} (-1)^{j+k}\binom{n}{j}\binom{j}{k} \qquad (\text{see [3]}).$$

We define the generalized Pascal matrix of the second kind, $Q_n[x]$, as

$$Q_n(x; i, j) = x^{i+j}\binom{i}{j}, \qquad i, j = 0, \dots, n.$$

Similarly, we define the $(n+1)\times(n+1)$ matrices $M_n[x]$, $N_n[x]$ by

$$M_n(x; i, j) = \begin{cases} x^{i+j} & \text{if } j \leqslant i, \\ 0 & \text{if } j > i, \end{cases}$$

$$N_n(x; i, i) = \frac{1}{x^{i+j}} \qquad \text{for } i = 0, \dots, n, \quad x \neq 0,$$

$$N_n(x; i+1, i) = \frac{1}{(-x)^{i+j}} \qquad \text{for } i = 0, \dots, n-1, \quad x \neq 0,$$

$$N_n(x; i, j) = 0 \qquad \text{if } j > i \text{ or } j < i - 1.$$

It is easy to see that

LEMMA 3.

$$M_n[x] = N_n^{-1}[x],$$

$$Q_n^{-1}[x] = Q_n\left[-\frac{1}{x}\right].$$

EXAMPLE.

$$M_3[x]N_3[x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & x^2 & 0 & 0 \\ x^2 & x^3 & x^4 & 0 \\ x^3 & x^4 & x^5 & x^6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{x} & \frac{1}{x^2} & 0 & 0 \\ 0 & -\frac{1}{x^3} & \frac{1}{x^4} & 0 \\ 0 & 0 & -\frac{1}{x^5} & \frac{1}{x^6} \end{bmatrix} = I_3,$$

$$Q_3[x]Q_3\left[-\frac{1}{x}\right]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & x^2 & 0 & 0 \\ x^2 & 2x^3 & x^4 & 0 \\ x^3 & 3x^4 & 3x^5 & x^6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{x} & \frac{1}{x^2} & 0 & 0 \\ \frac{1}{x^2} & -\frac{2}{x^3} & \frac{1}{x^4} & 0 \\ -\frac{1}{x^3} & \frac{3}{x^4} & -\frac{3}{x^5} & \frac{1}{x^6} \end{bmatrix} = I_3.$$

By the definition of $\bar{P}_k[x]$, we get

LEMMA 4.

$$M_k[x]\bar{P}_{k-1}\left[\frac{1}{x}\right] = Q_k[x] \qquad \textit{for} \quad k \geqslant 1.$$

Proof. Let $M_k[x]\bar{P}_{k-1}[1/x] = (C_k(x; i, j))$; then $C_k(x; i, 0) = x^i$ ($i = 0, \ldots, k$) and $C_k(x; i, j) = 0$ ($i < j$). When $i > j$ we have

$$C_k(x; i, j) = \sum_{h=0}^{k} x^{i+h} \binom{h-1}{j-1} \frac{1}{x^{n-j}}$$

$$= \sum_{h=0}^{i} \binom{h-1}{j-1} x^{i+j} = \binom{i}{j} x^{i+j}.$$

Thus,

$$M_k[x]\bar{P}_{k-1}\left[\frac{1}{x}\right] = Q_k[x].$$

■

EXAMPLE.

$$M_3[x]\bar{P}_2\left[\frac{1}{x}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & x^2 & 0 & 0 \\ x^2 & x^3 & x^4 & 0 \\ x^3 & x^4 & x^5 & x^6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{1}{x} & 1 & 0 \\ 0 & \frac{1}{x^2} & \frac{2}{x} & 1 \end{bmatrix} = Q_3[x].$$

An immediate consequence of Lemma 4 and the definition of the $G_k[x]$'s is

THEOREM 3. *The generalized Pascal matrix of the second kind, $Q_n[x]$, can be factorized by the summations $G_k[x]$ and $M_n[x]$:*

$$Q_n[x] = M_n[x]G_{n-1}\left[\frac{1}{x}\right]G_{n-2}\left[\frac{1}{x}\right]\cdots G_1\left[\frac{1}{x}\right].$$

EXAMPLE.

$$Q_3[x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & x^2 & 0 & 0 \\ x^2 & 2x^3 & x^4 & 0 \\ x^3 & 3x^4 & 3x^5 & x^6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & x^2 & 0 & 0 \\ x^2 & x^3 & x^4 & 0 \\ x^3 & x^4 & x^5 & x^6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{1}{x} & 1 & 0 \\ 0 & \frac{1}{x^2} & \frac{1}{x} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{x} & 1 \end{bmatrix}.$$

