Learning Marginal-Cost Pricing via Trial-and-Error Procedure with Day-to-Day Flow Dynamics

Hongbo Ye a, Hai Yang a, Zhijia Tan b,*

a Department of Civil and Environmental Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, PR China
b School of Management, Huazhong University of Science & Technology, Luoyu Road 1037, Wuhan, PR China

Abstract

This paper investigates the convergence of the trial-and-error procedure to achieve system optimum by incorporating day-to-day evolution of traffic flows. The path flows are assumed to follow a so-called ‘excess travel cost dynamics’ and evolve from dis-equilibrium states to the equilibrium day by day. With this consideration, the observed link flow pattern during the trial-and-error procedure is in disequilibrium. With certain assumptions on the flow evolution dynamics, we prove that the trial-and-error procedure is capable of learning the system optimum link tolls without requirement of explicit knowledge on the demand functions and flow evolution mechanism. A methodology is developed for updating the toll charges and choosing the inter-trial periods to assure convergence of the iterative approach towards the system optimum. Some numerical examples are conducted to support the theoretical findings.

1. Introduction

As rational road users selfishly minimize their own travel cost, the user equilibrium (UE) flow patterns usually deviate from the system optimum (SO), which describes the status of minimal total travel time in a traffic network. To achieve this system optimum state, researchers have dedicated to design appropriate mechanisms, among which is the well-known first-best road pricing scheme. It is increasingly believed that road pricing may offer an effective and efficient instrument to relieve traffic congestion, reduce vehicular emissions, manage travel demand and achieve transportation sustainability. The initial idea of road pricing can trace back to Pigou (1920) and the followers such as
Yang and Huang (2005), and de Palma and Lindsey (2011). In comparison with the booming development in the academia, the implementation of congestion pricing is only limited in a dozen of cities, the rejected proposals are far more than the cases in use. Besides the political reasons, from the perspective of practical implementation, there may be other barriers that impede the promotion of congestion pricing from a purely economic concept to a comprehensive and practical traffic regulation policy.

Exact calculation of the first-best tolls requires explicit and analytical demand functions, which are difficult to establish in practice (Walters, 1961), and the commonly-used linear or exponential demand functions are usually too arbitrary and not convincing (Li, 2002). Fortunately, this issue can be circumvented and the congestion pricing can still proceed on a trial-and-error basis without demand functions. This enlightened idea was proposed by Vickrey (1993) and Downs (1993) and accomplished for the first time when Li (1999, 2002) gave an iterative bi-section algorithm that can be applied to a homogeneous traffic stream along a single expressway. The trial-and-error method allows a traffic planner to estimate or update the tolls easily by using readily available traffic count data while requiring the travel cost functions only. In the same spirit, Yang et al. (2004) suggested an algorithm based on the method of successive average (MSA) (Powell and Sheffi, 1982) and presented a rigorous theoretical proof of its convergence in a general network, which was later modified by Han and Yang (2009) with a faster convergence. Yang et al. (2005) developed a sequential bi-level programming approach for iteratively estimating traffic demand information (demand matrix or demand functions) and optimizing link tolls to deal with the second-best road pricing problem with unknown demand functions. Meng et al. (2005) and Yang et al. (2010) employed the trial-and-error method in the traffic-restrained road pricing problems. Wang and Yang (2012) and Wang et al. (2013) fixed a non-convergence issue of the bisection method in Li (2002) and further adapted it to implement the tradable travel credit schemes for network mobility management. Xu et al. (2013) developed a trial-and-error pricing scheme on a network with multiple interacted vehicle types and multiple time periods with interdependent demands. Zhou et al. (2014) proposed a unified framework of the trial-and-error congestion pricing scheme for achieving capacity restraint and system optimum.

The trial-and-error method obviates the requirement for analytical demand functions and has been proved to be efficient and promising. A critical underlying assumption in most of the above-mentioned trial-and-error methods is the existence and immediate appearance of user equilibrium for any given toll charges (Yang et al., 2004), which is idealized and in some degree too restrictive in practice. The incontestable fact is that traffic flow on a certain road or path changes from day to day. Once a pricing scheme is imposed or altered, travelers take time to learn and adjust their trip-making decisions in a new pricing environment. More realistically, even if the road users can be quickly informed of altered toll charges and a new equilibrium is reachable, the network flows may tend to temporarily evolve towards a new stable state in response to the adjustment of road pricing schemes. As a result, it is very likely that the link flows observed by the planner may not be in equilibrium at any arbitrary time. In this situation, even if the trial-and-error procedure can also be adopted, its convergence should be reexamined. Thus there is a great need for the development of efficient road pricing methods in networks taking into account day-to-day flow dynamics. Yang and Zhang (2009) summarized a type of fixed-demand day-to-day dynamics as ‘rational behavior adjustment process’, which comprises some previous path-based models. The recent development on the day-to-day flow dynamics can be found in Watling and Cantarella (2013) and Ye and Yang (2013).

Yang and Szeto (2006) adopted a dynamic toll scheme in the network with the ‘rational behavior adjustment process’ to achieve SO by charging the marginal-cost tolls (Button, 1993) based on the instantaneous link flows. Yang et al. (2007) suggested that, imposing the tolls corresponding to the steepest descent direction of the total system cost could accelerate the system’s convergence to SO. Sandholm (2002) recommended a dynamic pricing mechanism to achieve the SO tolls without knowing the exact demand information in the network with the excess payoff dynamics. Guo (2013) proposed a toll strategy to achieve the target flow pattern in a network with boundedly rational user equilibrium. Yet, the above-mentioned limited number of dynamic pricing schemes under flow evolution either requires that the tolls can be adjusted (continuously or daily) in response to the change in road flows or that (at least part of) the explicit mechanisms of the networks flow evolution are known to the social planner. These assumptions are restrictive in actual implementation. A piecewise constant dynamic pricing scheme is a better
alternative for practical consideration, and the exact mechanism of the flow dynamics is usually unknown to the social planner, which are the basic considerations in this paper.

In this paper, we will incorporate the day-to-day flow evolution and dynamics into the trial-and-error procedure for the implementation of congestion pricing in a general network with unknown demand functions and an unknown flow evolution mechanism. The toll level will be changed at some time point, which is called a ‘trial’ moment, and keep unchanged till next trial; while during each inter-trial period, the path flows will evolve following some dynamic process given some toll level. The consequent toll pattern would be a piecewise-constant dynamic pricing mechanism. Appropriate toll updating strategy and time periods between two adjacent trials will be sought. The efficiency of the dynamic pricing scheme is important if the planning time for the whole toll-adjustment process is limited. Therefore, given either the total time horizon or the target error bound, the planner may select the combination of inter-trial periods and total number of trials to minimize the error or the time horizon.

The rest of this paper is organized as follows. Section 2 quickly reviews the trial-and-error procedure in Yang et al. (2004) and some other relevant concepts. In Section 3, the ‘excess travel cost dynamics’ is used to describe the flow dynamics. The adaptive toll charging and updating scheme is proposed and the convergence of the whole procedure under this flow dynamics is discussed. Numerical examples are presented in Section 4 and conclusions are drawn in Section 5.

2. Trial-and-error implementation without flow dynamics

In this section, we first introduce the trial-and-error procedure in a general network proposed by Yang et al. (2004). Consider a general network consisting of a set of directed links denoted by $A$. Let $W$ and $wR$ denote the sets of origin-destination (OD) pairs and paths between OD pair $w \in W$, respectively. Each link $a \in A$ is associated with a link travel time function $t_a(\nu_a)$, which is strictly increasing, convex and differentiable w.r.t. its own link flow $\nu_a$. Each OD pair $w \in W$ is associated with a demand function $d_w$ - $d_w(c_w)$, where $d_w(c_w)$ is invertible, strictly decreasing and differentiable w.r.t. the generalized travel cost $c_w$ between OD pair $w \in W$.

