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Letter to the Editor

Frame expansions with erasures: an approach through the non-commutative operator theory

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Abstract

In modern communication systems such as the Internet, random losses of information can be mitigated by oversampling the source. This is equivalent to expanding the source using overcomplete systems of vectors (frames), as opposed to the traditional basis expansions. Dependencies among the coefficients in frame expansions often allow for better performance compared to bases under random losses of coefficients. We show that for any n-dimensional frame, any source can be linearly reconstructed from only $O(n \log n)$ randomly chosen frame coefficients, with a small error and with high probability. Thus every frame expansion withstands random losses better (for worst case sources) than the orthogonal basis expansion, for which the $n \log n$ bound is attained. The proof reduces to M. Rudelson's selection theorem on random vectors in the isotropic position, which is based on the non-commutative Khinchine's inequality.

Keywords: Frames; Overcomplete representations; Source coding; Multiple descriptions; Linear reconstruction

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1. Introduction

Representation of signals using frames, which are overcomplete sets of vectors, is advantageous over basis expansions in a variety of practical applications. Dependencies among the coefficients of the overcomplete representations guarantee a better stability in presence of noise, quantization, erasures, as well as greater freedom of design compared to bases. This general paradigm is confirmed by many experiments and some theoretical work, see, e.g., [1,3–8] and the bibliography cited therein.

Of particular importance are the dependencies contained in frame expansions for design of communication systems. The redundancy of frames can mitigate random losses of expansion coefficients that occur in packet-based communication systems such as the Internet. Detection and retransmission of lost packets in such systems takes much longer than their original transmission. This is the main source of delays known to all network users. Such delays are unacceptable for many applications, such as the real-time video. It is thus desirable for the receiver to be able to approximately reconstruct the information sent to him from *whatever* packets he receives, despite the loss of some packets. There should exist certain dependencies among the packets, otherwise the information contained in a missing packet would be irrevocably lost. Then, what is the best way to distribute the information among the packets so that each packet is equally important? Equivalently, this is the problem of the multiple description coding (MDC) theory, where one wishes to communicate information over a set of parallel channels, each of which either works perfectly or not at all.

The idea originated in [6] was to use frame expansions to distribute the information among the packets with some dependencies. One can view this communication scheme as follows:

$$x \in \mathbb{R}^n \to \boxed{\begin{array}{c}\text{frame}\\ \text{expansion}\end{array}} \xrightarrow{y\in\mathbb{R}^m} \boxed{\begin{array}{c}\text{transmission}\\ \text{(losses)}\end{array}} \xrightarrow{\hat{y}\in\mathbb{R}^k} \boxed{\text{reconstruction}} \to \hat{x} \in \mathbb{R}^n. \tag{1}$$

The source information is viewed as a vector $x \in \mathbb{R}^n$. This vector is represented by its $m \geqslant n$ expansion coefficients with respect to some fixed frame. These coefficients are sent over the network in m packets, each in its own packet. Due to unpredictable communication losses, the user receives only a random subset of these packets, say k in average. The user applies the linear reconstruction to the received coefficients in hope that the reconstruction error would be small with graceful probability. The fundamental problem is[2]:

How many random coefficients of a frame expansion does the user need to receive to be able to linearly reconstruct the source vector with a small error and with large probability?

The work on this question, both theoretical and experimental, was initiated in [6] and continued in [8] and [3], see also a survey paper [5]. Both cases were considered: $k < n$, which clearly requires a statistical model of input vector x, and $k \geqslant n$. The performance of the frame representations was compared to that of the classical block channel-coded basis representations.

In the present paper we look for a best bound on k which works for *all* frames and *all* source vectors x. Does every frame necessarily perform better than the trivial frame, the orthonormal basis—or, more generally, an orthonormal basis in $\mathbb{R}^n$ each of whose elements is repeated s times? Communicating a source

[2] In this paper, we neglect the quantization issues, which are treated in [7] and [6].

vector x with the trivial frame is equivalent to sending each of the n coefficients of the orthonormal expansion of x precisely s times. To be able to reconstruct x, the user must receive each of the n coefficients at least once. This is possible with probability at least $1-\varepsilon$ only if the user receives $k \geqslant C(\varepsilon)n\log n$ random coefficients in total. This gives the lower bound on k in the question above. Remarkably, the upper bound matches.

