Fixed point theorems of upper semicontinuous multivalued mappings with applications in hyperconvex metric spaces

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Abstract

In the present paper, we establish two fixed point theorems for upper semicontinuous multivalued mappings in hyperconvex metric spaces and apply these to study coincidence point problems and minimax problems.

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1. Introduction and preliminaries

In 1956, Aronszajn and Panitchpakdi [1] introduced the notion of a hyperconvex metric space, and subsequently, Sine [18] and Soardi [19] independently proved that the fixed point property for nonexpansive mappings holds in bounded hyperconvex metric spaces. Since then, hyperconvex metric spaces have been widely studied and many interesting fixed point theorems for nonexpansive map-
pings have been established (see, for example, [2,3,6,7,10,11]). The corresponding linear theory is well developed and associated with the names of Gleason, Goodner, Kelley and Nachbin. For this linear theory the reader may refer to Lacey [13]. The nonlinear theory is still being developed. In particular, the study of multivalued analysis is just beginning (see [8,12,20]).

In 1999, Yuan [20] proved the following theorem, which improved on the corresponding fixed point theorems given by Horvath [5] and by Kirk and Shin [9], containing these results as special cases.

**Theorem 1.1.** Let \((M,d)\) be a hyperconvex metric space and \(X\) be a nonempty compact admissible subset of \(M\). Suppose that \(F: X \to 2^X\) is an upper semicontinuous multivalued mapping with nonempty closed admissible values. Then there is a point \(\bar{x} \in X\) such that \(\bar{x} \in F(\bar{x})\).

In the present paper, our purpose is to improve on the above theorem, and to apply our results to study coincidence point problems and minimax problems.

We begin by explaining Aronszajn and Panitchpakdi’s notion of a hyperconvex metric space and some related concepts.

Let \(N(x,r) = \{y \in M: d(x,y) < r\}\) and \(B(x,r) = \{y \in M: d(x,y) \leq r\}\).

**Definition 1.1.** A metric space \((M,d)\) is called hyperconvex if for any collection of points \(\{x_\alpha: \alpha \in I\}\) of \(M\) and any collection of nonnegative reals \(\{r_\alpha: \alpha \in I\}\) such that
\[
d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta
\]
for all \(\alpha, \beta \in I\), then
\[
\bigcap_{\alpha \in I} B(x_\alpha, r_\alpha) \neq \emptyset.
\]

**Definition 1.2.** Let \((M,d)\) be a metric space and \(A \subset M\) be a nonempty subset. Set
\[
co(A) = \bigcap \{B: B \text{ is a closed ball such that } A \subset B\}
\]
and
\[
A + r = \bigcup_{a \in A} B(a, r), \quad \forall r \geq 0.
\]
If \(A\) is an intersection of some closed balls, we will say \(A\) is an admissible subset of \(M\). \(A\) is called sub-admissible if for each finite subset \(D\) of \(A\), \(co(D) \subset A\). A function \(f: X \to R\) is said to be metric quasiconvex (or metric quasiconcave) if for each \(r \in R\), the set \(\{x \in X: f(x) \leq r\}\) (respectively, \(\{x \in X: f(x) \geq r\}\)) is sub-admissible.
Obviously, if $A$ an admissible subset of a metric space $(M, d)$, then $A$ must be sub-admissible.

Let $X$ be a topological space. We denote by $2^X$ the family of all subsets of $X$. If $A \subset X$ we shall denote by $\bar{A}$ the closure of $A$. If $X$ is a topological vector space we shall denote by $\text{conv}(A)$ and by $\text{conv}(A)$ the convex hull and the closed-convex hull of $A$, respectively.

**Definition 1.3.** Let $X, Y$ be two topological spaces and let $T : X \to 2^Y$ be a multivalued mapping. $T$ is said to be upper semicontinuous (respectively, lower semicontinuous) if for each $x \in X$ and each open set $V \subset Y$ with $T(x) \subset V$ (respectively, $T(x) \cap V \neq \emptyset$), there exists an open neighborhood $U$ of $x$ such that $T(z) \subset V$ (respectively, $T(z) \cap V \neq \emptyset$) for each $z \in U$.

