MATHEMATICS

ON SOME PROPERTIES OF LOCALLY COMPACT GROUPS

BY

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1. *Introduction and main result*

A locally compact group G is said to have the *property* (P_1) if to any compact set $K \subset G$ and any $\varepsilon > 0$ there is a function $s \in L^1(G)$ such that $s(x) \geqslant 0$ $(x \in G)$, $\int s(x)dx = 1$ and $\|L_y s - s\|_1 < \varepsilon$ for all $y \in K$. Here L_y is the left translation operator $[(L_y f)(x) = f(y^{-1}x)]$ and $\|\cdot\|_1$ is the norm in $L^1(G)$.

We say that G has the *property* (M) if there exists a left invariant mean on $C^\infty(G)$, the space of complex-valued, bounded, continuous functions on G, that is, a linear functional $\varphi \to M\{\varphi\}$, $\varphi \in C^\infty(G)$, such that $M\{\varphi\} \geqslant 0$ if φ is real and $\varphi \geqslant 0$, $M\{1\} = 1$ and $M\{L_y\varphi\} = M\{\varphi\}$ for all $y \in G$. The property (M) has been the subject of many investigations ever since the fundamental paper of VON NEUMANN [4], concerned with discrete groups; see the article by DIXMIER [2], also for references.

The property (M) implies the existence of a left invariant mean on $L^\infty(G)$, as is readily seen (and well-known). We also remark that, given a left invariant mean on $C^\infty(G)$ or $L^\infty(G)$, we can easily obtain a right invariant mean on $C^\infty(G)$ or $L^\infty(G)$, and conversely. We also note that $|M\{\varphi\}| \leqslant \|\varphi\|_\infty$ for all φ in $C^\infty(G)$ or $L^\infty(G)$.

It is a simple exercise in Functional Analysis to show that (P_1) implies (M). The purpose of this Note is to show that, conversely, (M) *implies* (P_1).

Thus the properties (M) and (P_1) are equivalent. As has recently been proved, (P_1) is also equivalent to another property which concerns the continuous positive definite functions on G [5d]. Thus we have gained some insight into these properties of locally compact groups.

2. *First part of the proof*

Proposition. *If G has the property (M), then for any $f \in L^1(G)$*

$$\inf \int \left| \sum c_n f(xy_n)\, \Delta(y_n)\right| dx = \left| \int f(x)\, dx \right|, \tag{1}$$

the infimum being taken for all finite sums such that $c_n > 0$, $\sum c_n = 1$, $y_n \in G$. Here Δ is the Haar modular function of G.

This proposition is a special case of a result of GLICKSBERG ([3], relation 4.7).

Remark. The relation (1) can also be proved directly, on the assumption that G has the property (P_1). The most general relation of this kind is as follows. Let H be any closed subgroup of G. If H has the property (P_1), then for every $f \in L^1(G)$

$$\inf \int | \sum c_n f(x\xi_n)\, \Delta(\xi_n)|\, dx = \int_{G/H} \left\{ \int_H \frac{f(x\xi)}{\varrho(x\xi)}\, d\xi \right\} d_\varrho \dot{x},$$

where the infimum is taken for all finite sums such that $c_n > 0$, $\sum c_n = 1$, $\xi_n \in H$. The function ϱ on the right is any strictly positive, continuous solution of the functional equation $\varrho(x\xi)/\varrho(x) = \Delta_H(\xi)/\Delta(\xi)$ $(x \in G,\ \xi \in H)$, where Δ_H is the Haar modular function of H, and $d_\varrho \dot{x}$ is the corresponding quasi-invariant measure on G/H determined by $\int_{G/H} \{ \int_H k(x\xi) d\xi \}\, d_\varrho \dot{x} = \int k(x)\, \varrho(x)\, dx$, $k \in \mathscr{K}(G)$.

The proof of the Proposition above is based, essentially, on a result of GLICKSBERG [3, § 2.5].

Let B be a real or complex Banach space, with norm $\|\cdot\|$. Suppose in B there acts a family of linear operators A_y, $y \in G$, satisfying the following conditions:

(i) G is a locally compact group possessing the property (M).[1])

(iia) A_y is a 'contraction operator' for every $y \in G$ (i.e. $\|A_y f\| \leqslant \|f\|$ for all $f \in B$).

