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The Heyneman–Radford Theorem for monoidal categories ☆

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Abstract

We prove Heyneman–Radford Theorem in the framework of monoidal categories.

Keywords: Monoidal categories; Wedge products; Colimits

Introduction

This paper is devoted to the proof of the Heyneman–Radford Theorem for monoidal categories. The original Heyneman–Radford's Theorem (see [HR, Proposition 2.4.2] or [Mo, Theorem 5.3.1, p. 65]) is a very useful tool in classical Hopf algebra theory. We also point out that our proof is pretty different from the classical one and hence might be of some interest even in the classical case.

Notations

A subobject of an object E in an abelian category $\mathcal{M}$, is an equivalence class of monomorphisms into E. By abuse of language, given a monomorphism $i_X = i_X^E : X \hookrightarrow E$, we will say that (X, i_X) is a subobject of E and we will write $(X, i_X) = (Y, i_Y)$ to mean that i_X and i_Y are

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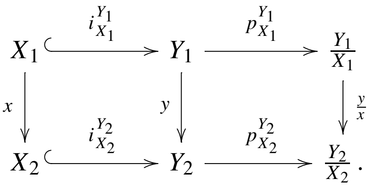
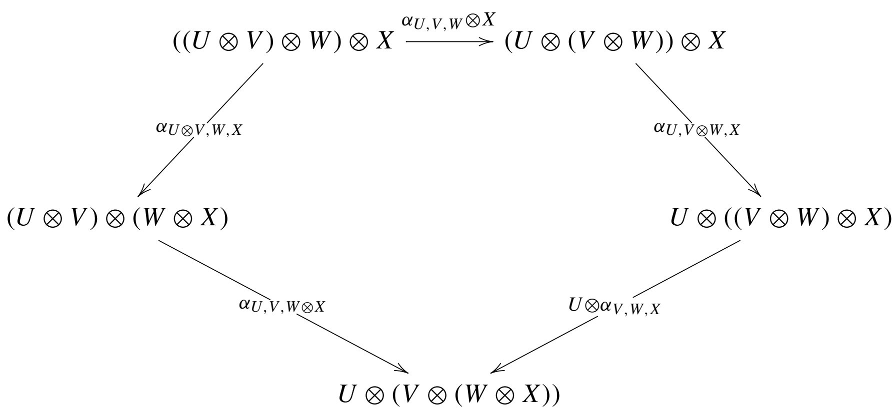
equivalent monomorphisms. The same convention applies to cokernels. If (X, i_X) is a subobject of E then we will write $(E/X, p_X) = \mathrm{Coker}(i_X)$, where $p_X = p_X^E : E \to E/X$.

Let $(X_1, i_{X_1}^{Y_1})$ be a subobject of Y_1 and let $(X_2, i_{X_2}^{Y_2})$ be a subobject of Y_2. Let $x : X_1 \to X_2$ and $y : Y_1 \to Y_2$ be morphisms such that $y \circ i_{X_1}^{Y_1} = i_{X_2}^{Y_2} \circ x$. Then there exists a unique morphism, which we denote by $y/x = \frac{y}{x} : Y_1/X_1 \to Y_2/X_2$, such that $\frac{y}{x} \circ p_{X_1}^{Y_1} = p_{X_2}^{Y_2} \circ y$:

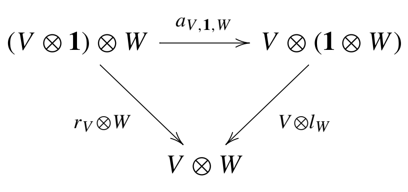


1. Wedge products in monoidal categories

1.1. Let us recall that a *monoidal category* is a category $\mathcal{M}$ that is endowed with a functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, an object $\mathbf{1} \in \mathcal{M}$ and functorial isomorphisms: $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $l_X : \mathbf{1} \otimes X \to X$ and $r_X : X \otimes \mathbf{1} \to X$. The functorial morphism a is called the *associativity constraint* and satisfies the *Pentagon Axiom*, that is the following diagram