Theorem 1.1. *For any uniform tight frame in $\mathbb{R}^n$ and any source vector x, the linear reconstruction from k random coefficients of x yields an approximation error at most ε with probability $1-\varepsilon$, provided $k \geqslant C(\varepsilon)n\log n$.*

Here $C(\varepsilon)$ is a constant that depends only on ε; this dependence is discussed in Corollary 2.2 below, which is a more explicit version of Theorem 1.1. Tightness of the frame is assumed only for simplicity.

Note that the optimal bound on k does not depend on the size m of the frame, so there may be many lost coefficients—in fact, most of them may be lost. Hence it is not the number of the lost coefficients that determines the performance but the number k of received coefficients.

As argued in [5], one advantage of frame representations over the traditional block channel-coded basis representations is that frames allow for a real time reconstruction of the source. The receiver can attempt to reconstruct a source vector—such as a still image or video—in real time as the packets arrive, starting from the very first successfully received coefficient. Within one communication session, the number of received coefficients k will thus grow in time from 1 to possibly m, and the quality of reconstruction will improve as more coefficients arrive. (In contrast to this, in the block channel-coded basis model the user must wait until n coefficients arrive.) Theorem 1.1 states that, with *any* frame design and *any* source, the reconstruction quality will reach a nearly optimal level as soon as $\sim n\log n$ coefficients are received, so one may stop the session then.

Theorem 1.1 shows that every frame must withstand random losses better than the trivial frame, the one formed by repeating the elements of the orthogonal basis. Of course, there exist frames that perform better than the trivial frame. The problem of optimal design of such frames is addressed in [6] and [3]. As noticed, e.g., in [7], a set of $m = sn$ random points (x_i) taken independently with the uniform distribution on the unit sphere S^{n-1} forms a frame which approaches a tight frame with large probability, provided the redundancy $s \to \infty$. Consequently, a random k-element subset of this set also forms an almost tight frame with large probability, provided $k \geqslant tn$ and t is large. Then one can linearly reconstruct any source vector x from using its k random coefficients with respect to the frame (x_i) with probability $1-\varepsilon$, provided $k \geqslant C(\varepsilon)n$. Hence for this frame, the logarithmic factor is not needed in the number received coefficients k.

Our proof of Theorem 1.1 is based on a result of M. Rudelson in the asymptotic convex geometry about vectors in the isotropic position [14]. There exists a remarkable equivalence of the theories. All of the following classes coincide in $\mathbb{R}^n$ (up to an appropriate rescaling), see [16]:

- the class of tight frames,
- the class of contact points of convex bodies,
- the class of John's decompositions of the identity,
- the class of vectors in the isotropic position.

The selection theorem of M. Rudelson [14] can thus be interpreted as a result about frames, which leads to Theorem 1.1. In order to obtain an exponentially large probability in Theorem 1.1 and because of a slightly different model of random selection in M. Rudelson's theorem, we will prove the latter with some necessary modifications. Two proofs of Rudelson's theorem are known. The one which was historically the first [13] uses majorizing measures, a deep technique in modern probability theory developed by M. Talagrand (see [15]). The other proof [14] is the one we follow in the present paper. It is based on the non-commutative operator theory, more precisely on the non-commutative Khinchine's inequality due to F. Lust-Piquard and G. Pisier (see [10,12,14]).

Section 2 relates frames to the decompositions of the identity and offers a precise form of Theorem 1.1. Section 3 discusses the non-commutative Khinchine's inequality and Pisier's proof of Rudelson's lemma. In Section 4 we show how Rudelson's lemma implies a precise form of Theorem 1.1.