2.1. User equilibrium, social optimum and marginal cost pricing

For a social planner, the SO link flows $\nu = (\nu_a, a \in A)$ can be obtained from the following optimization problem:

$$
\min_{\nu, d} L_1(\nu, d) = \sum_{a \in A} \nu_a t_a(\nu_a) - \sum_{w} \int_0^{\delta_w} D_w^{-1}(\omega) d\omega
$$

subject to

$$
\nu_a = \sum_{w, r : a \in r} \delta_{aw} f_r, \quad \forall a \in A
$$

$$
d_w = \sum_{r : w \in r} f_r, \quad \forall w \in W
$$

$$
f_r \geq 0, \quad \forall r \in R_w, \quad w \in W
$$

where $\nu = (\nu_a, a \in A)$ and $d = (d_w, w \in W)$ represent the link flow and demand vectors, respectively; $f_r$ is the path flow on path $r \in R_w$ between OD pair $w \in W$; $\delta_{aw}$ equals 1 if path $r$ uses link $a$ and 0 otherwise.

On the other hand, for any given link toll pattern $\tau = (\tau_a, a \in A)$, the user equilibrium link flows and OD demands can be acquired by solving the following minimization problem:

$$
\min_{\nu, d} L_2(\nu, d, \tau) = \sum_{a \in A} \int_0^{\tau_a} [t_a(\omega) + \tau_a d\omega] - \sum_{w} \int_0^{\delta_w} D_w^{-1}(\omega) d\omega
$$

subject to (2)-(4), or the following variational inequality (VI) problem (Yang and Huang, 2005): finding $\nu, d$ such
that for all feasible \( \mathbf{v}, \mathbf{d} \),
\[
\begin{bmatrix}
    t(\mathbf{v}) + \mathbf{\tau} \\
    -\mathbf{D}^{-1}(\mathbf{d})
\end{bmatrix}^{\top}
\begin{bmatrix}
    \mathbf{v} - \overline{\mathbf{v}} \\
    \mathbf{d} - \overline{\mathbf{d}}
\end{bmatrix} \geq 0.
\] (6)

Since \( D_w^{-1}() \), \( w \in W \), is strictly decreasing and \( t_a() \), \( a \in A \), is strictly increasing, the UE link flow and corresponding OD demand pattern is unique. With minor abuse of notation, given path flow vector \( \mathbf{f} \), we also write the objective function of (5) as
\[
L_z(\mathbf{f}, \mathbf{\tau}) = L_z(\mathbf{v}, \mathbf{d}, \mathbf{\tau}),
\] (7)
where path flow \( \mathbf{f} \) and link flow \( \mathbf{v} \) satisfy relationship (2), demand \( \mathbf{d} \) and path flow \( \mathbf{f} \) satisfy relationship (3).

From the marginal cost pricing principle, we know that the marginal-cost toll on each link is the difference between the marginal travel cost and the average travel cost, namely,
\[
\tilde{\tau}_a = \tilde{v}_a' t_a' (\tilde{v}_a), \quad \forall a \in A.
\] (8)

It is clear that the marginal-cost tolls, \( \tilde{\tau} = (\tilde{\tau}_a, a \in A) \), can decentralize the SO link flow pattern, which can be readily obtained by solving problem (1)-(4).

### 2.2. Learning the marginal-cost toll pattern via trial-and-error procedure

When the demand functions are unknown, the SO link flows cannot be obtained from the optimization problem described in the previous subsection. Under this circumstance, one simple method to obtain SO is to continuously adjust the link tolls according to the marginal-cost toll principle or set a day-to-day toll charge on each link based on the observed link flow (Sandholm, 2002; Yang and Szeto, 2006; Yang et al., 2007). However, it is hard to implement the continuously dynamic link tolls. An alternative method is to adopt the discrete toll adjustment strategy based on the trial-and-error procedure proposed by Yang et al. (2004). In their procedure, the toll on each link is updated based on a combination of the observed link flow pattern and the projected target SO link flow pattern. The method does not require a continuous adjustment of the tolls and thus allows the planner to observe the evolution of link flows after each toll adjustment. The trial-and-error procedure in Yang et al. (2004) is described as follows for later reference.

The trial-and-error procedure (Yang et al., 2004):

**Step 1.** Set \( k = 0 \), let \( \mathbf{v}^{(0)} \) be the initial link flows.

**Step 2.** Estimate the toll \( \mathbf{\tau}^{(i)} \) by
\[
\tau_a^{(i)} = \tilde{v}_a t_a' (\tilde{v}_a), \quad a \in A.
\] (9)

**Step 3.** Impose toll \( \mathbf{\tau}^{(i)} \) and observe the realized UE link flows \( \mathbf{v}^{(i)} \).

**Step 4.** Stop if \( \|\mathbf{v}^{(i)} - \mathbf{v}^{(i-1)}\| / \|\mathbf{v}^{(i)}\| < \eta \); otherwise, set \( k = k + 1 \) and go to Step 5.

**Step 5.** Update \( \mathbf{v}^{(i)} \) by
\[
\mathbf{v}^{(i)} = \mathbf{v}^{(i-1)} + \theta_i \left( \mathbf{v}^{(i-1)} - \mathbf{v}^{(i-1)} \right)
\] (10)
where
\[
0 < \theta_i \leq 1, \quad \sum_{i=1}^{\infty} \theta_i = +\infty, \quad \lim_{k \to \infty} \theta_i^2 < +\infty
\] (11)
Following assumptions are made in Yang et al. (2004) to guarantee the convergence of the above trial-and-error method and also required in the current study.

**Assumption 1.** \( \begin{pmatrix} t(v) \\ -D^{-1}(d) \end{pmatrix} \) is strongly monotone, i.e., there exists a positive number \( \rho \) such that for any feasible and distinct \( (v_1, d_1) \) and \( (v_2, d_2) \), we have

\[
\begin{pmatrix} t(v_1) - t(v_2) \\ -D^{-1}(d_1) - (-D^{-1}(d_2)) \end{pmatrix}^T \begin{pmatrix} v_1 - v_2 \\ d_1 - d_2 \end{pmatrix} \geq \rho \begin{pmatrix} v_1 - v_2 \\ d_1 - d_2 \end{pmatrix},
\]

where \( t(v) = (t_a(v_a), a \in A) \) and \( D^{-1}(d) = (D^+_w(d_w), w \in W) \).

**Assumption 2.** Assume the demand functions are bounded from above for all OD pairs, therefore all the feasible link flows, path flows and tolls are also bounded from above. Further, assume the Hessian matrix of \( \begin{pmatrix} t(v) \\ -D^{-1}(d) \end{pmatrix} \) is bounded from above as well. Denote a unified upper bound of all the terms as \( B \).

**Remark.** Mathematically speaking, the above-mentioned trial-and-error procedure is just another alternative for solving the mathematical programming problem (5) or VI problem (6). However, considering the background of our problem, the existing algorithms are not applicable here. On one hand, the demand functions, are not available to the social planner. At the same time, the values of modular \( \rho \) and upper bound \( B \) in the above assumptions are also unknown. However, this information is requisite in solving the VI problems (Facchinei and Pang, 2003), for both updating the solution and assuring the convergence of the algorithms. As a result, the rules of iterating \( \mathbf{v}^{(k)} \) in previous algorithms do not work in our case.