is commutative for every U, V, W, X in $\mathcal{M}$. The morphisms l and r are called the *unit constraints* and they are assumed to satisfy the *Triangle Axiom*, i.e. the following diagram



is commutative for every V, W in $\mathcal{M}$. The object $\mathbf{1}$ is called the *unit* of $\mathcal{M}$. For details on monoidal categories we refer to [Ka, Chapter XI] and [Maj]. A monoidal category is called *strict* if the associativity constraint and unit constraints are the corresponding identity morphisms.

As it is noticed in [Maj, p. 420], the Pentagon Axiom solves the consistency problem that appears because there are two ways to go from $((U \otimes V) \otimes W) \otimes X$ to $U \otimes (V \otimes (W \otimes X))$. The coherence theorem, due to S. Mac Lane, solves the similar problem for the tensor product of an arbitrary number of objects in $\mathcal{M}$. Accordingly with this theorem, we can always omit all brackets and simply write $X_1 \otimes \cdots \otimes X_n$ for any object obtained from $X_1, \ldots, X_n$ by using $\otimes$ and brackets. Also as a consequence of the coherence theorem, the morphisms a, l, r take care of themselves, so they can be omitted in any computation involving morphisms in $\mathcal{M}$.

The notions of algebra, module over an algebra, coalgebra and comodule over a coalgebra can be introduced in the general setting of monoidal categories. For more details, see [AMS2].

We quote the following definition from [AMS1, 2.4]. We remark that, in this context, we prefer to use the word "coabelian" instead of "abelian."

Definition 1.2. A monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$ will be called a ***coabelian monoidal category*** if:

(1) $\mathcal{M}$ is an abelian category;
(2) both the functors $X \otimes (-) : \mathcal{M} \to \mathcal{M}$ and $(-) \otimes X : \mathcal{M} \to \mathcal{M}$ are additive and left exact for every object $X \in \mathcal{M}$.

Let E be a coalgebra in a coabelian monoidal category $\mathcal{M}$.

Let us recall (see [Mo, p. 60]) the definition of wedge of two subobjects X, Y of E in $\mathcal{M}$:

$$\big(X \wedge_E Y, i^E_{X \wedge_E Y}\big) := \mathrm{Ker}\big[(p_X \otimes p_Y) \circ \Delta_E\big],$$

where $p_X : E \to E/X$ and $p_Y : E \to E/Y$ are the canonical quotient maps. In particular we have the following exact sequence:

$$0 \longrightarrow X \wedge_E Y \xrightarrow{i^E_{X \wedge_E Y}} E \xrightarrow{(p_X \otimes p_Y) \circ \Delta_E} E/X \otimes E/Y.$$

Consider the following commutative diagrams in $\mathcal{M}$

$$\begin{array}{ccc} X_1 & \xhookrightarrow{i^{E_1}_{X_1}} & E_1 \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle e} \\ X_2 & \xhookrightarrow[i^{E_2}_{X_2}]{} & E_2, \end{array} \qquad \begin{array}{ccc} Y_1 & \xhookrightarrow{i^{E_1}_{Y_1}} & E_1 \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle e} \\ Y_2 & \xhookrightarrow[i^{E_2}_{Y_2}]{} & E_2, \end{array}$$

where e is a coalgebra homomorphism. Now we have

$$\begin{aligned} &\big(p^{E_2}_{X_2} \otimes p^{E_2}_{Y_2}\big) \circ \Delta_{E_2} \circ e \circ i^{E_1}_{X_1 \wedge_{E_1} Y_1} \\ &\quad = \big(p^{E_2}_{X_2} \otimes p^{E_2}_{Y_2}\big) \circ (e \otimes e) \circ \Delta_{E_1} \circ i^{E_1}_{X_1 \wedge_{E_1} Y_1} \\ &\quad = \left(\frac{e}{x} \otimes \frac{e}{y}\right) \circ \big(p^{E_1}_{X_1} \otimes p^{E_1}_{Y_1}\big) \circ \Delta_{E_1} \circ i^{E_1}_{X_1 \wedge_{E_1} Y_1} = 0 \end{aligned}$$