2. Frames as decompositions of identity and their random parts

For an introduction to frames, see [4] and [2]. A system of vectors (x_i) finite or infinite, in a Hilbert space, is called a *frame* if there exist $A > 0$ and $B > 0$ (the *frame bounds*) such that

$$A\|x\|^2 \leqslant \sum_i \left|\langle x, x_i\rangle\right|^2 \leqslant B\|x\|^2 \quad \text{holds for all } x \in \mathbb{R}^n.$$

Our Hilbert space will be $\mathbb{R}^n$ with its canonical scalar product. We will specialize to *uniform frames*, those for which $\|x_i\| = 1$ for all i, and to *tight frames*, for which $A = B$. The reason for considering only tight frames is the simple fact that a frame has frame bounds (A, B) if and only if it is $\sqrt{AB}$-equivalent to some tight frame (see [2]). By being M-equivalent we mean that there exists a linear operator T that maps elements of one frame to the other with $\|T\|\|T^{-1}\| \leqslant M$.

We will view frame expansions as decompositions of identity. A pair of vectors (x, y) in $\mathbb{R}^n$ defines a one-dimensional linear operator $x \otimes y$ given by $(x \otimes y)(z) = \langle x, z\rangle y$. Then for any system of vectors $(x_i)_{i=1}^m$ with $\|x_i\| = 1$ and for the identity operator *id* on $\mathbb{R}^n$ one has

$$(x_i)_{i=1}^m \text{ is a uniform tight frame in } \mathbb{R}^n \text{ if and only if } id = \frac{n}{m}\sum_{i=1}^m x_i \otimes x_i. \tag{2}$$

Communication scheme (1) based on a uniform tight frame $(x_i)_{i=1}^m$ works as follows. A source vector $x \in \mathbb{R}^n$ is represented through the expansion (2), i.e.

$$x = \frac{n}{m}\sum_{i=1}^m \langle x_i, x\rangle x_i,$$

and the coefficients $y(i) := \langle x_i, x\rangle$, $i = 1, \dots, m$, are sent over the network. At each given time during the communication session, the user has received a random subset $\sigma \subset \{1, \dots, m\}$ of these coefficients. The user applies to them the linear reconstruction, computing

$$\hat{x} = \frac{n}{|\sigma|}\sum_{i\in\sigma} \langle x_i, x\rangle x_i \tag{3}$$

in hope that the error $\|x - \hat{x}\|$ would be small with large probability. The question is—how large should $|\sigma|$ for this to hold?

More formally, the random subset σ is realized by including each element of $\{1,\dots,m\}$ into σ independently with probability k/m, where $0<k<m$ is some fixed number. Then σ is a *random subset of* $\{1,\dots,m\}$ *of average size* k.

Theorem 2.1. *Let* $(x_i)_{i=1}^m$ *be a uniform tight frame in* $\mathbb{R}^n$*, and* $\varepsilon>0$*. Let* σ *be a random subset of* $\{1,\dots,m\}$ *of average size* $k \geqslant C\cdot(n/\varepsilon^2)\log(n/\varepsilon^2)$*. Then*

$$\mathbb{P}\left\{\left\| id - \frac{n}{|\sigma|}\sum_{i\in\sigma} x_i\otimes x_i\right\| > \varepsilon t\right\} \leqslant Ce^{-t^2}$$

in the (*only interesting*) *range* $0<t<1/\varepsilon$.

Here and thereafter $C, C_1, \dots,$ denote absolute constants, whose values for convenience may be different from line to line (but they do not depend on anything).

Theorem 2.1 gives an asymptotically optimal bound on the required number k of received coefficients in communication scheme (1).

Corollary 2.2. *Let* $(x_i)_{i=1}^m$ *be a uniform tight frame in* $\mathbb{R}^n$*. Let* $\varepsilon\in(0,1)$*,* $t>1$ *and* $k\geqslant C\times(n/\varepsilon^2)\log(n/\varepsilon^2)$*. With probability at least* $1-Ce^{-t^2}$*, the linear reconstruction* (3) *from a random subset* σ *of average size* k *gives the error*

$$\|x-\hat{x}\|<\varepsilon t \quad \textit{for all possible sources } x\in\mathbb{R}^n.$$

Thus any n-dimensional source can be reconstructed with error εt and with probability $1-Ce^{-t^2}$ from a random subset of $C\cdot(n/\varepsilon^2)\log(n/\varepsilon^2)$ frame coefficients.

