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ScienceDirect
Topology and its Applications 155 (2008) 1959–1969
Topology and its Applications
www.elsevier.com/locate/topol

Some covering properties of hyperspaces

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Received 12 October 2006; accepted 25 May 2007

Abstract

We study covering properties of hyperspaces with the Vietoris topology, as well as with its lower and upper parts. Considered covering properties are defined in terms of selection principles: the Menger, Hurewicz and Rothberger properties, and γ-sets and their relatives.

MSC: 54B20; 54D20

Keywords: Hyperspaces; Selection principles; Boundedness

1. Introduction

There are many results in the literature which show that properties of hyperspaces over a space X can be described by properties of the basic space X. We are interested to study such duality in connection with properties which are defined in terms of selection principles. Papers in this direction are, for instance, [1,3,4,13,16]; for a survey of related results see [15]. We continue this kind of investigation considering hyperspaces with the Vietoris topology.

Our notation and terminology are standard and follow the book [6]. The basic spaces over which we built hyperspaces are Hausdorff. We give necessary definitions about hyperspaces, selection principles and collections of open covers of a space.

1.1. Hyperspaces

Let X be a Hausdorff space. By $\mathsf{CL}(X)$, $\mathbb{K}(X)$, $\mathbb{F}(X)$ we denote the family of all nonempty closed, all nonempty compact, all nonempty finite subsets of X. For $n \in \mathbb{N}$, $\mathbb{F}_n(X)$ is the collection of sets $A \subset X$ with $1 \leqslant |A| \leqslant n$. If A is a subset of X and $\mathcal{A}$ a family of subsets of X, then we write

$$A^- = \{F \in \mathsf{CL}(X)\colon\ F \cap A \neq \emptyset\} \quad \text{and} \quad \mathcal{A}^- = \{A^-\colon\ A \in \mathcal{A}\},$$

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1 Supported by MURST-PRA 2000.
2 Supported by MNZŽS RS.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2007.05.025

$$A^+ = \{F \in \mathsf{CL}(X)\colon\ F \subset A\} \quad \text{and} \quad \mathcal{A}^+ = \{A^+\colon\ A \in \mathcal{A}\}.$$

The *upper Vietoris topology* V^+ on $\mathsf{CL}(X)$ is the topology whose base is the collection

$$\{U^+\colon\ U \text{ open in } X\},$$

while the *lower Vietoris topology* V^- is generated by all the sets U^-, $U \subset X$ nonempty, open. The *Vietoris topology*, denoted V, is defined by $\mathsf{V} = \mathsf{V}^+ \vee \mathsf{V}^-$. Recall that V-basic sets are of the form

$$U^+ \cap \Bigl(\bigcap_{i \leqslant m} V_i^-\Bigr), \quad U, V_1, \dots, V_m \text{ open in } X,$$

or equivalently, of the form

$$\langle U_1, \dots, U_m\rangle := \Bigl\{F \in \mathsf{CL}(X)\colon\ F \subset \bigcup_{i=1}^m U_i \text{ and } F \cap U_i \neq \emptyset\ \forall i \leqslant m\Bigr\}.$$

1.2. Selection principles

For more information on selection principles we refer the interested reader to [23,11], and to the survey papers [14,25,27].

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of sets of an infinite set X.

$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle: for each sequence $(A_n\colon n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n\colon n \in \mathbb{N})$ such that $b_n \in A_n$ for each $n \in \mathbb{N}$ and $\{b_n\colon n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathsf{S}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for each sequence $(A_n\colon n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n\colon n \in \mathbb{N})$ such that B_n is a finite subset of A_n for each $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}$.

When both $\mathcal{A}$ and $\mathcal{B}$ are the collection $\mathcal{O}$ of open covers of a space X, then $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ defines the classical *Rothberger covering property* (see [21]), while $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$ is the *Menger covering property* (see [19,9]).

In [16] the selection principles $\alpha_i(\mathcal{A}, \mathcal{B})$, $i = 1, 2, 3, 4$, were introduced; $\mathcal{A}$ and $\mathcal{B}$ are as above and $\mathcal{A}$ contains no finite elements.

Namely, $\alpha_i(\mathcal{A}, \mathcal{B})$, $i = 1, 2, 3, 4$, denotes the following selection hypothesis: for each sequence $(A_n\colon n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is an element $B \in \mathcal{B}$ such that

$\alpha_1(\mathcal{A}, \mathcal{B})$: for each $n \in \mathbb{N}$ the set $A_n \setminus B$ is finite;

$\alpha_2(\mathcal{A}, \mathcal{B})$: for each $n \in \mathbb{N}$ the set $A_n \cap B$ is infinite;

$\alpha_3(\mathcal{A}, \mathcal{B})$: for infinitely many $n \in \mathbb{N}$ the set $A_n \cap B$ is infinite;

$\alpha_4(\mathcal{A}, \mathcal{B})$: for infinitely many $n \in \mathbb{N}$ the set $A_n \cap B$ is nonempty.

We have,

$$\alpha_1(\mathcal{A}, \mathcal{B}) \;\Rightarrow\; \alpha_2(\mathcal{A}, \mathcal{B}) \;\Rightarrow\; \alpha_3(\mathcal{A}, \mathcal{B}) \;\Rightarrow\; \alpha_4(\mathcal{A}, \mathcal{B})$$

and

$$\mathsf{S}_1(\mathcal{A}, \mathcal{B}) \;\Rightarrow\; \alpha_4(\mathcal{A}, \mathcal{B}).$$

See also [28] for further study of these principles.

1.3. Open covers

An open cover $\mathcal{U}$ of a space X is called an *ω-cover* (a *k-cover*) if every finite (compact) subset of X is contained in a member of $\mathcal{U}$ and X is not a member of $\mathcal{U}$.

Recall that a space X is said to be *ω-Lindelöf* (*k-Lindelöf*) if every ω-cover (k-cover) of X contains a countable ω-cover (k-cover).