so that, being $(X_2 \wedge_{E_2} Y_2, i^{E_2}_{X_2 \wedge_{E_2} Y_2})$ the kernel of $(p^{E_2}_{X_2} \otimes p^{E_2}_{Y_2}) \circ \Delta_{E_2}$, there is a unique morphism

$$x \wedge_e y : X_1 \wedge_{E_1} Y_1 \to X_2 \wedge_{E_2} Y_2$$

such that the following diagram

$$\begin{array}{ccc} X_1 \wedge_{E_1} Y_1 & \xrightarrow{i^{E_1}_{X_1 \wedge_{E_1} Y_1}} & E_1 \\ {\scriptstyle x \wedge_e y} \downarrow & & \downarrow {\scriptstyle e} \\ X_2 \wedge_{E_2} Y_2 & \xrightarrow[i^{E_2}_{X_2 \wedge_{E_2} Y_2}]{} & E_2 \end{array}$$

commutes.

Lemma 1.3. *Consider the following commutative diagrams in $\mathcal{M}$*

$$\begin{array}{ccc} X_1 & \xrightarrow{i^{E_1}_{X_1}} & E_1 \\ {\scriptstyle x} \downarrow & & \downarrow {\scriptstyle e} \\ X_2 & \xrightarrow[i^{E_2}_{X_2}]{} & E_2 \\ {\scriptstyle x'} \downarrow & & \downarrow {\scriptstyle e'} \\ X_3 & \xrightarrow[i^{E_3}_{X_3}]{} & E_3, \end{array} \qquad \begin{array}{ccc} Y_1 & \xrightarrow{i^{E_1}_{Y_1}} & E_1 \\ {\scriptstyle y} \downarrow & & \downarrow {\scriptstyle e} \\ Y_2 & \xrightarrow[i^{E_2}_{Y_2}]{} & E_2 \\ {\scriptstyle y'} \downarrow & & \downarrow {\scriptstyle e'} \\ Y_3 & \xrightarrow[i^{E_3}_{Y_3}]{} & E_3, \end{array}$$

where e and e' are coalgebra homomorphisms. Then we have

$$(x' \wedge_{e'} y') \circ (x \wedge_e y) = (x'x \wedge_{e'e} y'y). \tag{1}$$

Proof. Straightforward. □

We now recall some definitions and some (standard) results established in [AMS1].

1.4. Let X be an object in a coabelian monoidal category $(\mathcal{M}, \otimes, \mathbf{1})$. Set

$$X^{\otimes 0} = \mathbf{1}, \qquad X^{\otimes 1} = X \quad \text{and} \quad X^{\otimes n} = X^{\otimes n-1} \otimes X, \quad \text{for every } n > 1,$$

and for every morphism $f : X \to Y$ in $\mathcal{M}$, set

$$f^{\otimes 0} = \mathrm{Id}_{\mathbf{1}}, \qquad f^{\otimes 1} = f \quad \text{and} \quad f^{\otimes n} = f^{\otimes n-1} \otimes f, \quad \text{for every } n > 1.$$

Let $(C, \Delta_C, \varepsilon_C)$ be a coalgebra in $\mathcal{M}$ and for every $n \in \mathbb{N}$, define the nth iterated comultiplication of C,

$$\Delta_C^n : C \to C^{\otimes n+1},$$

by

$$\Delta_C^0 = \mathrm{Id}_C, \qquad \Delta_C^1 = \Delta_C \quad \text{and} \quad \Delta_C^n = \left(\Delta_C^{\otimes n-1} \otimes C\right)\Delta_C, \quad \text{for every } n > 1.$$