(iib) $A_{y_1}(A_{y_2} f) = A_{y_1 y_2} f$ $(y_1, y_2 \in G)$, for $f \in B$.

(iic) The function $y \to \langle A_y f, \varphi \rangle$ is continuous on G, for every $f \in B$ and $\varphi \in B'$, the dual of B.[2])

For $f \in B$ let $C_G(f)$ be the set of all finite linear combinations $\sum c_n A_{y_n} f$, with $c_n > 0$, $\sum c_n = 1$ and $y_n \in G$; this is the smallest convex set containing f which is invariant under the operators A_y, $y \in G$. Let J_G be the closed linear subspace of B generated by all vectors of the form $A_y f - f$ $(f \in B$, $y \in G)$.

Glicksberg has shown [loc. cit.] that, *under the conditions* (i) *and* (iia, b, c) *above, the distance of $C_G(f)$ from the origin coincides with the distance of f from J_G.* We shall give a simple proof of this in § 4.

From Glicksberg's result we can obtain immediately the Proposition stated above: we take $B = L^1(G)$ and for A_y the family of operators $f \to A_y f$, $y \in G$, defined for $f \in L^1(G)$ by $(A_y f)(x) = f(xy)\Delta(y)$. Then the conditions (i) and (iia, b, c) are satisfied. Moreover, the distance of any $f \in L^1(G)$ from the linear subspace J_G defined above is precisely $|\int f(x) dx|$ which yields (1).

The calculation of this distance can be effected by the method in [5b, § 2], since J_G is clearly invariant under left translations. The only

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[1]) It is actually enough to assume that G is a topological semi-group, but we do not need this here.

[2]) $\langle h, \varphi \rangle$ denotes the value of $\varphi \in B'$ for $h \in B$.

continuous functions $\varphi \in L^\infty(G)$ orthogonal to J_G are the constants. Cf. also [5a] and the footnote in [5b, p. 260].

3. *Second part of the proof*

Suppose G has the property (M) and let a compact set $K \subset G$ and $\varepsilon > 0$ be given. Take any $h \in L^1(G)$ such that $h(x) \geqslant 0$ $(x \in G)$ and $\int h(x)dx = 1$. There is some open neighbourhood U of the neutral element of G such that

(2) $$\|L_y h - h\|_1 < \varepsilon \qquad \text{if } y \in U.$$

Next there are finitely many points of K, say $(a_m)_{1 \leqslant m \leqslant M}$, such that the translates $(a_m \cdot U)_{1 \leqslant m \leqslant M}$ together cover K. Now consider the M functions $L_{a_m} h - h$. We shall show: there are finitely many numbers $c_n > 0$, with $\sum c_n = 1$, and elements $y_n \in G$, such that

(3) $$\|\sum_n c_n A_{y_n}(L_{a_m} h - h)\|_1 < \varepsilon \qquad 1 \leqslant m \leqslant M,$$

where A_y is the operator defined by $A_y f(x) = f(xy)\Delta(y)$.

If $M = 1$, this is simply (1), with $f = L_{a_1} h - h$, so that $\int f(x)dx = 0$. Now we use induction. Put for simplicity of notation $L_{a_m} h - h = f_m$, $1 \leqslant m \leqslant M$, and suppose $\|\sum_j c_j' A_{y_j'} f_m\| < \varepsilon$ for $1 \leqslant m \leqslant M-1$, with $c_j' > 0$, $\sum c_j' = 1$, $y_j' \in G$. Put $g = \sum c_j' A_{y_j'} f_M$. Since $\int g(x)dx = 0$, there are finitely many numbers $d_k > 0$, with $\sum d_k = 1$, and elements $z_k \in G$, such that

$$\|\sum d_k A_{z_k} g\|_1 < \varepsilon.$$

Put $d_k \cdot c_j' = c_n$, so $c_n > 0$ and $\sum c_n = 1$; put correspondingly $z_k \cdot y_j' = y_n$. Then $\|\sum_n c_n A_{y_n} f_m\|_1 < \varepsilon$ for $1 \leqslant m \leqslant M$ (observe that $\|\sum_n c_n A_{y_n} f_m\|_1 \leqslant$ $\leqslant \|\sum_j c_j' A_{y_j'} f_m\|_1$ for $1 \leqslant m \leqslant M-1$).