An open cover $\mathcal{U}$ of X is said to be a *γ-cover* (*γ_k-cover*) if it is infinite, and for each finite (compact) subset A of X the set $\{U \in \mathcal{U}\colon A \not\subseteq U\}$ is finite.

Observe that each infinite subset of a γ-cover (γ_k-cover) is still a γ-cover (γ_k-cover). So, we may suppose that such covers are countable.

For a space X, $\mathcal{O}$, Ω, $\mathcal{K}$, Γ, Γ_k denote the collections of open, ω-covers, k-covers, γ-covers, γ_k-covers of X.

According to [7] spaces satisfying $\mathsf{S}_1(\Omega, \Gamma)$ are called *γ-sets*; spaces that satisfy $\mathsf{S}_1(\mathcal{K}, \Gamma)$ are called *k-γ-sets* [2], and spaces from the class $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$ are called *γ_k-sets* [13]. Let us recall that the classical *Hurewicz covering property* [9,10] has been characterized as the property $\mathsf{S}_{\mathrm{fin}}(\Omega, \mathcal{O}^{\mathrm{gp}})$ in [18]. In the same paper the *Gerlits–Nagy property* $\mathsf{GN}(*)$ [7] was characterized as the property $\mathsf{S}_1(\Omega, \mathcal{O}^{\mathrm{gp}})$.

An open cover $\mathcal{U}$ of X is said to be a *groupable* (*ω-groupable* [17,18], *k-groupable* [13,2,1]) if it is countable and can be represented as the union of countably many pairwise disjoint finite subfamilies $\mathcal{V}_n$, $n \in \mathbb{N}$, such that any $x \in X$ (any finite $F \subset X$, any compact $K \subset X$) is contained in a member of $\mathcal{V}_n$ for all but finitely many n. The symbols $\mathcal{O}^{\mathrm{gp}}$, Ω^{gp} and $\mathcal{K}^{\mathrm{gp}}$ denote the collections of groupable, ω-groupable and k-groupable covers of a space.

2. Properties of $(\mathsf{CL}(X), \mathsf{V}^-)$

We prove that for a space X the hyperspace $\mathsf{CL}(X)$ with the lower Vietoris topology satisfies certain selection properties if and only if X has the same properties.

Lemma 2.1. *For a space X the following are equivalent*:

(a) *$\mathcal{U}$ is an open cover of X*;
(b) *$\mathcal{U}^-$ is an open cover of* $(\mathsf{CL}(X), \mathsf{V}^-)$.

Proof. (a) $\Rightarrow$ (b): Let $\mathcal{U}$ be an open cover of X, F an element in $(\mathsf{CL}(X), \mathsf{V}^-)$ and x any element in F. There is a $U \in \mathcal{U}$ such that $x \in U$. This means $F \cap U \neq \emptyset$ and thus $F \in U^-$.

(b) $\Rightarrow$ (a): If $x \in X$, then $\{x\} \in U^-$ for some $U \in \mathcal{U}$, so that we have $x \in U$. □

Theorem 2.2. *For a space X the following are equivalent*:

(1) *X has the Rothberger property* $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$;
(2) $(\mathsf{CL}(X), \mathsf{V}^-)$ *has the Rothberger property.*

Proof. (1) $\Rightarrow$ (2): Let $(\mathcal{W}_n\colon n \in \mathbb{N})$ be a sequence of open covers of $(\mathsf{CL}(X), \mathsf{V}^-)$. Without loss of generality one can assume that for each n all elements W of $\mathcal{W}_n$ are basic sets of the form $W = O^-(\mathcal{U}_W)$, where $\mathcal{U}_W$ is a finite collection $\{U_{W,1}, \ldots, U_{W,m_W}\}$ of open subsets of X with $W = U^-_{W,1} \cap \cdots \cap U^-_{W,m_W}$. For each $n \in \mathbb{N}$ let

$$\mathcal{U}_n = \left\{\bigcap \mathcal{U}_W\colon W \in \mathcal{W}_n\right\}.$$

Let us show that $\mathcal{U}_n$ is an open cover of X. Let $x \in X$. Then $\{x\} \in W$ for some $W = O^-(\mathcal{U}_W)$, i.e. $x \in \bigcap \mathcal{U}_W$. Apply now (1) to the sequence $(\mathcal{U}_n\colon n \in \mathbb{N})$ of open covers of X. There are $U_n = \bigcap \mathcal{U}_{W_n}$, $n \in \mathbb{N}$, with $W_n \in \mathcal{W}_n$, such that $\mathcal{U} = \{U_n\colon n \in \mathbb{N}\}$ is an open cover of X. We claim that $\{W_n\colon n \in \mathbb{N}\}$ is an open cover of $(\mathsf{CL}(X), \mathsf{V}^-)$.

Let $F \in (\mathsf{CL}(X), \mathsf{V}^-)$ and let x be an element of F. Then $x \in U_j$ for some $j \in \mathbb{N}$. Take the corresponding W_j. Evidently, we have $F \in W_j^-$.

(2) $\Rightarrow$ (1): Let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence of open covers of X. Then, by Lemma 2.1, $(\mathcal{U}_n^-\colon n \in \mathbb{N})$ is a sequence of open covers of $(\mathsf{CL}(X), \mathsf{V}^-)$. Apply (2) to the last sequence and find a sequence $(U_n^-\colon n \in \mathbb{N})$ such that for each n, $U_n^- \in \mathcal{U}_n^-$ and the set $\{U_n^-\colon n \in \mathbb{N}\}$ is an open cover of $(\mathsf{CL}(X), \mathsf{V}^-)$. Apply again Lemma 2.1 to conclude that the sequence $(U_n\colon n \in \mathbb{N})$ is a selector for $(\mathcal{U}_n\colon n \in \mathbb{N})$ which shows that X has the Rothberger property. □

In a similar way one proves:

Theorem 2.3. *For a space X the following are equivalent*:

(1) *X has the Menger property* $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$;
(2) $(\mathsf{CL}(X), \mathsf{V}^-)$ *has the Menger property.*

Lemma 2.4. *For a space X the following are equivalent*:

(a) *$\mathcal{U}$ is a γ-cover of X*;
(b) *$\mathcal{U}^-$ is a γ-cover of* $(\mathsf{CL}(X), \mathsf{V}^-)$.