A topological space is said to be acyclic if all of its reduced Čech homology groups over the rationals vanish. In particular, any contractible space is acyclic, and thus any convex or star-shaped set is acyclic.

We need the following two propositions; Proposition 1.2 appeared in Sine [17].

**Proposition 1.2.** Let $A = \bigcap_{\alpha \in I} B(x_\alpha, r_\alpha) \neq \emptyset$ in a hyperconvex metric space $(M, d)$. Then for any $\varepsilon > 0$, the set $A + \varepsilon = \bigcap_{\alpha \in I} B(x_\alpha, r_\alpha + \varepsilon)$.

**Proposition 1.3.** If $A$ is a nonempty sub-admissible subset of a hyperconvex metric space $(M, d)$, then so is $\bar{A}$.

**Proof.** Let $x_1, \ldots, x_k$ be any finite collection of points in $\bar{A}$. Since $\bar{A} = \bigcap_{\varepsilon > 0}(A + \varepsilon)$, for each $\varepsilon > 0$ there exist $a_1, \ldots, a_k \in A$ such that $d(x_n, a_n) \leq \varepsilon$ for all $n \in \{1, 2, \ldots, k\}$. Hence

$$\{x_1, \ldots, x_k\} \subset \text{co}(\{a_1, \ldots, a_k\}) + \varepsilon.$$

By Proposition 1.2

$$\text{co}(\{x_1, \ldots, x_k\}) \subset \text{co}(\{a_1, \ldots, a_k\}) + \varepsilon \subset A + \varepsilon.$$

Consequently, $\text{co}(\{x_1, \ldots, x_k\}) \subset \bar{A}$. This shows that $\bar{A}$ is sub-admissible. \qed

**Proposition 1.4.** Every compact sub-admissible subset $A$ of a hyperconvex space $M$ is admissible.

**Proof.** For any $\varepsilon > 0$, by the compactness of $A$ there is a finite subset $D$ of $A$ such that $A = \bigcup_{x \in D} N(x, \varepsilon)$. Since $A$ is sub-admissible,

$$\text{co}(D) = \bigcap_{i \in I} B(x_i, r_i) \subset A.$$
For each \( z \in A \), there is a \( y \in D \) such that \( d(z, y) < \varepsilon \). Hence
\[
d(x_i, z) \leq d(x_i, y) + d(y, z) < r_i + \varepsilon
\]
for all \( i \in I \). Therefore
\[
A \subset \bigcap_{i \in I} B(x_i, r_i + \varepsilon) = co(D) + \varepsilon,
\]
and thus
\[
co(A) \subset co(D) + \varepsilon \subset A + \varepsilon.
\]
Since \( \varepsilon > 0 \) is arbitrary we conclude \( co(A) \subset \bar{A} = A \), proving that \( A \) is admissible. \( \Box \)

2. Main results

We now present our two new fixed point theorems. Later we will use these to establish our coincidence point and minimax results.

**Theorem 2.1.** Let \((M, d)\) be a hyperconvex metric space and \(X\) be a nonempty sub-admissible subset of \(M\). Suppose that \(F\) is an upper semicontinuous multivalued mapping with closed acyclic values from \(X\) into a compact subset \(Z\) of \(X\). Then there is a point \(\bar{x} \in Z\) such that \(\bar{x} \in F(\bar{x})\).

**Proof.** Step 1. For each \(\varepsilon > 0\) there are a compact convex set \(P\) and continuous mappings \(r: P \to X\) and \(g: Z \to P\) such that \(d(z, r[g(z)]) \leq \varepsilon\) for all \(z \in Z\).

Indeed, since \((M, d)\) be a hyperconvex metric space, by conclusion 1 of Proposition 1 in [8] there exists an index set \(I\) and an isometric embedding from \(M\) into \(l^\infty(I)\). We will identify \(M\) with the isometric embedding image set in \(l^\infty(I)\). Since hyperconvexity is preserved by isometry, by conclusion 4 of Proposition 1 in [8], there exists a nonexpansive retract \(r : l^\infty(I) \to M\).