Theorem 1.1 clearly follows from Corollary 2.2.

Remark. The proof also shows that the average approximation error in Theorem 2.2 is small, $\mathbb{E}\|x-\hat{x}\|<\varepsilon$.

3. Non-commutative Khinchine's inequality and Rudelson's theorem

The main ingredient in the proof of Theorem 2.1 is the following result of M. Rudelson [14].

Lemma 3.1 (M. Rudelson). *Let* (z_i) *be a finite collection of vectors in* $\mathbb{R}^d$*. Then*

$$\left(\mathbb{E}\left\|\sum_i \varepsilon_i z_i\otimes z_i\right\|^p\right)^{1/p} \leqslant C(p+\log d)^{1/2}\max_i\|z_i\|\cdot\left\|\sum_i z_i\otimes z_i\right\|^{1/2}.$$

G. Pisier ([12], see [14]) discovered an approach to this result via the non-commutative operator theory, which greatly simplified the original proof of M. Rudelson [13]. For completeness, we give a proof of Lemma 3.1 since only the case $p=1$ was treated explicitly in the literature.

Lemma 3.1 reduces to the non-commutative Khinchine inequality due to F. Lust-Piquard and G. Pisier (see [10,12,14]). In the non-commutative operator theory, the role of scalars is played by linear operators.

Beside the usual operator norm, an operator Z on $\mathbb{R}^d$ has the norm in the Schatten class C_p^d for $p \geqslant 1$, defined as follows. Let $s_i(Z)$ be the s-numbers of Z, that is the eigenvalues of Z^*Z. The norm in the Schatten class is then $\|Z\|_{C_p^d} = (\sum_{i=1}^d s_i(Z)^p)^{1/p}$.

Theorem 3.2 (Non-commutative Khinchine's inequality [10,12,14]). *Let $2 \leqslant p < \infty$. For any finite sequence (Z_i) in C_p^d one has*

$$R\big((Z_i)\big) \leqslant \left(\mathbb{E}\left\|\sum_i \varepsilon_i Z_i\right\|_{C_p^d}^p\right)^{1/p} \leqslant C\sqrt{p}\cdot R\big((Z_i)\big),$$

where

$$R\big((Z_i)\big) = \max\left(\left\|\left(\sum Z_i^* Z_i\right)^{1/2}\right\|_{C_p^d}, \left\|\left(\sum Z_i Z_i^*\right)^{1/2}\right\|_{C_p^d}\right).$$

In the scalar case, that is for $d = 1$, Theorem 3.2 is the classical Khinchine's inequality (see, e.g., [9] Lemma 4.1).

Proof of Lemma 3.1. Note that for every $r \geqslant 1$ and every operator $Z \in C_r^d$,

$$\|Z\|_{C_r^d} = \left(\sum_{i=1}^d s_i(Z)^r\right)^{1/r} \leqslant d^{1/r} \max_i s_i(Z).$$

Let $r = p + \log d$. Then $d^{1/r} \leqslant e$, hence

$$\|Z\| \leqslant \|Z\|_{C_r^d} \leqslant e\|Z\|. \tag{4}$$

We apply the non-commutative Khinchine's inequality for $Z_i = z_i \otimes z_i$. Note that $Z_i^* Z_i = Z_i Z_i^* = \|z_i\|^2 z_i \otimes z_i$. By (4),

$$\left(\mathbb{E}\left\|\sum_i \varepsilon_i z_i \otimes z_i\right\|^p\right)^{1/p} \leqslant \left(\mathbb{E}\left\|\sum_i \varepsilon_i z_i \otimes z_i\right\|_{C_r^d}^p\right)^{1/p} \leqslant C\sqrt{r}\left\|\left(\sum_i \|z_i\|^2 z_i \otimes z_i\right)^{1/2}\right\|_{C_r^d}^p$$
$$\leqslant Ce\sqrt{r}\max_i \|z_i\| \cdot \left\|\left(\sum_i z_i \otimes z_i\right)^{1/2}\right\|.$$