Proof. (a) $\Rightarrow$ (b): Let $\mathcal{U}$ be a γ-cover of X and assume that (b) does not hold. Then there is $S \in (\mathsf{CL}(X), \mathsf{V}^-)$ such that the set $\{U^- \in \mathcal{U}^-\colon S \notin U^-\}$ is infinite, i.e. for infinitely many $U \in \mathcal{U}$ we have $S \cap U = \emptyset$. Pick any point $x \in S$. Then $x \notin U$ for infinitely many $U \in \mathcal{U}$ which is a contradiction.

(b) $\Rightarrow$ (a): If $x \in X$, then $\{x\} \notin U^-$ for at most finitely many U^- in $\mathcal{U}^-$. It is the same as to say that $x \notin U$ for at most finitely many $U \in \mathcal{U}$, i.e. $\mathcal{U}$ is a γ-cover of X. □

Theorem 2.5. *For a space X the following are equivalent*:

(1) *X satisfies* $\mathsf{S}_1(\Gamma, \Gamma)$;
(2) $(\mathsf{CL}(X), \mathsf{V}^-)$ *satisfies* $\mathsf{S}_1(\Gamma, \Gamma)$.

Proof. (1) $\Rightarrow$ (2): Let $(\mathcal{W}_n\colon n \in \mathbb{N})$ be a sequence of γ-covers of $(\mathsf{CL}(X), \mathsf{V}^-)$. We may suppose that all elements of all covers are basic open sets of the form $U_1^- \cap \cdots \cap U_m^-$, with $U_1, \ldots, U_m$ open subsets of X. For each $n \in \mathbb{N}$ denote by $\mathcal{U}_n$ the set of all elements of the form $U_1 \cap \cdots \cap U_m$ such that $U_1^-, \ldots, U_m^-$ occur in the representation of a member of $\mathcal{W}_n$. It is not hard to check that each $\mathcal{U}_n$ is a γ-cover of X. Applying (1), for each n we can pick an element H_n in $\mathcal{U}_n$, say $H_n = U_1^{(n)} \cap \cdots \cap U_{m_n}^{(n)}$, such that the set $\{H_n\colon n \in \mathbb{N}\}$ is a γ-cover of X. One proves that the sequence $((U_1^{(n)})^- \cap \cdots \cap (U_{m_n}^{(n)})^-\colon n \in \mathbb{N})$ witnesses that $(\mathsf{CL}(X), \mathsf{V}^-)$ satisfies $\mathsf{S}_1(\Gamma, \Gamma)$.

(2) $\Rightarrow$ (1): Let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence of γ-covers of X. Then, by Lemma 2.4, $(\mathcal{U}_n^-\colon n \in \mathbb{N})$ is a sequence of γ-covers of $(\mathsf{CL}(X), \mathsf{V}^-)$. Apply (2) to this sequence. We find a sequence $(U_n^-\colon n \in \mathbb{N})$ such that for each n, $U_n^- \in \mathcal{U}_n^-$ and the set $\{U_n^-\colon n \in \mathbb{N}\}$ is a γ-cover of $(\mathsf{CL}(X), \mathsf{V}^-)$. Now Lemma 2.4 says that the sequence $(U_n\colon n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n\colon n \in \mathbb{N})$ that X satisfies $\mathsf{S}_1(\Gamma, \Gamma)$. □

3. Properties of $(\mathbb{K}(X), \mathsf{V}^+)$ and $(\mathbb{F}(X), \mathsf{V}^+)$

We begin this section with the following three technical lemmas that will be often used in what follows.

Lemma 3.1. *For a space X and an open cover $\mathcal{U}$ of X we have*:

(a) *$\mathcal{U}$ is an ω-cover of X if and only if $\mathcal{U}^+$ is an ω-cover of* $(\mathbb{F}(X), \mathsf{V}^+)$;
(b) *$\mathcal{U}$ is a k-cover of X if and only if $\mathcal{U}^+$ is an ω-cover of* $(\mathbb{K}(X), \mathsf{V}^+)$;
(c) *$\mathcal{U}$ is a γ-cover of X if and only if $\mathcal{U}^+$ is a γ-cover of* $(\mathbb{F}(X), \mathsf{V}^+)$;
(d) *$\mathcal{U}$ is a γ_k-cover of X if and only if $\mathcal{U}^+$ is a γ-cover of* $(\mathbb{K}(X), \mathsf{V}^+)$;
(e) *$\mathcal{U}$ is an ω-groupable cover of X if and only if $\mathcal{U}^+$ is an ω-groupable cover of* $(\mathbb{F}(X), \mathsf{V}^+)$;
(f) *$\mathcal{U}$ is a k-groupable cover of X if and only if $\mathcal{U}^+$ is an ω-groupable cover of* $(\mathbb{K}(X), \mathsf{V}^+)$.

Proof. (a) Let $\{F_1, \ldots, F_n\}$ be a finite subset of $(\mathbb{F}(X), \mathsf{V}^+)$. Then $F = F_1 \cup \cdots \cup F_n$ is a finite subset of X and thus there is an element U in $\mathcal{U}$ with $F \subset U$. It follows that $F_i \in U^+$ for each $i \leqslant n$, hence $\{F_0, \ldots, F_n\} \subset U^+$.

Conversely, if $\mathcal{U}^+$ is an ω-cover of $(\mathbb{F}(X), \mathsf{V}^+)$ and F is a finite subset of X, then $F \in U^+$ for some $U \in \mathcal{U}$, i.e., $F \subset U$. This means that $\mathcal{U}$ is an ω-cover of X.