As proved in Yang et al. (2004), the vector \( \begin{pmatrix} \mathbf{v}^{(i)} - \mathbf{v}^{(i)} \\ \mathbf{d}^{(i)} - \mathbf{d}^{(i)} \end{pmatrix} \) is a descent direction of the optimization problem (1) - (4) at \( \begin{pmatrix} \mathbf{v}^{(i)} \\ \mathbf{d}^{(i)} \end{pmatrix} \). According to Powell and Sheffi (1982) and Bazaraa et al. (2006), both \( \mathbf{v}^{(i)} \) and \( \mathbf{v}^{(i)} \) will converge to \( \tilde{\mathbf{v}} \) as \( k \to \infty \), and the tolls will approach \( \tilde{\mathbf{r}} \) as well. Furthermore, by Liu et al. (2009), the third condition in (11) can be relaxed and the process is still convergent, then we have a relaxed condition on \( \theta_k \) as follows,

\[
0 < \theta_k \leq 1, \quad \sum_{k=1}^{\infty} \theta_k = +\infty, \quad \lim_{k \to \infty} \theta_k = 0.
\]

Note that a critical assumption adopted in the above procedure (more precisely, in Step 3) is that the realized and observed link flow \( \mathbf{v}^{(i)} \) reacting on each pricing trial \( \mathbf{r}^{(i)} \) is a UE link flow pattern. However, with flow dynamics, the realized flow may not be exact UE but a non-equilibrium (or approximate equilibrium) and the vector...
\[
\left( \frac{\vec{v}^{(i)} - \hat{v}^{(i)}}{\vec{d}^{(i)} - \hat{d}^{(i)}} \right)
\] is not assured to be a descent direction of the optimization problem (1)-(4) at \( \vec{v}^{(i)} \), therefore the necessary condition for the convergence of MSA in Powell and Sheffi (1982) may not be satisfied and the convergence of the trial-and-error procedure should be reinvestigated, which will be conducted in the next section.

3. Trial-and-error procedure with flow dynamics

3.1. Excess travel cost dynamics with elastic demand

Define the excess travel cost \( ETC_r \) for path \( r \in R_w \) between OD pair \( w \in W \) as the difference between the generalized path travel cost \( C_r \) and the travel benefit of the corresponding OD pair \( D_w(d_w) \), i.e.,

\[
ETC_r(f) = C_r(f) - D_w(d_w), \quad \forall r \in R_w, \ w \in W.
\]

**Definition 1.** The path flow adjustment process

\[
\vec{f} = F(f)
\]

is the excess travel cost dynamics (ETCD) if the following three conditions (i)-(iii) are satisfied, where

\[
F(f) = (F_r(f), r \in R_w, w \in W)
\]

(i) Dynamic system (14) admits a unique solution trajectory for any given initial condition, and the trajectory is Lipschitz continuous;

(ii) \( ETC(f)F(f) < 0 \) whenever \( F(f) \neq 0 \), where

\[
ETC(f) = (ETC_r(f), r \in R_w, w \in W);
\]

(iii) \( f \) is a UE path flow pattern if \( F(f) = 0 \).

It is easy to verify that this dynamics is compatible with the rational behavior adjustment process with fixed demand (Yang and Zhang, 2009), as well as including the network tatonnement process (Friesz et al., 1994) and the projected dynamical system (Zhang and Nagurney, 1996; Nagurney and Zhang, 1997) with elastic demand as its special cases. The network tatonnement process and projected dynamical system with elastic demand are modelled as

\[
\dot{f}_r = \alpha \left( \max \{0, f_r - \beta ETC_r(f)\} - f_r \right), \quad \alpha > 0, \ \beta > 0, \ \forall r \in R_w, \ w \in W,
\]

and

\[
\dot{f}_r = \begin{cases} 
-\beta ETC_r(f) & f_r > 0 \\
\max \{-\beta ETC_r(f), 0\} & f_r = 0 \\
0 & \beta > 0, \ \forall r \in R_w, \ w \in W.
\end{cases}
\]

Without difficulty, the stability of the ETCD can be proved by adopting LaSalle’s invariant set theory (Khalil, 2003) (Li et al., 2012). The path flow dynamics with any initial state converges to the unique UE link flow and OD demand pattern. Furthermore, given any toll pattern \( \tau \), the objective function in (5) satisfies \( L_2(f, \tau) = ETC(f)\vec{f} < 0 \), therefore \( L_2(f, \tau) \) is strictly deceasing along time.

3.2. Convergence of the trial-and-error procedure with flow dynamics
We now investigate the convergence of the trial-and-error procedure described in Subsection 2.2. The main difference between the two procedures with and without flow dynamics lies in that the observed link flow pattern in the former case is not UE. Suppose the pricing experiment is implemented and adjusted on a periodic basis such as a month or a quarter, and during each inter-trial period $\Delta_i$, $k = 1, 2, \ldots$, the traffic flows on the network evolve according to the ETCD described above. Under this circumstance, at the end of each inter-trial period, the social planner can only observe the non-equilibrium link flows and calculate the new tolls based on this observation. After imposing toll $\tau^{(i)}$, the social planner will wait for time $\Delta_i$ and adjust the toll based on the observed flow $\tilde{v}^{(i)}$. Therefore, the trial-and-error procedure with flow dynamics is the same as that without flow dynamics described in Subsection 2.2 with the non-equilibrium link flow pattern $\tilde{v}^{(i)}$ in place of the UE link flow pattern $\bar{v}^{(i)}$ at each trial.

Denote $\tilde{d}^{(i)}$ and $\tilde{f}^{(i)}$ as the realized (but not observable) demand and path flow pattern at trial $k$. Correspondingly, let $\bar{d}^{(i)}$, $\bar{f}^{(i)}$ and $\bar{v}^{(i)}$ be the equilibrium demand, path flow and link flow patterns under toll $\tau^{(i)}$. During the inter-trial period $\Delta_i$, the path flows will evolve from $\tilde{f}^{(i-1)}$ to $\tilde{f}^{(i)}$ following path flow dynamics (14). Here we intend to establish the conditions, under which the symbiotic process of flow dynamics eventually converges to the desirable and unknown SO flow pattern, and thus the SO toll pattern can be effectively estimated.

Intuitively, under a convergent flow evolution process (converging to UE as time goes infinite, such as the aforementioned ETCD), if each inter-trial period $\Delta_i$, $k = 1, 2, \ldots$, is long enough, the non-UE flow and demand pattern $\begin{pmatrix} \tilde{v}^{(i)} \\ \tilde{d}^{(i)} \end{pmatrix}$ is sufficiently close to UE and forms a descent direction of the optimization problem (1)-(4), which is summarized in the following lemma.

**Lemma 1.** At each trial, given toll pattern $\tau^{(i)}$, if $\bar{v}^{(i)} \neq \tilde{v}^{(i)}$, then there always exists $\Delta_i > 0$ such that

$$
\begin{pmatrix}
\begin{pmatrix} \tilde{v}^{(i)} \\ \tilde{d}^{(i)} \end{pmatrix} & \tau^{(i)}
\end{pmatrix}^T 
\begin{pmatrix} \tilde{v}^{(i)} - \bar{v}^{(i)} \\ \tilde{d}^{(i)} - \bar{d}^{(i)} \end{pmatrix} < 0.
$$

**Proof.** First, note that

$$
\begin{pmatrix}
\begin{pmatrix} \tilde{v}^{(i)} \\ \tilde{d}^{(i)} \end{pmatrix} & \tau^{(i)}
\end{pmatrix}^T 
\begin{pmatrix} \tilde{v}^{(i)} - \bar{v}^{(i)} \\ \tilde{d}^{(i)} - \bar{d}^{(i)} \end{pmatrix} \leq \begin{pmatrix}
\begin{pmatrix} \tilde{v}^{(i)} \\ \tilde{d}^{(i)} \end{pmatrix} & \tau^{(i)}
\end{pmatrix}^T 
\begin{pmatrix} \tilde{v}^{(i)} - \bar{v}^{(i)} \\ \tilde{d}^{(i)} - \bar{d}^{(i)} \end{pmatrix} + \begin{pmatrix} \tilde{v}^{(i)} - \bar{v}^{(i)} \\ \tilde{d}^{(i)} - \bar{d}^{(i)} \end{pmatrix} \begin{pmatrix} \tilde{v}^{(i)} - \bar{v}^{(i)} \\ \tilde{d}^{(i)} - \bar{d}^{(i)} \end{pmatrix}.
$$