Let $\delta : D \to E$ be a monomorphism which is a homomorphism of coalgebras in $\mathcal{M}$. Denote by (L, p) the cokernel of δ in $\mathcal{M}$. Regard D as a E-bicomodule via δ and observe that L is a E-bicomodule and p is a morphism of bicomodules. Let

$$\left(D^{\wedge_E^n}, \delta_n\right) := \ker\left(p^{\otimes n} \Delta_E^{n-1}\right)$$

for any $n \in \mathbb{N} \setminus \{0\}$. Note that $(D^{\wedge_E^1}, \delta_1) = (D, \delta)$ and $(D^{\wedge_E^2}, \delta_2) = D \wedge_E D$.

In order to simplify the notations we set $(D^{\wedge_E^0}, \delta_0) = (0, 0)$.

Now, since $\mathcal{M}$ has left exact tensor functors and since $p^{\otimes n} \Delta_E^{n-1}$ is a morphism of E-bicomodules (as a composition of morphisms of E-bicomodules), we get that $D^{\wedge_E^n}$ is a coalgebra and $\delta_n : D^{\wedge_E^n} \to E$ is a coalgebra homomorphism for any $n > 0$ and hence for any $n \in \mathbb{N}$.

Proposition 1.5. [AMS1, Proposition 1.10] *Let $\delta : D \to E$ be a monomorphism which is a coalgebra homomorphism in a coabelian monoidal category $\mathcal{M}$. Then, for any $i \leqslant j$ in $\mathbb{N}$, there is a (unique) morphism $\xi_i^j : D^{\wedge_E^i} \to D^{\wedge_E^j}$ such that*

$$\delta_j \xi_i^j = \delta_i. \tag{2}$$

Moreover ξ_i^j is a coalgebra homomorphism and $((D^{\wedge_E^i})_{i\in\mathbb{N}}, (\xi_i^j)_{i,j\in\mathbb{N}})$ is a direct system in $\mathcal{M}$ whose direct limit, if it exists, carries a natural coalgebra structure that makes it the direct limit of $((D^{\wedge_E^i})_{i\in\mathbb{N}}, (\xi_i^j)_{i,j\in\mathbb{N}})$ as a direct system of coalgebras.

Proposition 1.6. [AMS1, Proposition 2.17] *Let $\delta : D \to E$ be a monomorphism which is a coalgebra homomorphism in a coabelian monoidal category $\mathcal{M}$. Then we have*

$$\left(D^{\wedge_E^m} \wedge_E D^{\wedge_E^n}, i^E_{D^{\wedge_E^m} \wedge_E D^{\wedge_E^n}}\right) = \left(D^{\wedge_E^{m+n}}, i^E_{D^{\wedge_E^{m+n}}}\right). \tag{3}$$

Notation 1.7. *Let $\delta : D \to E$ be a morphism of coalgebras in a cocomplete coabelian monoidal category $\mathcal{M}$ with left exact tensor functors. By Proposition* 1.5 *$((D^{\wedge_E^i})_{i\in\mathbb{N}}, (\xi_i^j)_{i,j\in\mathbb{N}})$ is a direct system in $\mathcal{M}$ whose direct limit carries a natural coalgebra structure that makes it the direct limit of $((D^{\wedge_E^i})_{i\in\mathbb{N}}, (\xi_i^j)_{i,j\in\mathbb{N}})$ as a direct system of coalgebras.*

From now on we will use the following notation

$$D^n := D^{\wedge_E n}, \quad \textit{for every } n \in \mathbb{N},$$

$$\left(\widetilde{D}_E, (\xi_i)_{i\in\mathbb{N}}\right) := \varinjlim \left(D^{\wedge_E^i}\right)_{i\in\mathbb{N}},$$

where $\xi_i : D^{\wedge_E^i} \to \widetilde{D}_E$ denotes the structural morphism of the direct limit. We note that, since $\widetilde{D}_E$ is a direct limit of coalgebras, the canonical (coalgebra) homomorphisms $(\delta_i : D^{\wedge_E^i} \to E)_{i\in\mathbb{N}}$, which are compatible by (2), factorize to a unique coalgebra homomorphism

$$\tilde{\delta} : \widetilde{D}_E \to E$$

such that $\tilde{\delta}\xi_i = \delta_i$, for any $i \in \mathbb{N}$.