For each \(\varepsilon > 0\) and each \(x \in Z\), let \(G(x)\) be the intersection of all sub-admissible subsets of \(M\) containing \(B(x, \varepsilon) \cap Z\). Then \(G : Z \to 2^X\) is a multivalued mapping with nonempty sub-admissible values. For each \(y \in X\),
\[
G^{-1}(y) = \{x \in Z: y \in G(x)\} \supset \{x \in Z: y \in N(x, \varepsilon) \cap Z\}.
\]

Obviously, the set,
\[
O_y = \begin{cases} 
N(y, \varepsilon) \cap Z, & \text{if } y \in Z, \\
\emptyset, & \text{if } y \notin Z,
\end{cases}
\]
is open in \(Z\), \(Z = \bigcup_{y \in X} O_y\) and \(O_y \subset G^{-1}(y)\).

Since \(Z\) is compact, there exists a finite subset \(\{y_1, \ldots, y_n\} \subset X\) such that \(Z = \bigcup_{k=1}^n O_{y_k}\). Let \(\{f_k: k = 1, 2, \ldots, n\}\) be a continuous partition of unity
corresponding to the covering \{O_{y_k}: k = 1, 2, \ldots, n\} of Z. Thus we can define a mapping \(g: Z \to l^\infty(I)\) by

\[
g(x) = \sum_{k=1}^{n} f_k(x) y_k, \quad \forall x \in Z.
\]

Then \(g(Z) \subset \text{conv}\{y_1, \ldots, y_n\} =: P \subset l^\infty(I)\). Now let \(f = r \circ g\). Then \(f: Z \to M\) is continuous. For each \(x \in Z\), let \(\sigma(x) = \{k \in \{1, \ldots, n\}: f_k(x) \neq 0\}\). Then

\[
f(x) = r\left[ g(x) \right] = r\left[ \sum_{k \in \sigma(x)} f_k(x) y_k \right] \in r\left( \text{conv}\{y_k: k \in \sigma(x)\} \right)
\]

and for every \(z \in Z\),

\[
d(z, r\left[ g(z) \right]) = d(r(z), r\left[ g(z) \right]) \leq d(z, g(z)) \leq \varepsilon,
\]

where \(\text{conv}(A)\) is the convex hull of \(A\) in \(l^\infty(I)\).

**Step 2.** There is a point \(\bar{x} \in Z\) such that \(\bar{x} \in F(\bar{x})\).

In fact, let \(S(u) = F[r(u)]\). Then \(S: P \to 2^Z\) is an upper semicontinuous multivalued mapping with compact acyclic values. By Lemma 2.1 in [15], there exists a point \(u \in P\) such that \(u \in g[S(u)]\). Consequently, there is a point \(x_\varepsilon \in S(u) = F[r(u)]\) such that \(u = g(x_\varepsilon)\). Let \(y_\varepsilon = r(u) = r[g(x_\varepsilon)] = f(x_\varepsilon)\).

Notice that \(G(x) \subset X\) for all \(x \in Z\) and that \(f\) is a continuous selection of \(G\). Then \(y_\varepsilon \in G(x_\varepsilon)\) and \(x_\varepsilon \in F(y_\varepsilon)\). Hence \((y_\varepsilon, x_\varepsilon) \in \text{Gr}(F)\) and \(d(y_\varepsilon, x_\varepsilon) < \varepsilon\).