For the first term in the right-hand-size of (18), executing Taylor’s expansion with respect to $\begin{pmatrix} \tilde{v}^{(i)} \\ \tilde{d}^{(i)} \end{pmatrix}$ at $\begin{pmatrix} \tilde{v}^{(i)} \\ \tilde{d}^{(i)} \end{pmatrix}$ yields

$$
\begin{pmatrix}
\begin{pmatrix} \tilde{v}^{(i)} \\ \tilde{d}^{(i)} \end{pmatrix} & \tau^{(i)}
\end{pmatrix}^T 
\begin{pmatrix} \tilde{v}^{(i)} - \bar{v}^{(i)} \\ \tilde{d}^{(i)} - \bar{d}^{(i)} \end{pmatrix} \leq \begin{pmatrix}
\begin{pmatrix} \tilde{v}^{(i)} \\ \tilde{d}^{(i)} \end{pmatrix} & \tau^{(i)}
\end{pmatrix}^T 
\begin{pmatrix} \tilde{v}^{(i)} - \bar{v}^{(i)} \\ \tilde{d}^{(i)} - \bar{d}^{(i)} \end{pmatrix} + \begin{pmatrix} \tilde{v}^{(i)} - \bar{v}^{(i)} \\ \tilde{d}^{(i)} - \bar{d}^{(i)} \end{pmatrix} \begin{pmatrix} \tilde{v}^{(i)} - \bar{v}^{(i)} \\ \tilde{d}^{(i)} - \bar{d}^{(i)} \end{pmatrix} \begin{pmatrix} \tilde{v}^{(i)} - \bar{v}^{(i)} \\ \tilde{d}^{(i)} - \bar{d}^{(i)} \end{pmatrix}.
$$
where \( \hat{\nu}_v^{(i)} = \xi \nu_v^{(i)} + (1 - \xi) \bar{\nu}_v^{(i)} \), \( \hat{d}_v^{(i)} = \xi d_v^{(i)} + (1 - \xi) \bar{d}_v^{(i)} \), \( \xi \in [0,1] \). Substituting (19) into (18) leads to

\[
\begin{aligned}
\left\{ t \left( \nu_v^{(i)} \right) + \tau_v^{(i)} \right\}^T \left( \nu_v^{(i)} - \nu_v^{(i)} \right) &\leq \left\{ t \left( \nu_v^{(i)} \right) + \tau_v^{(i)} \right\}^T \left( \hat{d}_v^{(i)} - \hat{d}_v^{(i)} \right) + \left\{ t \left( \nu_v^{(i)} \right) + \tau_v^{(i)} \right\}^T \left( \hat{d}_v^{(i)} - \hat{d}_v^{(i)} \right) \\
&\leq \left\| \hat{d}_v^{(i)} - \hat{d}_v^{(i)} \right\| \end{aligned}
\]

(20)

The first and third terms in the right hand side of the last inequality in (20) approach zero since \( \nu_v^{(i)} \rightarrow \nu_v^{(i)} \) and \( \hat{d}_v^{(i)} \rightarrow \hat{d}_v^{(i)} \) with \( \Delta_i \rightarrow \infty \); the second term is time-invariant since \( \nu_v^{(i)} \) and \( \tau_v^{(i)} \) are given at trial \( k \). From the UE condition, since \( \nu_v^{(i)} = \nu_v^{(i)} \), by (6), we also know

\[
\begin{aligned}
\left\{ t \left( \nu_v^{(i)} \right) + \tau_v^{(i)} \right\}^T \left( \nu_v^{(i)} - \nu_v^{(i)} \right) &< 0.
\end{aligned}
\]

(21)

Combining (20) and (21), we can conclude that there always exists \( \Delta_i > 0 \) such that condition (17) holds. This completes the proof.

Then we readily have the following proposition.

**Proposition 1.** Under Assumptions 1 and 2, there always exist inter-trial periods \( \{ \Delta_i \}_{i=0}^{\infty} \) such that the link flow and toll vectors, \( \{ \nu_v^{(i)} \}_{i=0}^{\infty} \) and \( \{ \tau_v^{(i)} \}_{i=0}^{\infty} \), converge to the SO link flow pattern \( \nu_v \) and SO toll vector \( \tau_v \).

**Proof.** Note that the gradient of the objective function \( L_i (v, d) \) in (1) is given by

\[
\nabla L_i (v, d) = (\cdots, t_a (v_a^{(i)}), \cdots, -D_{-1} (d_v^{(i)}), \cdots)^T.
\]

Then

\[
\nabla L_i (v, d)^T (\bar{\nu}_v^{(i)} - \nu_v^{(i)}) = \left( \begin{array}{c}
-t (\nu_v^{(i)}) - t (\hat{\nu}_v^{(i)}) \\
D_{-1} (\hat{d}_v^{(i)}) - D_{-1} (\bar{d}_v^{(i)})
\end{array} \right)^T \left( \nu_v^{(i)} - \nu_v^{(i)} \right) + \left( \begin{array}{c}
t (\nu_v^{(i)}) + \tau_v^{(i)} \\
D_{-1} (\hat{d}_v^{(i)}) - D_{-1} (\bar{d}_v^{(i)})
\end{array} \right)^T \left( \nu_v^{(i)} - \nu_v^{(i)} \right)
\]

(23)

where link toll \( \tau_v^{(i)} \) is determined by \( \nu_v^{(i)} \) via eqn. (9), \( \tau_v^{(i)} = (\tau_v^{(i)} = v_a^{(i)} t_a (v_a^{(i)}), a \in A) \). According to Assumption 1, the first term in the right-hand side of (23) can be bounded as

\[
\left\{ t (\nu_v^{(i)}) - t (\hat{\nu}_v^{(i)}) \right\}^T (\nu_v^{(i)} - \nu_v^{(i)}) \leq -\rho \left\| \nu_v^{(i)} - \nu_v^{(i)} \right\|
\]

Denote \( \nu_v^{(i)} \) as the UE flow and demand pattern under toll \( \tau_v^{(i)} \). If \( \nu_v^{(i)} = \nu_v^{(i)} \), then we have \( \nu_v^{(i)} = \nu_v \), since the UE link flow is derived by the marginal-cost link toll pattern. Without loss of generality, suppose \( \nu_v^{(i)} \neq \nu_v^{(i)} \) and condition (17) holds, adding up (24) and (17) yields
which implies that the vector \( \frac{v^{(i)}}{d^{(i)}} - \frac{v^{(i)}}{d^{(i)}} \) is a feasible descent direction of the objective function \( L_1(v, d) \) at \( \left( \frac{v^{(i)}}{d^{(i)}} \right) \). Since the SO problem is strictly convex and has a unique minimum, we have \( v^{(i)} \rightarrow \bar{v} \), and thus, \( \tau^{(i)} \rightarrow \bar{\tau} \), \( v^{(i)} \rightarrow \bar{v} \) with \( k \rightarrow \infty \). This completes the proof. \( \square \)

Proposition 1 indicates that once an appropriate sequence of inter-trial periods are selected, the convergence of dynamic link flow pattern to SO is guaranteed. However, how to select the appropriate sequence of inter-trial periods \( \{\Delta_k\}_k \) is still difficult since the exact UE link flow pattern at each trial is unknown. Choosing the inappropriate \( \Delta_k \) (which is likely to happen since the flow dynamics are not clear to the planner) will violate condition (17) and the proof above no longer stands. Thus the lengths of the inter-trial periods seem to play key roles in shaping the convergence of the dynamic system. Technically speaking, a long inter-trial period would ensure that the actual flow is ‘close enough’ to an equilibrium and thus tolls could be adjusted more appropriately each time; a short inter-trial period, on the other hand, would allow tolls to be adjusted more frequently based on a still evolving disequilibrium flow pattern. Hereinafter, we will show that, even though the inter-trial periods are not sufficiently long, the trial-and-error procedure can still converge. For simplicity, we only consider identical inter-trial periods, namely, \( \Delta_k = \Delta > 0 \) for all \( k \geq 0 \). It must be pointed out that the identical length of inter-trial periods is not necessary for the convergence of the procedure. However, it requires more information on the demand function and/or flow dynamics to set the adaptive inter-trial periods. We leave the topic for our future research.