(b) This is proved similarly to (a).

(c) Similar to (d).

(d) Let $\mathcal{U}$ be a γ_k-cover of X and let $K \in (\mathbb{K}(X), \mathsf{V}^+)$. The set $\{U \in \mathcal{U}\colon K \nsubseteq U\}$ is finite. But that means that the set $\{U^+ \in \mathcal{U}^+\colon K \notin U^+\}$ is finite, i.e. $\mathcal{U}^+$ is a γ-cover of $(\mathbb{K}(X), \mathsf{V}^+)$.

For the converse, let K be a compact subset of X. The set $\{U^+ \in \mathcal{U}^+\colon K \notin U^+\}$ is finite. Therefore, the set $\{U \in \mathcal{U}\colon K \nsubseteq U\}$ is finite which says that $\mathcal{U}$ is a γ_k-cover of X.

(e) Let $\mathcal{U}$ be an ω-groupable cover of X and let $\mathcal{U} = \bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ be a partition of $\mathcal{U}$ witnessing that fact. We prove that $\bigcup_{n \in \mathbb{N}} \mathcal{U}_n^+$ is the partition of $\mathcal{U}^+$ which shows that $\mathcal{U}^+$ is an ω-groupable cover of $(\mathbb{F}(X), \mathsf{V}^+)$. Let $\{F_1, \ldots, F_p\}$ be a finite subset of $\mathbb{F}(X)$. Then $F = F_1 \cup \cdots \cup F_p$ is a finite subset of X and thus there is $m \in \mathbb{N}$ such that for each $n \geqslant m$ F is contained in a member U of $\mathcal{U}_n$. But then for each $n \geqslant m$ the element U^+ in $\mathcal{U}_n^+$ contains the set $\{F_1, \ldots, F_p\}$, i.e., $\mathcal{U}^+$ is an ω-groupable cover of $(\mathbb{F}(X), \mathsf{V}^+)$.

The reverse implication may be easily proved.

(f) Similar to (e). □

The following two lemmas have been proved in [16].

Lemma 3.2. *For a space X and an open cover $\mathcal{W}$ of $(\mathbb{F}(X), \mathsf{V}^+)$ the following holds: $\mathcal{W}$ is an ω-cover of $(\mathbb{F}(X), \mathsf{V}^+)$ if and only if $\mathcal{U}(\mathcal{W}) := \{U \subset X\colon U$ is open in X and $U^+ \subset W$ for some $W \in \mathcal{W}\}$ is an ω-cover of X.*

Lemma 3.3. *For a space X and an open cover $\mathcal{W}$ of $(\mathbb{K}(X), \mathsf{V}^+)$ the following holds: $\mathcal{W}$ is an ω-cover of $(\mathbb{K}(X), \mathsf{V}^+)$ if and only if $\mathcal{U}(\mathcal{W}) := \{U \subset X\colon U$ is open in X and $U^+ \subset W$ for some $W \in \mathcal{W}\}$ is a k-cover of X.*

We need also the following assertion from [16].

Proposition 3.4. *A space X is ω-Lindelöf if and only if $(\mathbb{F}(X), \mathsf{V}^+)$ is ω-Lindelöf.*

The following theorem is given without proof in [16]; for the reader convenience we provide it.

Theorem 3.5. *For an ω-Lindelöf space X the following are equivalent:*

(1) $(\mathbb{F}(X), \mathsf{V}^+)$ *satisfies* $\alpha_2(\Omega, \Gamma)$;
(2) $(\mathbb{F}(X), \mathsf{V}^+)$ *satisfies* $\alpha_3(\Omega, \Gamma)$;
(3) $(\mathbb{F}(X), \mathsf{V}^+)$ *satisfies* $\alpha_4(\Omega, \Gamma)$;
(4) $(\mathbb{F}(X), \mathsf{V}^+)$ *satisfies* $\mathsf{S}_1(\Omega, \Gamma)$;
(5) X *satisfies* $\mathsf{S}_1(\Omega, \Gamma)$.

Proof. (1) ⇒ (2) ⇒ (3) ⇒ (4) hold for any space.

(4) ⇒ (1): By Proposition 3.4 the space $(\mathbb{F}(X), \mathsf{V}^+)$ is ω-Lindelöf. It remains to apply Theorem 2 from [16] asserting that for any ω-Lindelöf space $\mathsf{S}_1(\Omega, \Gamma)$ is equivalent to $\alpha_2(\Omega, \Gamma)$.

(4) ⇒ (5): Let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence of ω-covers of X. Then, by Lemma 3.1, $(\mathcal{U}_n^+\colon n \in \mathbb{N})$ is a sequence of ω-covers of $(\mathbb{F}(X), \mathsf{V}^+)$. By (4) for each n choose an element U_n^+ in $\mathcal{U}_n^+$ such that the set $\mathcal{U}^+ = \{U_n^+\colon n \in \mathbb{N}\}$ is a γ-cover of $(\mathbb{F}(X), \mathsf{V}^+)$. Due to Lemma 3.1 we have that $\{U_n\colon n \in \mathbb{N}\}$ is a γ-cover of X and so (5) holds.