In view of our choice of r, this completes the proof of Lemma 3.1. □

4. Proof of Theorem 2.1

4.1. Moments and tails

The tail probability in Theorem 2.1 can be computed by estimating the moments. This is described in the following standard lemma. For any $\alpha \geqslant 1$, the ψ_α-norm of a random variable Z is defined as

$$\|Z\|_{\psi_\alpha} = \inf\big\{\lambda > 0\colon \mathbb{E}\exp|Z/\lambda|^\alpha \leqslant e\big\}.$$

Lemma 4.1 (See [9] Lemmae 3.7 and 4.10). *Let Z be a nonnegative random variable, and let $\alpha = d/2$ for some positive integer d. The following are equivalent*:

(i) *there exists a constant $K > 0$ such that*
$$\left(\mathbb{E}Z^p\right)^{1/p} \leqslant Kp^\alpha \quad \textit{for all } p \geqslant 2;$$
(ii) *there exists a constant $K > 0$ such that*
$$\mathbb{P}\{Z > Kt\} \leqslant 2\exp\left(-t^{1/\alpha}\right) \quad \textit{for all } t > 0;$$
(iii) *there exists a constant $K > 0$ such that*
$$\|Z\|_{\psi_\alpha} \leqslant K.$$

Furthermore, the constants in (i), (ii), *and* (iii) *depend only on α and on each other.*

Corollary 4.2. *Let Z be a nonnegative random variable and let $p \geqslant 2$. Then*
$$\left(\mathbb{E}Z^p\right)^{1/p} \leqslant Cp\log(\mathbb{E}\exp Z)$$
for all $p \geqslant 1$.

Proof. Let $M = \|Z\|_{\psi_1}$. Assume first that $M \geqslant 1$. We have
$$\mathbb{E}\exp(Z/M) = e.$$
By Lemma 4.1, $(\mathbb{E}(Z/M)^p)^{1/p} \leqslant Cp$. Then by Jensen's inequality
$$\left(\mathbb{E}Z^p\right)^{1/p} \leqslant CpM = CpM\log\left(\mathbb{E}\exp(Z/M)\right) = Cp\log\left(\mathbb{E}\exp(Z/M)\right)^M \leqslant Cp\log(\mathbb{E}\exp Z).$$
For a general nonnegative variable Z, note that $\|1+Z\|_{\psi_1} \geqslant 1$, hence by the previous argument
$$\left(\mathbb{E}Z^p\right)^{1/p} \leqslant \left(\mathbb{E}(1+Z)^p\right)^{1/p} \leqslant Cp\log\left(\mathbb{E}\exp(1+Z)\right) = Cep\log(\mathbb{E}\exp Z).$$
This completes the proof. □

4.2. Symmetrization

We start our proof of Theorem 2.1 with decomposition (2),
$$x = \frac{n}{m}\sum_{i=1}^m \langle x_i, x\rangle x_i.$$
To realize a random subset σ, we introduce selectors $(\delta_i)_{i=1}^m$, that is independent $\{0,1\}$-valued random variables with means $\mathbb{E}\delta_i = \delta$, where $\delta = k/m$. Then $\sigma = \{i\colon \delta_i = 1\}$ is a random subset of $\{1,\dots,m\}$ of average size k.

Disregarding for a moment a difference between the random size $|\sigma|$ and its mean k, thanks to Lemma 4.1 we can compute the probability estimate in Theorem 2.1 by estimating the moments
$$E_p = \left(\mathbb{E}\left\|id - \frac{n}{k}\sum_{i\in\sigma} x_i\otimes x_i\right\|^p\right)^{1/p} = \left(\mathbb{E}\left\|id - \frac{n}{k}\sum_{i=1}^m \delta_i x_i\otimes x_i\right\|^p\right)^{1/p}$$

for $p \geqslant 2$. This will be done in several steps.