To ensure the convergence of the trial-and-error procedure, we first introduce the following assumption on the flow dynamics.

**Assumption 3.** Given any feasible initial path flow and link toll pattern, the relative decreasing rate along the path flow trajectory determined by dynamical system (14) is not greater than a decreasing function \( \delta(s) \) with \( \delta(0) = 1 \) and \( \lim_{s \to \infty} \delta(s) = 0 \),

\[
\frac{L_2(f(s), \tau) - L_2(\bar{f}, \bar{\tau})}{L_2(f_0, \tau) - L_2(\bar{f}, \bar{\tau})} \leq \delta(s),
\]

(26)

where \( f_0 \) is an initial path flow and \( \bar{f} \) is the resultant UE path flow pattern under link toll pattern \( \bar{\tau} \).

It is clear to see that, when the demand function is unknown, the mathematical programming problem (5) or VI problem (6) cannot be solved by many classical algorithms, which require exact information on the demand (Facchinei and Pang, 2003). In addition, different from Yang et al. (2004), the observed link flow pattern is in disequilibrium and given by the unknown flow dynamics (14). Characteristics of the flow dynamics essentially affect the performance of the trial-and-error procedure. Assumption 3 or inequality (26) is a stronger assumption on the dynamical system, which requires that the adjustment process of the path flows to UE should not be too “slow”. Note that, \( \delta(s) \) is independent of the initial path flows and toll charges, and thus, travelers should be ‘smart enough’ to learn their user optimum. It must be pointed out that, given a dynamical system, such as the network tatonnement process (15) and the projected dynamical system (16), the relative decreasing rate defined by the left-hand-side term of inequality (26) would approach one when the initial path flow pattern \( f_0 \) is very close to
UE \( \widetilde{T} \). As a result, it is not easy to find function \( \delta(s) \). For the exponentially stable system regarding link flow and OD demand pattern, \( \delta(s) \) can be simply estimated by the Lipschitz constant of the link travel time functions, inverse demand functions and the exponential power. Note that, the objective function \( L_+(f, \tau) \) can be viewed as the potential energy with elastic demand (Sandholm, 2002; Peeta and Yang, 2003; Xiao et al., 2014). Assumption 3 implies that, the potential energy should be strictly decreasing along the dynamical trajectory, which must hold for many commonly used dynamic systems in transportation system analysis.

The rest of this section will show that, based on Assumption 3, we can relax Proposition 1, while keeping the effectiveness of the trial-and-error scheme. Proposition 2 shows that, width fixed-length inter-trial periods, the realized non-UE flows will eventually approach the UE flows. However, under this circumstance, the results in Section 2.2 or Proposition 1 cannot be directly applied. Thus in Proposition 3, we further prove the convergence of our scheme to SO.

**Proposition 2.** Under Assumptions 1-3, the Euclidean distances \( \|\mathbf{v}^{(i)} - \mathbf{v}^{(0)}\| \) and \( \|\mathbf{d}^{(i)} - \mathbf{d}^{(0)}\| \) both approach zero when \( k \to \infty \).

**Proof.** To begin with, from (6), we have
\[
\begin{pmatrix}
  t(V_{(i-1)}) \\
  -D^{-1}(\mathbf{d}^{(i-1)})
\end{pmatrix}^T
\begin{pmatrix}
  V_{(i)} - V_{(i-1)} \\
  \mathbf{d}^{(i)} - \mathbf{d}^{(i-1)}
\end{pmatrix} \geq 0
\]
(27)
and
\[
\begin{pmatrix}
  t(V_{(i)}) + \tau^{(i)} \\
  -D^{-1}(\mathbf{d}^{(i)})
\end{pmatrix}^T
\begin{pmatrix}
  V_{(i)} - V_{(i)} \\
  \mathbf{d}^{(i)} - \mathbf{d}^{(i)}
\end{pmatrix} \geq 0.
\]
(28)
Adding up inequalities (27) and (28) gives rise to
\[
\begin{pmatrix}
  t(V_{(i-1)}) \\
  -D^{-1}(\mathbf{d}^{(i-1)})
\end{pmatrix}^T
\begin{pmatrix}
  V_{(i-1)} - V_{(i-1)} \\
  \mathbf{d}^{(i-1)} - \mathbf{d}^{(i-1)}
\end{pmatrix} \geq 0
\]
(29)
or, equivalently,
\[
\begin{pmatrix}
  \tau^{(i-1)} - \tau^{(i)} \\
  0
\end{pmatrix}^T
\begin{pmatrix}
  V_{(i)} - V_{(i)} \\
  \mathbf{d}^{(i)} - \mathbf{d}^{(i)}
\end{pmatrix} \geq \begin{pmatrix}
  t(V_{(i-1)}) \\
  -D^{-1}(\mathbf{d}^{(i-1)})
\end{pmatrix}^T
\begin{pmatrix}
  V_{(i-1)} - V_{(i-1)} \\
  \mathbf{d}^{(i-1)} - \mathbf{d}^{(i-1)}
\end{pmatrix} \geq \rho \left\| \mathbf{v}^{(i)} - \mathbf{v}^{(i-1)} \right\| \left\| \mathbf{d}^{(i)} - \mathbf{d}^{(i-1)} \right\|,
\]
(30)
where the second inequality comes from Assumption 1. Therefore we have
\[
\left\| \mathbf{v}^{(i)} - \mathbf{v}^{(i-1)} \right\| \leq \rho \left\| \mathbf{d}^{(i)} - \mathbf{d}^{(i-1)} \right\| \cdot \left\| \mathbf{v}^{(i)} - \mathbf{v}^{(i-1)} \right\|.
\]
(31)
and thus
\[
\left\| \mathbf{v}^{(i)} - \mathbf{v}^{(i-1)} \right\| \leq \frac{1}{\rho} \left\| \mathbf{v}^{(i)} - \mathbf{v}^{(i-1)} \right\|.
\]
(32)
Note that, the marginal-cost toll function \( \tau(v) = (\tau_a = v_a t'_a(v_a), a \in A) \) is Lipschitz continuous if the link travel time functions \( t_a(v_a), a \in A \), are smooth. Therefore, there exist a Lipschitz constant \( M_\tau > 0 \), such that the right-hand side of (32) can be bounded as
\[
\left\| \mathbf{v}^{(i)} - \mathbf{v}^{(i-1)} \right\| \leq M_\tau \left\| \mathbf{v}^{(i)} - \mathbf{v}^{(i-1)} \right\| = M_\tau \theta_i \left\| \mathbf{v}^{(i-1)} - \mathbf{v}^{(i-1)} \right\| \leq M_i B_\theta_i,
\]
(33)
where the equality follows eqn. (10) and the last inequality follows Assumption 2.