2. The Heyneman–Radford Theorem for monoidal categories

Definition 2.1. Let E be a coalgebra and let $\delta : X \to E$ be a monomorphism in a coabelian monoidal category $\mathcal{M}$. Define the morphism

$$\alpha_X^E : E \to \frac{E}{X} \otimes \frac{E}{X}$$

by setting

$$\alpha_X^E = \left(p_X^E \otimes p_X^E\right) \circ \Delta_E.$$

Observe that $(X \wedge_E X, i_{X\wedge_E X}^E) = \operatorname{Ker}(\alpha_X^E)$.

Lemma 2.2. *Let $\delta : D \to E$ and let $f : E \to C$ be coalgebra homomorphisms in a coabelian monoidal category $\mathcal{M}$. Assume that both δ and $f \circ \delta$ are monomorphism. Then the following diagram*

$$\begin{array}{ccc} E & \xrightarrow{\quad f \quad} & C \\ \downarrow{\scriptstyle \alpha_D^E} & & \downarrow{\scriptstyle \alpha_D^C} \\ \frac{E}{D} \otimes \frac{E}{D} & \xrightarrow{\quad \frac{f}{D} \otimes \frac{f}{D} \quad} & \frac{C}{D} \otimes \frac{C}{D} \end{array}$$

is commutative.

Proof. Note that the notations E/D and C/D make sense as both δ and $f \circ \delta$ are monomorphisms. We have

$$\begin{aligned} \left(\frac{f}{D} \otimes \frac{f}{D}\right) \circ \alpha_D^E &= \left(\frac{f}{D} \otimes \frac{f}{D}\right) \circ \left(p_D^E \otimes p_D^E\right) \circ \Delta_E \\ &= \left(p_D^C \otimes p_D^C\right) \circ (f \otimes f) \circ \Delta_E \\ &= \left(p_D^C \otimes p_D^C\right) \circ \Delta_C \circ f = \alpha_D^C \circ f. \qquad \square \end{aligned}$$

Lemma 2.3. *Let D and E be coalgebras in a coabelian monoidal category $\mathcal{M}$. Let $\delta : D \to E$ be a monomorphism which is a coalgebra homomorphism in $\mathcal{M}$. Then, for every $n \in \mathbb{N}$, there exists a unique morphism $\tau_n : D^{n+1} \to D^n/D \otimes D^n/D$ such that the following diagram*

$$\begin{array}{ccc} & & D^{n+1} \\ & \overset{\tau_n}{\swarrow} & \Big\downarrow \alpha_D^{D^{n+1}} \\ \frac{D^n}{D} \otimes \frac{D^n}{D} & \xrightarrow[\frac{\xi_n^{n+1}}{D} \otimes \frac{\xi_n^{n+1}}{D}]{} & \frac{D^{n+1}}{D} \otimes \frac{D^{n+1}}{D} \end{array}$$

is commutative.