Since \(Z\) is compact, we may assume that \(x_\varepsilon \to \bar{x}\) as \(\varepsilon \to 0\). Consequently, \(y_\varepsilon \to \bar{x}\) as \(\varepsilon \to 0\). Since \(F: X \to 2^Z\) is an upper semicontinuous multivalued mapping with nonempty compact values, \(\bar{x} \in F(\bar{x})\). \(\Box\)

**Theorem 2.2.** Let \((M, d)\) be a hyperconvex metric space and \(X\) be a nonempty closed sub-admissible subset of \(M\). Suppose that \(F\) is an upper semicontinuous multivalued mapping with nonempty closed sub-admissible values from \(X\) into a compact subset \(Z\) of \(X\). Then there is a point \(\bar{x} \in Z\) such that \(\bar{x} \in F(\bar{x})\).

First, we prove the following lemma.

**Lemma.** If \(A\) is a nonempty closed sub-admissible subset of a hyperconvex metric space \((M, d)\) and \(r: l^\infty(I) \to M\) is a nonexpansive retract as in the proof of Theorem 2.1, then \(r(y) \in A\) for all \(y \in \text{conv}(A)\).

**Proof.** For each \(y \in \text{conv}(A)\), there is a sequence \(\{y_n\}\) in \(\text{conv}(A)\) such that \(y_n \to y\) so that \(r(y_n) \to r(y)\). Let \(y_n = \sum_{i=1}^{k} \lambda_i a_i\), where \(\{a_1, a_2, \ldots, a_k\} \subset A\), every \(\lambda_i \geq 0\) and \(\sum_{i=1}^{k} \lambda_i = 1\). Then every closed ball \(B(a, \rho)\) in \(l^\infty(I)\) with
center in $A$ containing $\{a_1, a_2, \ldots, a_k\}$ contains $y_n$ too since it is convex. And since
\[
\|r(y_n) - a\| \leq \|y_n - a\| \leq \rho, \quad r(y_n) \in B(a, \rho).
\]
Thus $r(y_n) \in \text{co}((a_1, a_2, \ldots, a_k)) \subset A$ since $A$ is convex. And $r(y) \in A$.

**Proof of Theorem 2.2.** Define $r$ as in the proof of Theorem 2.1. Let $X_1 = \text{conv}(X)$ and $Z_1 = \text{conv}(Z)$. Then $X_1$ is a closed convex subset of $l^\infty(I)$ and $Z_1$ is a compact convex subset of $X_1$. For each $x \in X_1$, let $F_1(x) = \text{conv}(F[r(x)])$. Since $X$ is a nonempty closed sub-admissible subset of $M$, $F_1 : X_1 \to 2^{Z_1}$ is a well-defined upper semicontinuous multivalued mapping with nonempty compact convex values by Lemma 2 in [16].

By the Himmelberg’s fixed point theorem (see [4]) there exists a point $\tilde{y} \in Z_1$ such that $\tilde{y} \in F_1(\tilde{y})$. Consequently, there exists a sequence $\{y_n\}$ in $\text{conv}(F[r(\tilde{y})])$ such that $y_n \to \tilde{y}$. Let $\tilde{x} = r(\tilde{y})$. Then
\[
\tilde{x} = \lim_{n \to \infty} r(y_n) \in F[r(\tilde{y})] = F(\tilde{x}) \subset Z
\]
by the above Lemma. This completes the proof.

**Remark.** Theorems 2.1 and 2.2 improve on Theorem 1.1.

As an application of our fixed point theorems we have the following two coincidence point results.

**Theorem 2.3.** Let $X$ be a nonempty closed sub-admissible subset of a hyperconvex metric space $(M, d)$, let $Y$ be a nonempty subset of a hyperconvex metric space $(N, p)$, and let $Z \subset X$ be a nonempty compact subset. Suppose that $F : X \to 2^Y$ is a lower semicontinuous multivalued mapping with nonempty closed sub-admissible values and that $T : Y \to 2^Z$ is an upper semicontinuous multivalued mapping with nonempty closed sub-admissible values. Then there is a point $\tilde{x} \in Z$ and a point $\tilde{y} \in Y$ such that $\tilde{x} \in T(\tilde{y})$ and $\tilde{y} \in F(\tilde{x})$.