Now, according to (26), we have

\[
L_2 \left( \mathbf{f}^{(i)}(t), \mathbf{v}^{(i)} \right) - L_2 \left( \mathbf{f}^{(i)}(t), \mathbf{v}^{(i)} \right) \leq \delta(\Delta) \left[ L_2 \left( \mathbf{f}^{(i)}(t), \mathbf{v}^{(i)} \right) - L_2 \left( \mathbf{f}^{(i)}(t), \mathbf{v}^{(i)} \right) \right]
\]

\[
= \delta(\Delta) \left\{ \left[ L_2 \left( \mathbf{f}^{(i)}(t), \mathbf{v}^{(i)} \right) - L_2 \left( \mathbf{f}^{(i)}(t), \mathbf{v}^{(i)} \right) \right] \right\} + \left[ L_2 \left( \mathbf{f}^{(i)}(t), \mathbf{v}^{(i)} \right) - L_2 \left( \mathbf{f}^{(i)}(t), \mathbf{v}^{(i)} \right) \right]
\]

\[
\leq \delta(\Delta) \left\{ \left[ \mathbf{v}^{(i)}(t) - \mathbf{v}^{(i+1)}(t) \right] \right\} + \delta(\Delta) \left[ B \left[ \mathbf{v}^{(i)}(t) - \mathbf{v}^{(i+1)}(t) \right] + \delta(\Delta) \left[ B \left[ \mathbf{v}^{(i)}(t) - \mathbf{v}^{(i+1)}(t) \right] \right]
\]

\[
\leq \delta(\Delta) \left\{ \left[ \mathbf{v}^{(i)}(t) - \mathbf{v}^{(i+1)}(t) \right] \right\} + \delta(\Delta) \left[ B \left[ \mathbf{v}^{(i)}(t) - \mathbf{v}^{(i+1)}(t) \right] \right]
\]

\[
\leq \delta(\Delta) \left\{ \left[ \mathbf{v}^{(i)}(t) - \mathbf{v}^{(i+1)}(t) \right] \right\} + \delta(\Delta) \left[ M_i B^2 \left[ 1 + \frac{1}{\rho} \left( \sqrt{\mu + 1} \right) \right] \right]
\]

where the second inequality is derived based on (32), and the third inequality holds due to (33). For simplicity and without confusion, \( \delta(\Delta) \) is written as \( \delta \). From (36), we further have

\[
L_2 \left( \mathbf{f}^{(i)}(t), \mathbf{v}^{(i)} \right) - L_2 \left( \mathbf{f}^{(i)}(t), \mathbf{v}^{(i)} \right) \leq \delta(\Delta) \left[ L_2 \left( \mathbf{f}^{(i)}(t), \mathbf{v}^{(i)} \right) - L_2 \left( \mathbf{f}^{(i)}(t), \mathbf{v}^{(i)} \right) \right] + M_i B^2 \left[ 1 + \frac{1}{\rho} \left( \sqrt{\mu + 1} \right) \right] \sum_{i=1}^{\infty} \theta_i \delta^{i-1}. \]  

For the first term in the right-hand side of inequality (37), we know

\[
\lim_{i \to \infty} \left\{ \delta(\Delta) \left[ L_2 \left( \mathbf{f}^{(i)}(t), \mathbf{v}^{(i)} \right) - L_2 \left( \mathbf{f}^{(i)}(t), \mathbf{v}^{(i)} \right) \right] \right\} = 0. \]  

We move to consider the second term in the right-hand side of inequality (37). If \( 0 < \delta < 1 \), \( \delta^{-k} \) is strictly increasing with \( k \) and \( \lim_{k \to \infty} \delta^{-k} = +\infty \). Denote \( b_k = \sum_{i=1}^{\theta_i} \theta_i \delta^{-i} \), if \( \lim_{k \to \infty} \theta_i = 0 \), then

\[
\lim_{i \to \infty} \frac{b_i - b_{i+1}}{\delta^{-i} - \delta^{-i-1}} = \lim_{i \to \infty} \frac{\theta \delta^{-i}}{\delta^{-i} - \delta^{-i-1}} = \lim_{i \to \infty} \frac{\theta}{1 - \delta} = 0
\]

By the Stolz–Cesàro Theorem (Theorem 1.22, Muresan, 2009),
\[
\lim_{k \to \infty} \sum_{i=1}^k \theta \delta^{i-1} = \lim_{k \to \infty} \left\{ \delta^{i-1} \sum_{i=1}^k \theta \delta^{i-1} \right\} = \lim_{k \to \infty} \frac{b}{\delta^{i-1}} = \lim_{k \to \infty} \frac{b_k - b_{k+1}}{\delta^{i-1} - \delta^{(i-1)}} = 0.
\]

If \( \delta = 0 \), then \( \| f^{(i)} - \bar{T}^{(i)} \| = 0 \) for all \( k \geq 1 \) and \( \Delta > 0 \), the problem degenerates to the case without flow dynamics and \( \sum_{i=1}^k \theta \delta^{i-1} = 0 \). Therefore when \( 0 \leq \delta ( \Lambda < 1 ) \) with \( \theta \) satisfying (13), we have
\[
\lim_{k \to \infty} \left[ L_2 \left( f^{(i)}, \tau^{(i)} \right) - L_2 \left( \bar{T}^{(i)}, \tau^{(i)} \right) \right] = 0
\]

or, equivalently,
\[
\lim_{k \to \infty} \left[ L_2 \left( \bar{v}^{(i)}, \bar{d}^{(i)}, \tau^{(i)} \right) - L_2 \left( \bar{v}^{(i)}, \bar{d}^{(i)}, \tau^{(i)} \right) \right] = 0.
\]

Since \( \left( \bar{v}^{(i)}, \bar{d}^{(i)} \right) \) is the unique minimum of \( L_2 \left( v, d, \tau^{(i)} \right) \), eqn. (40) indicates that \( \lim_{k \to \infty} \| v^{(i)} - \bar{v}^{(i)} \| = 0 \) and \( \lim_{k \to \infty} \| d^{(i)} - \bar{d}^{(i)} \| = 0 \). This completes the proof.

Proposition 2 depicts that, under certain assumptions, the gap between the observed non-equilibrium link flow and OD demand pattern and the corresponding UE pattern approaches zero as the trial-and-error procedure goes on. No matter how short the identical inter-trial period \( \Delta \) is, as long as the dynamical system satisfies Assumption 3, the observed flow will eventually become UE. Naturally, the trial-and-error procedure eventually stops since the observed link flow pattern is stable, but it doesn’t mean that the stable flow pattern will be SO. The following result concludes that the observed link flow pattern do converge to SO. The basic idea is to separate the sequence \( \{ v^{(i)} \}_{i=1}^{\infty} \) into two subsequences, based on whether condition (25) is satisfied or not. By showing that both sequences converge to SO, we prove the convergence of the trial-and-error procedure.

**Proposition 3.** With Assumptions 1-2, if \( v^{(i)} \to \bar{v}^{(i)} \) and \( d^{(i)} \to \bar{d}^{(i)} \) as \( k \to \infty \), then the toll pattern \( \{ \tau^{(i)} \}_{i=1}^{\infty} \) converges to the system optimum link toll pattern \( \bar{\tau} \).