(5) ⇒ (4): Let $(\mathcal{W}_n : n \in \mathbb{N})$ be a sequence of ω-covers of $(\mathbb{F}(X), \mathsf{V}^+)$. For each n let

$$\mathcal{U}_n = \{U \subset X\colon U \text{ is open in } X \text{ and } U^+ \subset W \text{ for some } W \in \mathcal{W}_n\}.$$

By Lemma 3.2 each $\mathcal{U}_n$ is an ω-cover of X. By (5) applied to the sequence $(\mathcal{U}_n\colon n \in \mathbb{N})$ one can find a sequence $(U_n\colon n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and the set $\mathcal{U} = \{U_n\colon n \in \mathbb{N}\}$ is a γ-cover of X. For each $U_n \in \mathcal{U}$ pick an element $W_n \in \mathcal{W}_n$ so that $U_n^+ \subset W_n$. We claim that $\{W_n\colon n \in \mathbb{N}\}$ is a γ-cover of $(\mathbb{F}(X), \mathsf{V}^+)$ and so it witnesses for $(\mathcal{W}_n\colon n \in \mathbb{N})$ that (4) is satisfied. Let $F \in \mathbb{F}(X)$. Then there is n_0 such that for each $n \geqslant n_0$, $F \subset U_n$, i.e. $F \in U_n^+ \subset W_n$. □

The following theorem was shown in [16, Theorem 13].

Theorem 3.6. *For a k-Lindelöf space X the following are equivalent:*

(1) $(\mathbb{K}(X), \mathsf{V}^+)$ *satisfies* $\alpha_2(\Omega, \Gamma)$;
(2) $(\mathbb{K}(X), \mathsf{V}^+)$ *satisfies* $\alpha_3(\Omega, \Gamma)$;
(3) $(\mathbb{K}(X), \mathsf{V}^+)$ *satisfies* $\alpha_4(\Omega, \Gamma)$;
(4) $(\mathbb{K}(X), \mathsf{V}^+)$ *satisfies* $\mathsf{S}_1(\Omega, \Gamma)$;
(5) X *satisfies* $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$.

Theorem 3.7. *For a space X we have*:

(1) $(\mathbb{F}(X), \mathsf{V}^+)$ *satisfies* $\mathsf{S}_1(\Omega, \Omega^{\mathrm{gp}})$ *if and only if* X *satisfies* $\mathsf{S}_1(\Omega, \Omega^{\mathrm{gp}})$;
(2) $(\mathbb{K}(X), \mathsf{V}^+)$ *satisfies* $\mathsf{S}_1(\Omega, \Omega^{\mathrm{gp}})$ *if and only if* X *satisfies* $\mathsf{S}_1(\mathcal{K}, \mathcal{K}^{\mathrm{gp}})$.

Proof. (2): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of k-covers of X. Then, by Lemma 3.1 $(\mathcal{U}_n^+ \colon n \in \mathbb{N})$ is a sequence of ω-covers of $(\mathbb{K}(X), \mathsf{V}^+)$ so that there is a sequence $(U_n^+ \colon n \in \mathbb{N})$ such that for each n, $U_n^+ \in \mathcal{U}_n^+$ and the set $\mathcal{V}^+ := \{U_n^+ \colon n \in \mathbb{N}\}$ is an ω-groupable cover of $(\mathbb{K}(X), \mathsf{V}^+)$. Let $\mathcal{V}^+ = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n^+$ shows the last fact. Put for each $n \in \mathbb{N}$, $\mathcal{H}_n = \{V \colon V^+ \in \mathcal{V}_n^+\}$. Then $\mathcal{H} = \bigcup_{n\in\mathbb{N}} \mathcal{H}_n$ is a k-groupable cover for X which shows that X satisfies $\mathsf{S}_1(\mathcal{K}, \mathcal{K}^{\mathrm{gp}})$.

Conversely, let $(\mathcal{W}_n \colon n \in \mathbb{N})$ be a sequence of ω-covers of $(\mathbb{K}(X), \mathsf{V}^+)$ and let for each n

$$\mathcal{U}(\mathcal{W}_n) = \{U \colon U \subset X \text{ open and } U^+ \subset W \text{ for some } W \in \mathcal{W}_n\}.$$

Then for each n, $\mathcal{U}(\mathcal{W}_n)$ is a k-cover of X (Lemma 3.3). From each $\mathcal{U}(\mathcal{W}_n)$ choose U_n so that the set $\mathcal{U} := \{U_n \colon n \in \mathbb{N}\}$ is a k-groupable cover of X. Let $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ witnesses this fact. Further, for each n fix an element $W_n \in \mathcal{W}_n$ such that $U_n^+ \subset W_n$ and consider $\mathcal{W} = \{W_n \colon n \in \mathbb{N}\}$. For each n set $\mathcal{H}_n = \{W \in \mathcal{W} \colon W \text{ contains a member of } \mathcal{V}_n\}$. Then $\mathcal{H} = \bigcup_{n\in\mathbb{N}} \mathcal{H}_n$ is an ω-groupable cover for $(\mathbb{K}(X), \mathsf{V}^+)$ that shows that $(\mathbb{K}(X), \mathsf{V}^+)$ satisfies $\mathsf{S}_1(\Omega, \Omega^{\mathrm{gp}})$. □

4. Properties of $(\mathbb{F}(X), \mathsf{V})$

In this section we discuss duality between a space X and its hyperspace $(\mathbb{F}(X), \mathsf{V})$.

Theorem 4.1. *For a space X* (1) ⇒ (2) ⇔ (3) *holds*:

(1) X *satisfies* $\mathsf{S}_1(\Omega, \Omega)$;
(2) *For each n,* $(\mathbb{F}_n(X), \mathsf{V})$ *satisfies* $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$;
(3) $(\mathbb{F}(X), \mathsf{V})$ *satisfies* $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$.

Proof. (1) ⇒ (2): Recall that (1) is equivalent to the statement: each finite power of X has the Rothberger property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ (see [22,11]). Fix n and consider the mapping

$$\varphi : X^n \to (\mathbb{F}_n(X), \mathsf{V}), \quad \varphi(x_1, \ldots, x_n) = \{x_1, \ldots, x_n\}.$$

If $W = U^+ \cap (\bigcap_{j=1}^m V_j^-)$ is a basic set in $(\mathbb{F}_n(X), \mathsf{V})$, then

$$\varphi^{\leftarrow}(W) = \bigcap_{i\leqslant n} p_i^{\leftarrow}(U) \cap \left(\bigcap_{j\leqslant m}\left(\bigcup_{i\leqslant n} p_i^{\leftarrow}(V_j)\right)\right)$$

(where $p_i : X^n \to X$ is the ith projection) so that the mapping φ is continuous. The Rothberger property is an invariant of continuous mappings, and thus $(\mathbb{F}_n(X), \mathsf{V})$ has the property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$.