Now put

$$s = \sum c_n A_{y_n} h.$$

Then $s(x) \geqslant 0$ $(x \in G)$, $\int s(x)dx = 1$ and, as we can verify,

(4) $$\|L_y s - s\|_1 < 2\varepsilon \qquad \text{for all } y \in K.$$

Indeed, if we put $A = \sum c_n A_{y_n}$, then A is also a contraction operator ($\|Af\|_1 \leqslant \|f\|_1$ for $f \in L^1(G)$) and we can write, since the operators A and L commute,

$$L_{a_m} L_y A h - A h = L_{a_m} A(L_y h - h) + A(L_{a_m} h - h), \qquad 1 \leqslant m \leqslant M,\ y \in U.$$

Hence $\|L_{a_m y} s - s\|_1 < \varepsilon + \varepsilon$ $(1 \leqslant m \leqslant M,\ y \in U)$, by (2) and (3), which proves (4), since $(a_m \cdot U)_{1 \leqslant m \leqslant M}$ is a covering of K.

The compact set $K \subset G$ and $\varepsilon > 0$ were arbitrary, so G has the property (P_1).

4. *A proof of Glicksberg's theorem*

First we establish the following lemma.

Lemma A.[3]) *Let C be a convex set in a real or complex Banach space B and put* $d = \text{dist}\,\{C, 0\} = \inf \|v\|$ $(v \in C)$. *If* $d > 0$, *then there is a functional* φ *in* B', *the dual of B, such that*

(5a) $$\text{Re}\,\langle v, \varphi\rangle \geqslant 1 \textit{ for all } v \in C;$$

(5b) $$\|\varphi\|_{B'} = 1/d.$$

Here $\langle v, \varphi\rangle$ is the value of φ at the 'point' v, Re denotes the real part, and $\|\varphi\|_{B'}$ is the norm of φ.

Proof. Suppose first B is real. Put $B_d{}^0 = \{b | b \in B, \|b\| < d\}$. Then the set $A = C - B_d{}^0$ is open, convex and does not contain the vector zero. Hence there exists $\psi \in B'$ such that $\langle a, \psi\rangle > 0$ for all $a \in A$ (cf. [1], p. 70, Remarque). Then $\langle c, \psi\rangle > \langle b, \psi\rangle$ for all $c \in C$, $b \in B_d{}^0$. Thus for each $c \in C$ we have $\langle c, \psi\rangle \geqslant \sup_{b \in B_d{}^0} \langle b, \psi\rangle = d \cdot \|\psi\|_{B'}$. Put $\varphi = \psi/(d \cdot \|\psi\|_{B'})$: then $\langle c, \varphi\rangle \geqslant 1$ for all $c \in C$, and $\|\varphi\|_{B'} = 1/d$.

If B is complex, we can consider B also as a *real* Banach space. Thus, by the preceding, there is a real-linear continuous functional φ_r on B such that $\langle c, \varphi_r\rangle \geqslant 1$ for all $c \in C$ and φ_r has norm $1/d$. As is well-known, there is then a (complex) linear continuous functional φ on B such that $\text{Re}\,\varphi = \varphi_r$ and $\|\varphi\|_{B'} = 1/d$ (it is defined by $\langle v, \varphi\rangle = \langle v, \varphi_r\rangle - i\langle iv, \varphi_r\rangle$, for $v \in B$). This completes the proof of the lemma.

Now suppose the family of operators $(A_y)_{y \in G}$ acting on B satisfies the conditions (i) and (iia, b, c) of § 2. Let us call a functional $\dot\varphi \in B'$ *invariant* with respect to the family $(A_y)_{y \in G}$ if, for each $v \in B$, $\langle A_y v, \dot\varphi\rangle = \langle v, \dot\varphi\rangle$ for all $y \in G$. We prove next:

Lemma B.[4]) *Suppose the conditions* (i) *and* (iia, b, c) *of* § 2 *hold. Let* $f \in B$ *be such that* $\inf \| \sum c_n A_{y_n} f\| = d > 0$, *the infimum being taken for all finite sums with* $c_n > 0$, $\sum c_n = 1$, $y_n \in G$. *Then there is an* INVARIANT *functional* $\dot\varphi \in B'$ *such that* $\langle f, \dot\varphi\rangle = 1$ *and* $\|\dot\varphi\|_{B'} = 1/d$.