**Proof.** By Proposition 2, \( \lim_{k \to \infty} \| v^{(i)} - \bar{v}^{(i)} \| = 0 \) and \( \lim_{k \to \infty} \| d^{(i)} - \bar{d}^{(i)} \| = 0 \), then there exists a non-negative sequence \( \{ e_i \}_{i=1}^{\infty} \) such that \( \lim_{k \to \infty} e_i = 0 \), \( \| v^{(i)} - \bar{v}^{(i)} \| \leq e_i \) and \( \| d^{(i)} - \bar{d}^{(i)} \| \leq e_i \) for all \( k \geq 1 \). According to (23) and (24), we have
\[
\nabla L_i \left( v^{(i)}, d^{(i)} \right)^T \left( \bar{v}^{(i)} - v^{(i)} \right) = \nabla L_i \left( v^{(i)}, d^{(i)} \right)^T \left( \bar{v}^{(i)} - d^{(i)} \right) + \nabla L_i \left( v^{(i)}, d^{(i)} \right)^T \left( \bar{d}^{(i)} - \bar{v}^{(i)} \right)
\]
\[
\leq B e_i - \rho \| \bar{v}^{(i)} - d^{(i)} \|.
\]

If the following inequality always holds
\[ B e_i \leq \rho \frac{\|v^{(i)} - v^{(i)}\|}{d^{(i)} - d^{(i)}} \leq -\frac{\rho}{2} \frac{\|v^{(i)} - v^{(i)}\|}{d^{(i)} - d^{(i)}} \]  

(42)

or, equivalently,

\[ \frac{\|v^{(i)} - v^{(i)}\|}{d^{(i)} - d^{(i)}} \geq \frac{2 B e_i}{\rho}, \]  

(43)

then \( \begin{pmatrix} \hat{v}^{(i)} - v^{(i)} \\ \hat{d}^{(i)} - d^{(i)} \end{pmatrix} \) is the descent direction of \( L_i \left( v^{(i)}, d^{(i)} \right) \). Similar to the proof of Proposition 1, we immediately know that the Proposition 3 holds. On the other hand, if (42) is violated for all \( k \geq 1 \), then \( \lim_{k \to \infty} \frac{\|v^{(i)} - v^{(i)}\|}{d^{(i)} - d^{(i)}} = 0 \) since \( \lim_{k \to \infty} e_i = 0 \). Therefore Proposition 3 is also true.

We now discuss the case that inequality (42) is neither always true nor always violated. Suppose inequality (42) is violated at \( k = k_i \), i.e.,

\[ \frac{\|v^{(i)} - v^{(i)}\|}{d^{(i)} - d^{(i)}} < \frac{2 B e_i}{\rho}, \]

then by (32) and (33), we know

\[ \frac{\|v^{(i)} - v^{(i)}\|}{d^{(i)} - d^{(i)}} \leq \frac{\|v^{(i)} - v^{(i)}\|}{d^{(i)} - d^{(i)}} + \frac{\|v^{(i)} - v^{(i)}\|}{d^{(i)} - d^{(i)}} + \frac{\|v^{(i)} - v^{(i)}\|}{d^{(i)} - d^{(i)}} < \frac{M B}{\rho} \theta_{k_i} + B \theta_{k_i} + \sqrt{\frac{2 B e_i}{\rho}}. \]

(44)

Assume that (42) holds for all \( i \in \{k_i + 1, k_i + 2, \ldots, k_j \} \), then by Taylor’s expansion and eqn. (10),

\[ L_i \left( v^{(i)}, d^{(i)} \right) - L_i \left( v^{(i)}, d^{(i)} \right) = \theta_{i} \nabla L_i \left( v^{(i)}, d^{(i)} \right)^T \begin{pmatrix} v^{(i)} - v^{(i)} \\ d^{(i)} - d^{(i)} \end{pmatrix} \]

\[ + \frac{1}{2} \theta_{i} \nabla^2 L_i \left( v^{(i)}, d^{(i)} \right) \begin{pmatrix} v^{(i)} - v^{(i)} \\ d^{(i)} - d^{(i)} \end{pmatrix} \]

\[ \leq \frac{\rho}{2} \theta_{i} \frac{\|v^{(i)} - v^{(i)}\|}{d^{(i)} - d^{(i)}} \leq \frac{1}{2} B \theta_{i} \frac{\|v^{(i)} - v^{(i)}\|}{d^{(i)} - d^{(i)}} \]

\[ \leq \frac{1}{2} \theta_{i} \left( B \theta_{i} - \rho \right) \frac{\|v^{(i)} - v^{(i)}\|}{d^{(i)} - d^{(i)}} + 2 B \theta_{i} e_i \frac{\|v^{(i)} - v^{(i)}\|}{d^{(i)} - d^{(i)}} + B \theta_{i} e_i^2, \]  

(45)

where \( \xi \in [0,1] \), and the first inequality follows (41), (42) and Assumption 2. Note that \( \lim_{k \to \infty} e_i = 0 \) and
lim_{k \to \infty} \theta_k = 0$, thus when $l$ is large enough, we always have $B \theta_{i+1} - \rho < 0$, and
\[
\sqrt{\frac{2B \epsilon}{\rho}} - \frac{\sqrt{B \theta_{i+1}}}{\sqrt{\rho - B \theta_{i+1}}} = \sqrt{B \epsilon} \left( \frac{2}{\sqrt{\rho}} - \frac{\sqrt{\epsilon_1}}{\sqrt{\rho - B \theta_{i+1}}} \right) > 0.
\]
Then by (43),
\[
\frac{\| \mathbf{v}^{(i)} - \mathbf{v}^{(i)} \|}{\| \mathbf{d}^{(i)} - \mathbf{d}^{(i)} \|} \geq \sqrt{\frac{2B \epsilon}{\rho}} - \frac{\sqrt{B \theta_{i+1}}}{\sqrt{\rho - B \theta_{i+1}}}.
\]
Since we are only interested in the limit of the trial-and-error procedure, we can select $k_i$ large enough such that
inequality (46) holds. With this treatment, we get
\[
\left( B \theta_{i+1} - \rho \right) \left( \| \mathbf{v}^{(i)} - \mathbf{v}^{(i)} \| + 2B \theta_{i+1} \epsilon \| \mathbf{d}^{(i)} - \mathbf{d}^{(i)} \| + B \theta_{i+1} \epsilon_1^2 < 0. \right.
\]
From (45) and (47), we immediately know that $\{L_i \left( \mathbf{v}^{(i)}, \mathbf{d}^{(i)} \right) \}_{i=k_i+1}^{\infty}$ is strictly decreasing and bounded from above by $L_i \left( \mathbf{v}^{(k_i)}, \mathbf{d}^{(k_i)} \right)$. By (44), $\| \mathbf{v}^{(k_i+1)} - \mathbf{v}^{(k_i+1)} \| \to 0$ and $\| \mathbf{d}^{(k_i+1)} - \mathbf{d}^{(k_i+1)} \| \to 0$ as $k_i \to \infty$, thus $\mathbf{v}^{(k_i)} \to \mathbf{v}$, $\mathbf{d}^{(k_i)} \to \mathbf{d}$, and then $L_i \left( \mathbf{v}^{(i)}; \mathbf{d}^{(i)} \right) \to L_i \left( \mathbf{v}, \mathbf{d} \right)$, therefore $\{L_i \left( \mathbf{v}^{(i)}, \mathbf{d}^{(i)} \right) \}_{i=k_i+1}^{\infty} \to L_i \left( \mathbf{v}, \mathbf{d} \right)$. Therefore we conclude that $\mathbf{v}^{(i)}$ will approach $\mathbf{v}$ as proved above. Therefore we conclude that $\mathbf{v}^{(i)}$ will approach $\mathbf{v}$ and the SO tolls will be obtained. This completes the proof. □

Proposition 3 shows that, with certain conditions in Assumptions 1 and 2, as long as the observed flow could gradually approach UE during the trial-and-error procedure, the link flow trajectory with any initial state resulted from the trial-and-error procedure will approach the SO link flow pattern. The procedure is still efficient when the path flows follow a day-to-day adjustment process and the link tolls need not be adjusted day by day. Specially, convergence of the observed flow to UE is assured by Assumption 3, regardless of the identical length of inter-trial period $\Delta$. However, Assumption 3 requires that all the travelers are very sensitive to their travel cost and adjust their routes quickly. It is natural that the trial-and-error procedure would be inefficient to obtain the SO link tolls if some of the travelers are inertia and do not change their routes even experiencing an excess travel cost.