**Proof.** By Proposition 1 in [8] there exists an index set $I$ and an isometric embedding from $N$ into $l^\infty(I)$ and a nonexpansive retract $r : l^\infty(I) \to N$ (we identify $N$ with the isometric embedding image set in $l^\infty(I)$). For each $x \in X$, let $F_1(x) = \text{conv}(F(x))$. By Propositions 2.3 and 2.6 in [14], it follows that $F_1 : X \to 2^{l^\infty(I)}$ is a lower semicontinuous multivalued mapping with nonempty closed convex values. By virtue of Theorem 3.2” in [14] there exists a continuous mapping $f_1 : X \to l^\infty(I)$ such that $f_1(x) \in F_1(x)$ for all $x \in X$. Let $f = r \circ f_1$. Then the mapping $f : X \to N$ is continuous. For each $x \in X$, since $f_1(x) \in F_1(x)$ and $F(x)$ is a nonempty closed sub-admissible subset of $(N, p)$, $f(x) \in F(x) \subset Y$. Hence $T \circ f : X \to 2^Z$ is an upper semicontinuous
multivalued mapping with nonempty closed sub-admissible values. Consequently, by Theorem 2.2 there exists a point \( \bar{x} \in Z \) such that \( \bar{x} \in T[f(\bar{x})] \). Let \( \bar{y} = f(\bar{x}) \). Then \( \bar{y} \in F(\bar{x}) \) and \( \bar{x} \in T(\bar{y}) \). □

As an application of Theorem 2.1 we have the following result. The proof is similar to that of Theorem 2.3.

**Theorem 2.4.** Let \( X \) be a nonempty sub-admissible subset of a hyperconvex metric space \((M, d)\), let \( Y \) be a nonempty subset of a hyperconvex metric space \((N, p)\), and let \( Z \subset X \) be a nonempty compact subset. Suppose that \( F : X \to 2^Y \) is a lower semicontinuous multivalued mapping with nonempty closed sub-admissible values and that \( T : Y \to 2^Z \) is an upper semicontinuous multivalued mapping with closed acyclic values. Then there is a point \( \bar{x} \in Z \) and a point \( \bar{y} \in Y \) such that \( \bar{x} \in T(\bar{y}) \) and \( \bar{y} \in F(\bar{x}) \).

As applications of our fixed point theorem and coincidence point theorem, we have the following two minimax results.

**Theorem 2.5.** Let \((M, d)\) be a hyperconvex metric space, let \( Y \) be a nonempty closed sub-admissible subset of \( M \), and let \( X \) be a compact subset of \( Y \). Suppose that \( \phi : X \times Y \to R \) is an upper semicontinuous function such that for each \( y \in Y \), \( \phi(x, y) \) is metric quasiconcave in \( x \), then

\[
\inf_{y \in Y} \sup_{x \in X} \phi(x, y) \leq \sup_{x \in X} \phi(x, x).
\]

If \( X = Y \) is a compact admissible subset of \( M \) and in addition, \( \phi(x, \cdot) \) is lower semicontinuous on \( X \) for each \( x \in X \), then there exists a point \( \bar{y} \in X \) such that

\[
\sup_{x \in X} \phi(x, \bar{y}) \leq \sup_{x \in X} \phi(x, x).
\]

**Proof.** Let \( \lambda = \sup_{x \in X} \phi(x, x) \). Then \( \lambda < +\infty \) by the compactness of \( X \) and the upper continuity of \( \phi \). If

\[
\inf_{y \in Y} \sup_{x \in X} \phi(x, y) > \lambda,
\]

we may take a real number \( r \in R \) such that

\[
\inf_{y \in Y} \sup_{x \in X} \phi(x, y) > r > \lambda.
\]

Define a multivalued mapping \( T : Y \to 2^X \) by

\[
T(y) = \{ x \in X : \phi(x, y) \geq r \}, \quad \forall y \in Y.
\]