Proof. Consider the following exact sequence

$$0 \longrightarrow \frac{D^n}{D} \xrightarrow{\frac{\xi_n^{n+1}}{D}} \frac{D^{n+1}}{D} \xrightarrow{\frac{D^{n+1}}{\xi_1^n}} \frac{D^{n+1}}{D^n}. \tag{4}$$

By applying the functor $D^{n+1}/D \otimes (-)$ we get

$$0 \longrightarrow \frac{D^{n+1}}{D} \otimes \frac{D^n}{D} \xrightarrow{\frac{D^{n+1}}{D} \otimes \frac{\xi_n^{n+1}}{D}} \frac{D^{n+1}}{D} \otimes \frac{D^{n+1}}{D} \xrightarrow{\frac{D^{n+1}}{D} \otimes \frac{D^{n+1}}{\xi_1^n}} \frac{D^{n+1}}{D} \otimes \frac{D^{n+1}}{D^n}.$$

We have

$$\begin{aligned} &\left(\frac{\delta_{n+1}}{D} \otimes \frac{\delta_{n+1}}{D^n}\right) \circ \left(\frac{D^{n+1}}{D} \otimes \frac{D^{n+1}}{\xi_1^n}\right) \circ \alpha_D^{D^{n+1}} \\ &= \left(\frac{\delta_{n+1}}{D} \otimes \frac{\delta_{n+1}}{D^n}\right) \circ \left(\frac{D^{n+1}}{D} \otimes \frac{D^{n+1}}{\xi_1^n}\right) \circ \left(p_D^{D^{n+1}} \otimes p_D^{D^{n+1}}\right) \circ \Delta_{D^{n+1}} \\ &= \left(\frac{\delta_{n+1}}{D} \otimes \frac{\delta_{n+1}}{D^n}\right) \circ \left(p_D^{D^{n+1}} \otimes p_{D^n}^{D^{n+1}}\right) \circ \Delta_{D^{n+1}} \\ &= \left(p_D^E \otimes p_{D^n}^E\right) \circ (\delta_{n+1} \otimes \delta_{n+1}) \circ \Delta_{D^{n+1}} \\ &= \left(p_D^E \otimes p_{D^n}^E\right) \circ \Delta_E \circ \delta_{n+1} = 0. \end{aligned}$$

In fact, by Proposition 1.6, $D^{n+1} = D \wedge_E D^n$. Since $\frac{\delta_{n+1}}{D} \otimes \frac{\delta_{n+1}}{D^n}$ is a monomorphism, we obtain

$$\left(\frac{D^{n+1}}{D} \otimes \frac{D^{n+1}}{\xi_1^n}\right) \circ \alpha_D^{D^{n+1}} = 0 \tag{5}$$

so that, as the above sequence is exact, by the universal property of kernels, there exists a unique morphism

$$\beta_n : D^{n+1} \to \frac{D^{n+1}}{D} \otimes \frac{D^n}{D}$$

such that

$$\left(\frac{D^{n+1}}{D}\otimes\frac{\xi_n^{n+1}}{D}\right)\circ\beta_n=\alpha_D^{D^{n+1}}. \tag{6}$$

By applying the functor $(-)\otimes D^n/D$ to (4), we get

$$0\longrightarrow\frac{D^n}{D}\otimes\frac{D^n}{D}\xrightarrow{\frac{\xi_n^{n+1}}{D}\otimes\frac{D^n}{D}}\frac{D^{n+1}}{D}\otimes\frac{D^n}{D}\xrightarrow{\frac{D^{n+1}}{\xi_1^n}\otimes\frac{D^n}{D}}\frac{D^{n+1}}{D^n}\otimes\frac{D^n}{D}.$$

We have

$$\begin{aligned}&\left(\frac{D^{n+1}}{D^n}\otimes\frac{\xi_n^{n+1}}{D}\right)\circ\left(\frac{D^{n+1}}{\xi_1^n}\otimes\frac{D^n}{D}\right)\circ\beta_n\\&=\left(\frac{D^{n+1}}{\xi_1^n}\otimes\frac{D^{n+1}}{D}\right)\circ\left(\frac{D^{n+1}}{D}\otimes\frac{\xi_n^{n+1}}{D}\right)\circ\beta_n\\&\overset{(6)}{=}\left(\frac{D^{n+1}}{\xi_1^n}\otimes\frac{D^{n+1}}{D}\right)\circ\alpha_D^{D^{n+1}}=0,\end{aligned}$$