4. A numerical example

In this section, the same example used in Yang et al. (2004) is adopted to illustrate the proposed trial-and-error scheme with day-to-day path flow dynamics. The network, as shown in Figure 1, consists of 7 nodes, 11 links and 4 OD pairs ($1 \to 7, 2 \to 7, 3 \to 7$ and $6 \to 7$).

The true but unknown inverse demand functions are given as below:
\[
D^{-1}_{1 \to 7} \left( d_{1 \to 7} \right) = - \frac{1}{0.04} \ln \frac{d_{1 \to 7}}{600}, D^{-1}_{2 \to 7} \left( d_{2 \to 7} \right) = - \frac{1}{0.03} \ln \frac{d_{2 \to 7}}{500}
\]
The link travel time functions follow the BPR (Bureau of Public Roads) form with the free travel time, $t_a^0$, and link capacity, $c_a$, given in Table 1.

$$t_a (v_a) = t_a^0 \left[ 1 + 0.15 \left( \frac{v_a}{c_a} \right)^4 \right], \quad a \in A. \quad (48)$$

The day-to-day path flow dynamics is assumed to be given by (16) with $\beta = 1$.

**Table 1. Parameters of the link travel time functions**

<table>
<thead>
<tr>
<th>Link no., $a$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Free travel time, $t_a^0$</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>6</td>
<td>1</td>
<td>5</td>
<td>10</td>
<td>11</td>
<td>11</td>
<td>15</td>
</tr>
<tr>
<td>Link capacity, $c_a$</td>
<td>200</td>
<td>200</td>
<td>200</td>
<td>200</td>
<td>100</td>
<td>100</td>
<td>150</td>
<td>150</td>
<td>200</td>
<td>200</td>
<td>200</td>
</tr>
</tbody>
</table>

4.1. **Convergence of the dynamic link flow trajectory to SO link flow pattern**

The initial path flows are identically 50 on all paths, the lengths of inter-trial periods are identically 10, and it is assumed that $\theta_i = 1/k$. Figure 2 shows the evolution of the link flows. The link flows converge to a stable flow pattern after 15 trials.

The changing of the Euclidean distance between the realized flows $\tilde{v}(i)$ and the SO link flow pattern $\tilde{v}$ is shown in Figure 3. It is clearly shown that the Euclidean distance approaches zero very quickly after 15 trials.

4.2. **Trial-and-error procedure with different $\theta_i$**

We now examine the convergence of link flows under different sequences of $\theta_i$. The initial path flows are identically 50 in all paths and the lengths of inter-trial periods are identically 5. Figure 4 shows the Euclidean distance between the realized flows and the SO link flows with: (a) $\theta_i = k^{-0.5}$; (b) $\theta_i = k^{-1}$; (c) $\theta_i = k^{-1.1}$. Sequences (a) and (b) satisfy (13) while sequence (3) does not. All the three sequences lead to the convergence of the realized flows towards SO. Further, in this example, for the form of $\theta_i = k^{-\alpha}$, $\alpha > 0$, a small $\alpha$ may correspond to a faster converging speed.
Figure 2. Evolution of realized link flows.

Figure 3. Euclidean distance between realized link flows $\tilde{v}(k)$ and SO link flows $\tilde{v}$.

Figure 4. Euclidean distance between the realized and the SO flows with different $\theta$: (a) $\theta = k^{-0.5}$; (b) $\theta = k^{-1}$; (c) $\theta = k^{-1.1}$.
4.3. Trial-and-error with different time horizons and inter-trial periods

If we consider the toll-adjustment cost in practical implementation, the inter-trial periods \( \Delta \) could not be arbitrarily short. We have to make a trade-off between the adjustment cost and the convergence rate, it is thus useful to consider how to choose a proper time horizon for the trial-and-error implementation and how to determine the sequence of all inter-trial periods during the entire implementation time horizon. In this section we will investigate how the length of constant inter-trial periods \( \Delta \) and the number of total trials will affect the convergence speed. Therefore it is meaningful to select the optimal \( \Delta \) and the total number of trials to achieve a target error while minimize the total implementation time horizon, or to achieve the minimal error with a predetermined finite time horizon or cost budget.

As above, the error or efficiency of the trial-and-error procedure could be assessed by the Euclidean distance between the observed and the system optimal link flows in each iteration. Given a time horizon \( T \), changing the total number of trials \( K \) will change the inter-trial periods accordingly as \( \Delta = T/K \). For each combination of time horizon \( T \) and total number of trials \( K \), we can calculate the Euclidean distance between the realized flows and the SO flows at the end of the whole time horizon, as depicted in Figure 5.

In Figure 5, each line corresponds to an identical \( T \) and various value of \( K \). Among different lines, the time horizon \( T \) varies from 50 to 400. Some observations can be made from the figures. Firstly, generally speaking, given the time horizon \( T \) and the form of \( \theta_k \), shorter inter-trial periods and more trials may be a better choice. Secondly, in this example, no matter how long the time horizon is, the minimum error can be approximately achieved within 10 to 20 trials. Thirdly, with the form of \( \theta_k = k^{-\alpha} \), \( \alpha > 0 \), a smaller \( \alpha \) seems to perform better; as one can see that \( \alpha = 0.5 \) always has the minimal error while \( \alpha = 1.1 \) the maximum by comparing the three figures for the same time horizon and the same number of total trials. Finally, in the sense of reducing the system error, once the inter-trial periods are chosen properly, a longer time horizon is always preferred. In summary, in order to achieve the minimal error associated with the trial-and-error procedure, a longer time horizon is always preferred; once the time horizon is determined, \( \theta_k = k^{-0.5} \) is recommended, and a total number of 10 to 20 trials is enough for practical implementation in view of the implementation cost.
5. Conclusions

This paper investigated the trial-and-error implementation of road pricing for achieving system optimum under day-to-day flow dynamics. Neither the demand functions nor the flow evolution mechanism are needed explicitly. The day-to-day flow evolution follows the ‘excess travel cost dynamics’ and the iterative updating procedure in Yang et al. (2004) is adopted. The convergence of the iterative method is guaranteed under certain conditions on the flow evolution process. The inter-trial periods can be identical and their identical length could be arbitrary, and thus the time scheme of implementation is not restrictive in practice. A numerical example was presented and the convergence of the trial-and-error was demonstrated, the performance of the iterative implementation procedure was examined under different choices of the parameter of MSA and under different time horizons and different lengths of inter-trial periods. It was shown that the parameter values of MSA affect the convergence speed as well as the system error. To reduce the system error, more trials and shorter inter-trial periods are preferred at the expense of higher implementation cost. To choose a proper combination of the total trials and the length of each inter-trial period would be a tradeoff between system error and implementation cost.

In summary, the pricing design problem is analyzed in a politically and socially acceptable manner and under more realistic conditions of traffic flow evolution. The results further show the availability and robustness of the trial-and-error congestion pricing schemes that allows for unknown demand functions in a dynamic and non-equilibrium network. The results also demonstrate the potential of the trial-and-error method in assisting in the design, implementation and evaluation of various urban road pricing schemes. Particularly, the traffic planner can easily estimate or update the socially optimal congestion tolls by using readily available traffic count data.

It is interesting yet challenging to study the optimal combination of the trial number and inter-trial period for a given finite time horizon and theoretically investigate the error bound of the trial-and-error procedure. Other research directions include designing the system optimum anonymous link toll scheme in a transportation network with heterogeneous users and extended the trial-and-error procedure with bounded rationality of traversers’ route choice.

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References


Xu, W., Yang, H., Han, D., 2013. Sequential experimental approach for congestion pricing with multiple vehicle types and multiple time periods. Transportmetrica B 1 (2), 136-152.