Then \( T(y) \neq \emptyset \) for all \( y \in Y \). By the upper semicontinuity of \( \phi \), \( T \) has a closed graph in \( Y \times X \) so that \( T \) is upper semicontinuous since \( X \) is compact. For
each \( y \in X \), since \( \phi(x, y) \) is metric quasiconcave in \( x \), \( T(y) \) is sub-admissible. By Theorem 2.2 there exists a point \( y_0 \in X \) such that \( y_0 \in T(y_0) \), that is, \( \phi(y_0, y_0) \geq r > \lambda = \sup_{x \in X} \phi(x, x) \). From this contradiction it follows that

\[
\inf_{y \in Y} \sup_{x \in X} \phi(x, y) \leq \lambda = \sup_{x \in X} \phi(x, x).
\]

If \( X = Y \) is a compact admissible subset of \( M \) and in addition, \( \phi(x, \cdot) \) is lower semicontinuous on \( X \) for each \( x \in X \), then the function \( g(y) = \sup_{x \in X} \phi(x, y) \) is also lower semicontinuous on \( X \), and hence there exists a point \( \bar{y} \in X \) such that

\[
\sup_{x \in X} \phi(x, \bar{y}) = \inf_{y \in X} \sup_{x \in X} \phi(x, y) \leq \sup_{x \in X} \phi(x, x). \quad \square
\]

**Theorem 2.6.** Let \( X \) be a nonempty compact admissible subset of a hyperconvex metric space \((M, d)\) and let \( Y \) be a nonempty closed sub-admissible subset of a hyperconvex metric space \((N, p)\). Suppose the functions \( f, g : X \times Y \to \mathbb{R} \) are such that

(i) \( f \) is upper semicontinuous and \( f(x, y) \leq g(x, y) \) for all \((x, y) \in X \times Y\),

(ii) for each \( x \in X \), \( g(x, \cdot) \) is lower semicontinuous and metric quasiconvex, and

(iii) for each \( y \in Y \), \( g(\cdot, y) \) is upper semicontinuous and \( f(\cdot, y) \) metric quasiconcave.

Then

\[
\inf_{y \in Y} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} \inf_{y \in Y} g(x, y).
\]

**Proof.** If the conclusion is false, then there exist real numbers \( \alpha \) and \( \beta \) such that

\[
\inf_{y \in Y} \sup_{x \in X} f(x, y) > \alpha > \beta > \sup_{x \in X} \inf_{y \in Y} g(x, y).
\]

For each \( x \in X \), let

\[
G(x) = \{ y \in Y : g(x, y) < \beta \} \quad \text{and} \quad F(x) = \{ y \in Y : g(x, y) \leq \beta \}.
\]

For each \( y \in Y \), let

\[
T(y) = \{ x \in X : f(x, y) \geq \alpha \}.
\]

Then \( F : X \to 2^Y \) and \( T : Y \to 2^X \) are multivalued mappings with nonempty closed sub-admissible values. From the upper semicontinuity of \( f \) and the compactness of \( X \) it follows that \( T \) is upper semicontinuous. Since \( g(\cdot, y) \) is upper semicontinuous it follows that \( G^{-1}(y) = \{ x \in X : g(x, y) < \beta \} \) is open in \( X \) for each \( y \in Y \). Hence \( G : X \to 2^Y \) is lower semicontinuous. From Proposition 2.3 in [14] it follows that the multivalued mapping \( \overline{G} : X \to 2^Y \) defined by \( \overline{G}(x) = \overline{G(x)} \) and
is also lower semicontinuous. Since $g(x, \cdot)$ is metric quasiconvex, $G(x)$ is a subadmissible subset of $Y$ for each $x \in X$. Hence $G(x)$ is a subadmissible subset of $Y$ for each $x \in X$ by Proposition 1.3. By Theorem 2.3 there is a point $\bar{x} \in X$ and a point $\bar{y} \in Y$ such that $\bar{x} \in T(\bar{y})$ and $\bar{y} \in G(\bar{x}) \subset F(\bar{x})$. Hence
\[ g(\bar{x}, \bar{y}) \leq \beta < \alpha \leq f(\bar{x}, \bar{y}). \]
This contradicts the assumption (i). The result follows. \[\square\]

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