where the last equality can be proved similarly to (5). Since $\frac{D^{n+1}}{D^n}\otimes\frac{\xi_n^{n+1}}{D}$ is a monomorphism we get

$$\left(\frac{D^{n+1}}{\xi_1^n}\otimes\frac{D^n}{D}\right)\circ\beta_n=0$$

so that, as the previous sequence is exact, by the universal property of kernels there exists a unique morphism

$$\tau_n:D^{n+1}\to\frac{D^n}{D}\otimes\frac{D^n}{D}$$

such that

$$\left(\frac{\xi_n^{n+1}}{D}\otimes\frac{D^n}{D}\right)\circ\tau_n=\beta_n.$$

Finally we have

$$\begin{aligned}\left(\frac{\xi_n^{n+1}}{D}\otimes\frac{\xi_n^{n+1}}{D}\right)\circ\tau_n&=\left(\frac{D^{n+1}}{D}\otimes\frac{\xi_n^{n+1}}{D}\right)\circ\left(\frac{\xi_n^{n+1}}{D}\otimes\frac{D^n}{D}\right)\circ\tau_n\\&=\left(\frac{D^{n+1}}{D}\otimes\frac{\xi_n^{n+1}}{D}\right)\circ\beta_n=\alpha_D^{D^{n+1}}.\qquad\square\end{aligned}$$

Theorem 2.4. *Let D and E be coalgebras in a cocomplete coabelian monoidal category $\mathcal{M}$ satisfying* AB5*. Let $\delta : D \to E$ be a monomorphism which is a coalgebra homomorphism in $\mathcal{M}$ and keep the notations introduced in Notation* 1.7*.*

Let $f : E \to C$ be a coalgebra homomorphism and assume that

$$f \circ \delta_2 : D \wedge_E D \to C$$

is a monomorphism. Then the coalgebra homomorphism

$$f \circ \tilde{\delta} : \widetilde{D}_E \to C$$

is a monomorphism.

Proof. Since $\mathcal{M}$ satisfies AB5, it is enough to prove that $f \circ \tilde{\delta} \circ \xi_n = f \circ \delta_n$ is a monomorphism for every $n \in \mathbb{N}$.

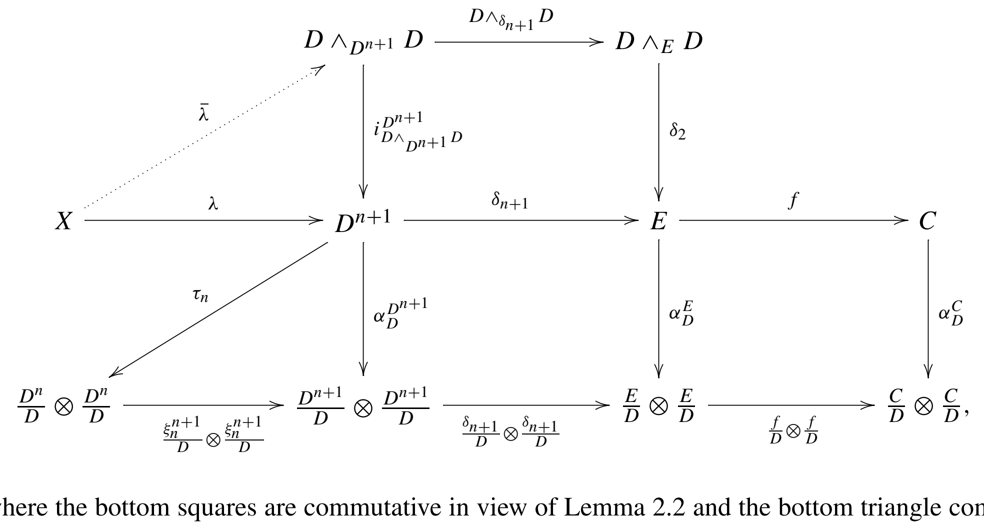
For $n = 0$, we have $f \circ \delta_0 = f \circ 0 = 0$ which is a monomorphism as $D^0 = 0$.