For the inverse of the generalized Pascal matrix of the second kind, $Q_n[x]$, we get

$$Q_n^{-1}[x] = G_1^{-1}\left[\frac{1}{x}\right]G_2^{-1}\left[\frac{1}{x}\right] \cdots G_{n-1}^{-1}\left[\frac{1}{x}\right]M_n^{-1}[x]$$

$$= F_1\left[\frac{1}{x}\right]F_2\left[\frac{1}{x}\right] \cdots F_{n-1}\left[\frac{1}{x}\right]N_n[x].$$

Using Lemma 3, we have

THEOREM 4.

$$Q_n^{-1}[x] = Q_n\left[-\frac{1}{x}\right] = F_1\left[\frac{1}{x}\right]F_2\left[\frac{1}{x}\right] \cdots F_{n-1}\left[\frac{1}{x}\right]N_n[x].$$

In particular

$$Q_n^{-1}[x] = J_n^* Q_n[x] J_n^*$$

where $J_n^* = \operatorname{diag}(1, -\frac{1}{x^2}, \frac{1}{x^4}, -\frac{1}{x^6}, \dots, (-1)^n \frac{1}{x^{2n}}) \in R^{(n+1)\times(n+1)}$.

We define the symmetric generalized Pascal matrix $R_n[x]$ as

$$R_n(x; i, j) = x^{i+j}\binom{i+j}{j}, \qquad i, j = 0, \dots, n.$$

THEOREM 5. *One has*

$$F_1[x]F_2[x] \cdots F_{n-1}[x]F_n[x]R_n[x] = Q_n^T[x],$$

$$F_1\left[\frac{1}{x}\right]F_2\left[\frac{1}{x}\right] \cdots F_{n-1}\left[\frac{1}{x}\right]N_n[x]R_n[x] = P_n^T[x],$$

and the Cholesky factorization [4] *of* $R_n[x]$ *is given by*

$$R_n[x] = Q_n[x]P_n^T[x] = P_n[x]Q_n^T[x].$$

Proof. Let $Q_n[x]P_n^T[x] = (C_n(x; i, j))$. Then

$$C_n(x;i,j) = \begin{cases} \sum_{k=0}^{j} \binom{i}{k}\binom{j}{k} x^{i+j}, & i \geqslant j, \\ \sum_{k=0}^{i} \binom{i}{k}\binom{j}{k} x^{i+j}, & i < j, \end{cases}$$

$$\sum_{k=0}^{i} \binom{i}{k}\binom{j}{k} = \sum_{k=0}^{i} \binom{i}{k}\binom{j}{j-k} = \binom{i+j}{j},$$

$$\sum_{k=0}^{j} \binom{i}{k}\binom{j}{k} = \sum_{k=0}^{j} \binom{i}{i-k}\binom{j}{k} = \binom{i+j}{j}$$

(Vandermonde identities). Thus, we have

$$Q_n[x]P_n^T[x] = R_n[x].$$

Similarly

$$P_n[x]Q_n^T[x] = R_n[x].$$

EXAMPLE.

$$R_3[x] = \begin{bmatrix} 1 & x & x^2 & x^3 \\ x & 2x^2 & 3x^3 & 4x^4 \\ x^2 & 3x^3 & 6x^4 & 10x^5 \\ x^3 & 4x^4 & 10x^5 & 20x^6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & x^2 & 0 & 0 \\ x^2 & 2x^3 & x^4 & 0 \\ x^3 & 3x^4 & 3x^5 & x^6 \end{bmatrix} \begin{bmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 1 & 3x \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using Lemmas 1 and 3, we have

THEOREM 6.

$$R_n^{-1}[x] = P_n^T[-x]Q_n\left[-\frac{1}{x}\right] = Q_n^T\left[-\frac{1}{x}\right]P_n[-x].$$

Using Theorems 2 and 5, we get

THEOREM 7.

$$R_n^{-1}[x] = J_n P_n^T[x]J_n J_n^* Q_n[x]J_n^* = J_n^* Q_n^T[x]J_n^* J_n P_n[x]J_n.$$

For the previous three kinds of generalized Pascal matrix, we also can get

THEOREM 8.

$$\det P_n[x] = \det P_n^{-1}[x] = 1,$$
$$\det Q_n[x] = x^{n(n+1)},$$
$$\det Q_n^{-1}[x] = x^{-n(n+1)},$$
$$\det R_n[x] = \det R_n^{-1}[x] = x^{n(n+1)}.$$

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