At the first step, we apply the classical symmetrization technique (see [9] 6.2). We look at $Y = id - (n/k)\sum_{i=1}^m \delta_i x_i \otimes x_i$ as a random variable (random operator) and consider its independent copy Y'. Since $\mathbb{E}Y' = 0$, Jensen's inequality yields $\mathbb{E}\|Y\|^p \leqslant \mathbb{E}\|Y - Y'\|^p$, hence

$$E_p \leqslant \left(\mathbb{E}\left\| \frac{n}{k}\sum_{i=1}^m (\delta_i - \delta_i')x_i \otimes x_i \right\|^p\right)^{1/p},$$

where $(\delta_i')_{i=1}^m$ is an independent copy of $(\delta_i)_{i=1}^m$. Let (ε_i) be a sequence of independent symmetric $\{-1,1\}$-valued random variables, independent of both (δ_i) and (δ_i'). Since $\delta_i - \delta_i'$ is a symmetric random variable, it is distributed identically to $\varepsilon_i(\delta_i - \delta_i')$. By Minkowski's inequality,

$$E_p \leqslant \left(\mathbb{E}\left\| \left(\frac{n}{k}\sum_{i=1}^m \varepsilon_i\delta_i x_i \otimes x_i\right) - \left(\frac{n}{k}\sum_{i=1}^m \varepsilon_i\delta_i' x_i \otimes x_i\right) \right\|^p\right)^{1/p} \leqslant 2\left(\mathbb{E}\left\| \frac{n}{k}\sum_{i=1}^m \varepsilon_i\delta_i x_i \otimes x_i \right\|^p\right)^{1/p}. \tag{5}$$

4.3. *Bounding the moments*

Let us fix a realization of the selectors (d_i) (hence a set σ) and denote by $\mathbb{E}_\varepsilon$ the expectation with respect to (ε_i). The number of nonzero elements among $z_i = \delta_i x_i$, $i = 1, \ldots, m$, is $d = |\sigma| = \sum_{i=1}^m \delta_i$. Consequently, we can view z_i as vectors in $\mathbb{R}^d$. Applying Lemma 3.1 to them, we obtain

$$\left(\mathbb{E}_\varepsilon\left\| \frac{n}{k}\sum_{i=1}^m \varepsilon_i\delta_i x_i \otimes x_i \right\|^p\right)^{1/p} = \frac{n}{k}\left(\mathbb{E}_\varepsilon\left\| \sum_{i\in\sigma} \varepsilon_i z_i \otimes z_i \right\|^p\right)^{1/p}$$
$$\leqslant \frac{Cn}{k}\big(p + \log|\sigma|\big)^{1/2}\left\| \sum_{i=1}^m \delta_i x_i \otimes x_i \right\|^{1/2}.$$

By (5) and the Cauchy–Schwartz inequality, we get

$$E_p \leqslant 2\left(\mathbb{E}\mathbb{E}_\varepsilon\left\| \frac{n}{k}\sum_{i=1}^m \varepsilon_i\delta_i x_i \otimes x_i \right\|^p\right)^{1/p} \leqslant 2C\sqrt{\frac{n}{k}}\big[\mathbb{E}\big(p + \log|\sigma|\big)^p\big]^{1/2p}\left[\mathbb{E}\left\| \frac{n}{k}\sum_{i=1}^m \delta_i x_i \otimes x_i \right\|^p\right]^{1/2p}. \tag{6}$$

The first expectation in (6) is estimated by Minkowski's inequality and Corollary 4.2 as

$$\big[\mathbb{E}\big(p + \log|\sigma|\big)^p\big]^{1/2p} \leqslant \big[p + \big(\mathbb{E}\log^p|\sigma|\big)^{1/p}\big]^{1/2} \leqslant \big[p + Cp\log\mathbb{E}|\sigma|\big]^{1/2} = [p + Cp\log k]^{1/2}$$
$$\leqslant C(p\log k)^{1/2}.$$