For $n = 1$, we have $f \circ \delta_1 = f \circ \delta_2 \circ \xi_1^2$ which is a monomorphism.

Let $n \geqslant 2$ and let us assume that $f \circ \delta_n$ is a monomorphism. Let us prove that $f \circ \delta_{n+1}$ is a monomorphism. Let $\lambda : X \to D^{n+1}$ be a morphism such that

$$f \circ \delta_{n+1} \circ \lambda = 0$$

and consider the following diagram



where the bottom squares are commutative in view of Lemma 2.2 and the bottom triangle commutes in view of Lemma 2.3. We have

$$\left(\frac{f\delta_n}{D} \otimes \frac{f\delta_n}{D}\right) \circ \tau_n \circ \lambda$$
$$= \left(\frac{f\delta_{n+1}\xi_n^{n+1}}{D} \otimes \frac{f\delta_{n+1}\xi_n^{n+1}}{D}\right) \circ \tau_n \circ \lambda$$

$$= \left(\frac{f}{D} \otimes \frac{f}{D}\right) \circ \left(\frac{\delta_{n+1}}{D} \otimes \frac{\delta_{n+1}}{D}\right) \circ \left(\frac{\xi_n^{n+1}}{D} \otimes \frac{\xi_n^{n+1}}{D}\right) \circ \tau_n \circ \lambda$$
$$= \alpha_D^C \circ f \circ \delta_{n+1} \circ \lambda = 0.$$

Since $f \circ \delta_n$ is a monomorphism, we get that also $f\delta_n/D \otimes f\delta_n/D$ is a monomorphism so that we obtain

$$\tau_n \circ \lambda = 0$$

and we have

$$\alpha_D^{D^{n+1}} \circ \lambda = \left(\frac{\xi_n^{n+1}}{D} \otimes \frac{\xi_n^{n+1}}{D}\right) \circ \tau_n \circ \lambda = 0.$$

Thus, since $(D \wedge_{D^{n+1}} D, i^{D^{n+1}}_{D\wedge_{D^{n+1}} D}) = \ker(\alpha_D^{D^{n+1}})$, by the universal property of the kernel, there exists a unique morphism, $\bar{\lambda} : X \to D \wedge_{D^{n+1}} D$ such that

$$\lambda = i^{D^{n+1}}_{D\wedge_{D^{n+1}} D} \circ \bar{\lambda}.$$

Now we have

$$f \circ \delta_2 \circ (D \wedge_{\delta_{n+1}} D) \circ \bar{\lambda} = f \circ \delta_{n+1} \circ \lambda = 0.$$

Since $f \circ \delta_2$ and $D \wedge_{\delta_{n+1}} D$ are monomorphisms, we get that $\bar{\lambda} = 0$ and hence $\lambda = 0$. □

Corollary 2.5. (Heyneman–Radford [HR, Proposition 2.4.2] or [Mo, Theorem 5.3.1, p. 65].) *Let K be a field. Let E and C be K-coalgebras and let $f : E \to C$ be a coalgebra homomorphism such that $f|_{D\wedge_E D}$ is injective, where D is the coradical of E. Then f is injective.*

Proof. Since D is the coradical of E is well known that $(E, \mathrm{Id}_E) = (\widetilde{D}_E, \tilde{\delta})$ (see, e.g., [Sw, Corollary 9.0.4, p. 185]). The conclusion follows by Theorem 2.4 applied in the case when $\mathcal{M}$ is the category of vector spaces over K. Observe that in this case "monomorphism" is equivalent to "injective." □

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