The proof is based on the property (M) of the group G. Let M be a *right* invariant mean on $C^\infty(G)$; we denote its value for $\alpha \in C^\infty(G)$ by $M\{\alpha\}$ or $M_x\{\alpha(x)\}$. Let $\varphi \in B'$ be given. Then for every $v \in B$ the function $x \to \langle A_x v, \varphi\rangle$ is continuous on G, by condition (iic) in § 2, and $|\langle A_x v, \varphi\rangle| \leqslant \|v\| \cdot \|\varphi\|_{B'}$ for all $x \in G$, by (iia). Thus $v \to M_x\{\langle A_x v, \varphi\rangle\}$ is a linear functional on B which is bounded: $|M_x\{\langle A_x v, \varphi\rangle\}| \leqslant \|\varphi\|_{B'} \cdot \|v\|$. Hence there is a $\dot\varphi \in B'$ such that

(6a) $$M_x\{\langle A_x v, \varphi\rangle\} = \langle v, \dot\varphi\rangle \qquad \text{for all } v \in B;$$

(6b) $$\|\dot\varphi\|_{B'} \leqslant \|\varphi\|_{B'}.$$

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[3]) Cf. [3], p. 100, relation (2.6) and footnote 6.

[4]) Cf. [3], Lemma 2.1. The proof given there is based on a theorem of M. M. Day.

Moreover, if we replace v by $A_y v$ in (6a), we obtain by condition (iib) in § 2 and the right invariance of M

(6c) $$\langle A_y v, \dot{\varphi}\rangle = \langle v, \dot{\varphi}\rangle \quad \text{for all } y \in G,$$

that is, $\dot{\varphi}$ is invariant.

We may express (6a) by saying that '$\dot{\varphi}$ is the mean value of φ along the orbit $A_x v$, $x \in G$'.

By Lemma A there is a $\varphi \in B'$ satisfying (5a, b) with $C = C_G(f)$, as defined in § 2. Applying the preceding to this φ, we obtain a functional $\dot{\varphi} \in B'$ with the properties

(7a) $$\mathrm{Re}\,\langle c, \dot{\varphi}\rangle \geqslant 1 \text{ for all } c \in C_G(f);$$

(7b) $$\|\dot{\varphi}\|_{B'} \leqslant 1/d;$$

(7c) $$\dot{\varphi} \text{ is invariant.}$$

This follows respectively from (5a) [with $C = C_G(f)$] and (6a), (5b) and (6b), and (6c).

Now, if $c \in C_G(f)$, we have

$$1 \leqslant \mathrm{Re}\,\langle c, \dot{\varphi}\rangle \leqslant |\langle c, \dot{\varphi}\rangle| \leqslant \|c\| \cdot 1/d.$$

But $\|c\|$ can be arbitrarily close to d, while $\langle c, \dot{\varphi}\rangle$ is constant on $C_G(f)$. It follows that $1 = \mathrm{Re}\,\langle c, \dot{\varphi}\rangle = |\langle c, \dot{\varphi}\rangle|$ and thus even $\langle c, \dot{\varphi}\rangle = 1$ for $c \in C_G(f)$; moreover, we must actually have equality in (7b). Thus Lemma B is proved.

From Lemma B we now obtain Glicksberg's result almost at once. Let (with an obvious notation) dist $\{C_G(f), 0\} = d \geqslant 0$ and put dist $\{f, J_G\} = d' \geqslant 0$. We want to show $d = d'$. Now clearly dist $\{f, J_G\} =$ dist $\{f + J_G, 0\}$; also $C_G(f) \subset f + J_G$, since $\sum c_n A_{y_n} f = f + \sum c_n (A_{y_n} f - f)$ (observe $\sum c_n = 1$). Thus $d \geqslant d'$ and it remains to show $d \leqslant d'$, where we may even assume $d > 0$. But if $d > 0$, there is an *invariant* $\dot{\varphi} \in B'$ such that $\langle f, \dot{\varphi}\rangle = 1$ and $\|\dot{\varphi}\|_{B'} = 1/d$, by Lemma B. Since $\dot{\varphi}$ is invariant, it vanishes on J_G, so $\langle f + v, \dot{\varphi}\rangle = 1$ for all $v \in J_G$. But $\|f + v\|$ can be arbitrarily close to d' (by proper choice of $v \in J_G$), hence $d' \cdot (1/d) \geqslant 1$ or $d \leqslant d'$. Thus $d = d'$ and Glicksberg's theorem is proved.

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