(2) ⇒ (3): The Rothberger property is σ-additive, and since each $(\mathbb{F}_n(X), \mathsf{V})$ has that property, the space $(\mathbb{F}(X), \mathsf{V}) = \bigcup_{n\in\mathbb{N}} (\mathbb{F}_n(X), \mathsf{V})$ also satisfies $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$.

(3) ⇒ (2): It follows from the fact that $(\mathbb{F}_n(X), \mathsf{V})$, $n \in \mathbb{N}$, is a closed subspace of $(\mathbb{F}(X), \mathsf{V})$ and that Rothberger's property is hereditary for closed subspaces. □

It is known that the Menger, Hurewicz and Gerlits–Nagy properties are preserved by closed subspaces, continuous images and countable unions, and that all finite powers of an ω-Lindelöf space X have the Menger (Hurewicz, Gerlits–Nagy) property if and only if X satisfies $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$ [11], $(\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega^{\mathrm{gp}})$ [18], $\mathsf{S}_1(\Omega, \Omega^{\mathrm{gp}}))$ [18]. So, similarly to the proof of Theorem 4.1 we can prove the following three theorems.

Theorem 4.2. *For an ω-Lindelöf space X we have* (1) ⇒ (2) ⇔ (3):

(1) X *satisfies* $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$;
(2) *For each n,* $(\mathbb{F}_n(X), \mathsf{V})$ *satisfies* $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$;

(3) $(\mathbb{F}(X), \mathsf{V})$ *satisfies* $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$.

Theorem 4.3. *For an ω-Lindelöf space X* $(1) \Rightarrow (2) \Leftrightarrow (3)$ *holds*:

(1) *X satisfies* $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega^{\mathrm{gp}})$;
(2) *For each n,* $(\mathbb{F}_n(X), \mathsf{V})$ *satisfies* $\mathsf{S}_{\mathrm{fin}}(\Omega, \mathcal{O}^{\mathrm{gp}})$;
(3) $(\mathbb{F}(X), \mathsf{V})$ *satisfies* $\mathsf{S}_{\mathrm{fin}}(\Omega, \mathcal{O}^{\mathrm{gp}})$.

Theorem 4.4. *For an ω-Lindelöf space X we have* $(1) \Rightarrow (2) \Leftrightarrow (3)$:

(1) *X satisfies* $\mathsf{S}_1(\Omega, \Omega^{\mathrm{gp}})$;
(2) *For each n,* $(\mathbb{F}_n(X), \mathsf{V})$ *satisfies* $\mathsf{S}_1(\Omega, \mathcal{O}^{\mathrm{gp}})$;
(3) $(\mathbb{F}(X), \mathsf{V})$ *satisfies* $\mathsf{S}_1(\Omega, \mathcal{O}^{\mathrm{gp}})$.

But for the class $\mathsf{S}_1(\Omega, \Gamma)$ the situation is a bit different. We have: X satisfies $\mathsf{S}_1(\Omega, \Gamma)$ if and only if all finite powers of X satisfies $\mathsf{S}_1(\Omega, \Gamma)$ [7]. Therefore, the following theorem holds:

Theorem 4.5. *For a space X the following are equivalent*:

(1) *X satisfies* $\mathsf{S}_1(\Omega, \Gamma)$;
(2) *For each n,* $(\mathbb{F}_n(X), \mathsf{V})$ *satisfies* $\mathsf{S}_1(\Omega, \Gamma)$.

If $(A_n\colon n \in \mathbb{N})$ is a sequence of subsets of a space X, then $\underline{\mathrm{Lim}} A_n = \bigcup_m \bigcap_{n \geqslant m} A_n$. If $\mathcal{A}$ is a family of subsets of X, then $L(\mathcal{A})$ is the closure of $\mathcal{A}$ with respect to the operator $\underline{\mathrm{Lim}}$. According to [7] a space X is said to be a δ-set if for each ω-cover $\mathcal{U}$ of X we have $X \in L(\mathcal{U})$. Every γ-set is a δ-set.

Combining Theorem 4.5, the fact that $(\mathbb{F}(X), \mathsf{V})$ is the union of the increasing sequence $(\mathbb{F}_n(X), \mathsf{V})$, $n \in \mathbb{N}$, of γ-sets, and the fact that unions of increasing sequences of γ-sets are δ-sets, we have this corollary.

Corollary 4.6. *If a space X is a γ-set, then* $(\mathbb{F}(X), \mathsf{V})$ *is a δ-set.*

Problem 4.7. *Is the converse in the previous corollary true*?

5. Properties of $(\mathbb{K}(X), \mathsf{V})$

In this section we deal with some boundedness conditions in the hyperspace $(\mathbb{K}(X), \mathsf{V})$. Notice that because we suppose that the basic space X is Hausdorff, the space $(\mathbb{K}(X), \mathsf{V})$ is also Hausdorff.

Recall that a Hausdorff space X is *H-closed* if for every open cover $\mathcal{U}$ of X there is a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $X = \bigcup\{\overline{V}\colon V \in \mathcal{V}\}$. (Clearly, a regular H-space is compact.) A subset A of a Hausdorff space X is an *H-set* if for any cover $\mathcal{U}$ of A by sets open in X there is a finite $\mathcal{V} \subset \mathcal{U}$ such that $A \subset \bigcup\{\overline{V}\colon V \in \mathcal{V}\}$ (closures are in X).