The second expectation in (6) is estimated by Minkowski's inequality as

$$\left[\mathbb{E}\left\| \frac{n}{k}\sum_{i=1}^m \delta_i x_i \otimes x_i \right\|^p\right]^{1/2p} \leqslant (1 + E_p)^{1/2}.$$

Summarizing, (6) becomes

$$E_p^2 \leqslant Cp\left(\frac{n\log k}{k}\right)(1+E_p).$$

Denoting $a = (n\log k)/k$ and solving for E_p, we have

$$E_p \leqslant C(ap + \sqrt{ap}),$$

thus

$$\min(E_p, 1) \leqslant C\sqrt{ap}.$$

Since $E_p = (\mathbb{E}Z^p)^{1/p}$ for $Z = \|id - (n/k)\sum_{i\in\sigma} x_i \otimes x_i\|$, we have

$$\left[\mathbb{E}\big(\min(Z,1)\big)^p\right]^{1/p} \leqslant \min(E_p, 1) \leqslant C\sqrt{ap}.$$

By Corollary 4.1,

$$\mathbb{P}\big\{\min(Z,1) > C_1\sqrt{a}t\big\} \leqslant 2\exp\big(-t^2\big) \quad \text{for all } t > 0. \tag{7}$$

Now recall the restriction on k in Theorem 2.1, $k \geqslant C(n/\varepsilon^2)\log(n/\varepsilon^2)$. By choosing C large enough, we can make

$$C_1\sqrt{a} = C_1\sqrt{\frac{n\log k}{k}} \leqslant \frac{\varepsilon}{10}.$$

In view of the definition of Z, (7) implies

$$\mathbb{P}\left\{\left\|id - \frac{n}{k}\sum_{i\in\sigma} x_i \otimes x_i\right\| > \frac{\varepsilon t}{10}\right\} \leqslant 2\exp\big(-t^2\big) \quad \text{for all } 0 < t < \frac{10}{\varepsilon}. \tag{8}$$

4.4. Replacing the average size of the random set by its actual size

It remains to replace k by $|\sigma|$ in (8). Indeed, since $|\sigma| = \sum_{i=1}^m \delta_i$ is a sum of m independent $\{0,1\}$-valued random variables δ_j with $\mathbb{E}\delta_j = \delta = k/m$, Bernstein's inequality (see [11]) shows that for $s \leqslant 2\delta m = 2k$ one has

$$\text{Prob}\big\{\big||\sigma| - k\big| > s\big\} \leqslant 2\exp\left(-\frac{s^2}{8\delta m}\right) \leqslant 2\exp\left(-\frac{s^2}{8k}\right).$$

Then for $s = (\varepsilon t k)/10$,

$$\text{Prob}\left\{\left|\frac{|\sigma|}{k} - 1\right| > \frac{\varepsilon t}{10}\right\} \leqslant 2\exp\left(-\frac{\varepsilon^2 t^2 k}{800}\right) \leqslant 2\exp\big(-t^2\big).$$

If both events $||\sigma|/k - 1| \leqslant (\varepsilon t)/10$ and $\|id - (n/k)\sum_{i\in\sigma} x_i \otimes x_i\| \leqslant (\varepsilon t)/10$ hold, which happens with probability at least $1 - 4\exp(-t^2)$, then by the triangle inequality $\|(n/k)\sum_{i\in\sigma} x_i \otimes x_i\| \leqslant 1 + (\varepsilon t)/10 < 2$, hence

$$\left\|id - \frac{n}{|\sigma|}\sum_{i\in\sigma} x_i \otimes x_i\right\| \leqslant \left\|id - \frac{n}{k}\sum_{i\in\sigma} x_i \otimes x_i\right\| + \left\|\left(1 - \frac{k}{|\sigma|}\right)\frac{n}{k}\sum_{i\in\sigma} x_i \otimes x_i\right\| \leqslant \frac{\varepsilon t}{10} + \frac{4\varepsilon t}{10} < \varepsilon t.$$

Thus k may be replaced by $|\sigma|$ in (8) at the cost of replacing $(\varepsilon t)/10$ by εt. This completes the proof of Theorem 2.1. □

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