Theorem 5.1. *Let X be a Hausdorff non-regular space. If* $\mathcal{A} \subset (\mathbb{K}(X), \mathsf{V})$ *is an H-set in* $(\mathbb{K}(X), \mathsf{V})$*, then* $A = \bigcup \mathcal{A}$ *is an H-set in X.*

Proof. Let $\mathcal{U}$ be an open cover of A by sets open in X. For each $K \in \mathcal{A}$ there is a finite subfamily $\mathcal{U}_K = \{U_{K,1}, \ldots, U_{K,m_K}\}$ of $\mathcal{U}$ such that $K \subset \bigcup \mathcal{U}_K$. This means that $K \in \langle \mathcal{U}_K \rangle := \langle U_{K,1}, \ldots, U_{K,m_K} \rangle$. Therefore,

$$\mathcal{G} = \big\{\langle \mathcal{U}_K \rangle\colon K \in \mathcal{A}\big\}$$

is an open cover of $\mathcal{A}$ by sets open in $(\mathbb{K}(X), \mathsf{V})$. Choose finitely many $K_1, \ldots, K_p \in \mathcal{A}$ such that $\mathcal{A} \subset \overline{\langle \mathcal{U}_{K_1} \rangle} \cup \cdots \cup \overline{\langle \mathcal{U}_{K_p} \rangle}$. Note that for each $i \leqslant p$ we have

$$\overline{\langle U_{K_i,1}, \ldots, U_{K_i,m_{K_i}} \rangle} = \langle \overline{U_{K_i,1}}, \ldots, \overline{U_{K_i,m_{K_i}}} \rangle.$$

We claim that the set

$$\mathcal{V}=\{U_{K_1,1},\ldots,U_{K_1,m_{K_1}};\ldots,U_{K_p,1},\ldots,U_{K_p,m_{K_p}}\}$$

witnesses for $\mathcal{U}$ that A is an H-set in X.

Let $x\in A$. Choose a $C\in\mathcal{A}$ such that $x\in C$. But $C\in\langle\overline{U_{K_j,1}},\ldots,\overline{U_{K_j,m_{K_j}}}\rangle$ for some $j\in\{1,2,\ldots,p\}$, i.e. $C\subset\overline{U_{K_j,1}}\cup\cdots\cup\overline{U_{K_j,m_{K_j}}}$ and so $x\in\overline{U_{K_j,s}}$ for some $s\leqslant m_{K_j}$. □

It is known [6] that H-closedness is not hereditary with respect to closed subspaces. But from the previous theorem we have the following.

Corollary 5.2. *If X is a space such that $(\mathbb{K}(X),\mathsf{V})$ is an H-closed space, then X is also H-closed.*

Using a similar proof we have something more. For a space (X,τ) let $\overline{\mathcal{O}}$ denote the collection of all $\mathcal{U}\subset\tau$ such that $X=\bigcup\{\overline{U}\colon U\in\mathcal{U}\}$. Spaces satisfying the selection principle $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\overline{\mathcal{O}})$ were called *almost Menger* in [12]. (Clearly, a regular almost Menger space is Menger.) The class $\overline{\mathcal{O}}$ of covers has been already considered in connection with games and selection principles [26,24]. In [24] it was shown that for a subset A of real numbers properties $\mathsf{S}_1(\overline{\mathcal{O}},\overline{\mathcal{O}})$ (which is stronger than Rothberger's property) and $\mathsf{S}_{\mathrm{fin}}(\overline{\mathcal{O}},\overline{\mathcal{O}})$ are equivalent, and either of these properties for A is equivalent to the fact that A is a Lusin set.

In what follows we use the following notation. If Y is a subspace of a space (X,τ), then $\mathcal{O}_{Y,X}$ denotes the collection of all $\mathcal{U}\subset\tau$ with $Y\subset\bigcup U$. By $\overline{\mathcal{O}}_{Y,X}$ we denote the collection of all $\mathcal{U}\subset\tau$ such that $Y\subset\bigcup\{Cl_X(U)\colon U\in\mathcal{U}\}$. Similar notation we use for other classes of covers.

Theorem 5.3. *Let X be a Hausdorff non-regular space. If $\mathcal{A}\subset(\mathbb{K}(X),\mathsf{V})$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_{\mathcal{A},\mathbb{K}(X)},\overline{\mathcal{O}}_{\mathcal{A},\mathbb{K}(X)})$, then $A=\bigcup\mathcal{A}$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_{A,X},\overline{\mathcal{O}}_{A,X})$.*

Proof. Let $(\mathcal{U}_n\colon n\in\mathbb{N})$ be a sequence of open covers of A by sets open in X. For each $K\in\mathcal{A}$ and each $n\in\mathbb{N}$ there is a finite set $\mathcal{U}_{n,K}\subset\mathcal{U}_n$ which covers K. Therefore, for each n

$$\mathcal{G}_n=\{\langle\mathcal{U}_{n,K}\rangle\colon K\in\mathcal{A}\}$$

is an open cover of $\mathcal{A}$ by sets open in $(\mathbb{K}(X),\mathsf{V})$. Apply assumption to the sequence $(\mathcal{G}_n\colon n\in\mathbb{N})$ and choose a sequence $(\mathcal{H}_n\colon n\in\mathbb{N})$ such that for each n, $\mathcal{H}_n$ is a finite subset of $\mathcal{G}_n$ and $\bigcup\{\overline{H}\colon H\in\mathcal{H}_n,\ n\in\mathbb{N}\}\supset\mathcal{A}$. Working as in the proof of Theorem 5.1 we conclude that the closures of all U that occur in representation of some element from $\mathcal{H}_n$, $n\in\mathbb{N}$, cover A and show that A satisfies $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_{A,X},\overline{\mathcal{O}}_{A,X})$. □

It would be interesting to study $\mathsf{S}_1(\overline{\mathcal{O}},\overline{\mathcal{O}})$ and $\mathsf{S}_{\mathrm{fin}}(\overline{\mathcal{O}},\overline{\mathcal{O}})$ in the context of hyperspaces.

Theorem 5.4. *If $\mathcal{A}\subset(\mathbb{K}(X),\mathsf{V})$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_{\mathcal{A},\mathbb{K}(X)},\mathcal{O}_{\mathcal{A},\mathbb{K}(X)})$, then $A=\bigcup\mathcal{A}\subset X$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_{A,X},\mathcal{O}_{A,X})$.*

Proof. Let $(\mathcal{U}_n : n\in\mathbb{N})$ be a sequence of elements of $\mathcal{O}_{A,X}$. Fix n. For every $K\in\mathcal{A}$ there is a finite set $\mathcal{U}_{n,K}\subset\mathcal{U}_n$ such that $K\subset\bigcup\mathcal{U}_{n,K}$ and $K\cap U\neq\emptyset$ for each $U\in\mathcal{U}_{n,K}$. Therefore, $\langle\mathcal{U}_{n,K}\rangle$ is a basic neighborhood of K in $(\mathbb{K}(X),\mathsf{V})$ and we have that the set $\mathcal{W}_n=\{\langle\mathcal{U}_{n,K}\rangle\colon K\in\mathcal{A}\}$ is an open cover of $\mathcal{A}$ by sets open in $(\mathbb{K}(X),\mathsf{V})$. Since $\mathcal{A}$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathcal{O}_{\mathcal{A},\mathbb{K}(X)},\mathcal{O}_{\mathcal{A},\mathbb{K}(X)})$, from each $\mathcal{W}_n$ we can choose a finite subfamily $\mathcal{H}_n$ so that $\bigcup_{n\in\mathbb{N}}\mathcal{H}_n$ is an open cover of $\mathcal{A}$. For each n let

$$\mathcal{V}_n=\{U\colon U \text{ occurs in the above representation of some } W\in\mathcal{H}_n\}.$$

Then $\mathcal{V}_n$ is a finite subfamily of $\mathcal{U}_n$ for each $n\in\mathbb{N}$. We are going to prove that $\bigcup_{n\in\mathbb{N}}\mathcal{V}_n$ is an open cover of A (by sets open in X).

Let $x\in A$. Then x belongs to a $C\in\mathcal{A}$. There is some $m\in\mathbb{N}$ and a $\langle\mathcal{U}_{m,C}\rangle\in\mathcal{H}_m$ for which $C\in\langle\mathcal{U}_{m,C}\rangle$, i.e. $C\subset\bigcup\mathcal{U}_{m,C}$. Thus x is a member of some $U\in\mathcal{V}_m$. □

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of open covers of a space X. Call a space X $\mathsf{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$-*bounded* if each $\mathsf{S}_{\mathrm{fin}}(\mathcal{A}_{Y,X},\mathcal{B}_{Y,X})$ set Y in X is contained in a compact subset of X. Similarly one defines a $\mathsf{S}_1(\mathcal{A},\mathcal{B})$-*bounded* space.

Theorem 5.5. *A space X is $\mathsf{S}_{\text{fin}}(\mathcal{O}, \mathcal{O})$-bounded if and only if $(\mathbb{K}(X), \mathsf{V})$ is $\mathsf{S}_{\text{fin}}(\mathcal{O}, \mathcal{O})$-bounded.*

Proof. ($\Rightarrow$): Let $\mathcal{A}$ be an $\mathsf{S}_{\text{fin}}(\mathcal{O}_{\mathcal{A},\mathbb{K}(X)}, \mathcal{O}_{\mathcal{A},\mathbb{K}(X)})$ set in $(\mathbb{K}(X), \mathsf{V})$. Then by Theorem 5.4 $A = \bigcup \mathcal{A}$ is an $\mathsf{S}_{\text{fin}}(\mathcal{O}_{A,X}, \mathcal{O}_{A,X})$ set in X and so $\overline{A}$ is compact. It follows that $(\mathbb{K}(\overline{A}), \mathsf{V})$ is a compact, hence closed subset of $(\mathbb{K}(X), \mathsf{V})$. Obviously, $\mathcal{A} \subset \mathbb{K}(\overline{A})$. So we have $\overline{\mathcal{A}} \subset \mathbb{K}(\overline{A})$ and thus $\overline{\mathcal{A}}$ is compact.

($\Leftarrow$): X is embedded in $(\mathbb{K}(X), \mathsf{V})$ as a closed subspace. Since evidently $\mathsf{S}_{\text{fin}}(\mathcal{O}, \mathcal{O})$-boundedness is hereditary for closed subspaces, we have that X is $\mathsf{S}_{\text{fin}}(\mathcal{O}, \mathcal{O})$-bounded. □

Theorem 5.6. *If $\mathcal{A} \subset (\mathbb{K}(X), \mathsf{V})$ satisfies $\mathsf{S}_1(\Gamma_{\mathcal{A},\mathbb{K}(X)}, \Gamma_{\mathcal{A},\mathbb{K}(X)})$, then $A = \bigcup \mathcal{A}$ satisfies $\mathsf{S}_1((\Gamma_k)_{A,X}, \Gamma_{A,X})$.*

Proof. Let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence of γ_k-covers of A by sets open in X. One can suppose that all covers are countable, say $\mathcal{U}_n = \{U_{n,m}\colon m \in \mathbb{N}\}$. Let $K \in \mathcal{A}$. For each $n \in \mathbb{N}$ there is $m_0 \in \mathbb{N}$ such that $K \subset U_{n,m}$ for each $m \geqslant m_0$. That means that for a fixed n the set $\mathcal{U}_n^+$ is a γ-cover for $\mathcal{A}$ by sets open in $(\mathbb{K}(X), \mathsf{V})$. Apply assumption to the sequence $(\mathcal{U}_n^+\colon n \in \mathbb{N})$ of γ-covers of $\mathcal{A}$ and choose a sequence $(U_n^+\colon n \in \mathbb{N})$ such that for each n, $U_n^+ \in \mathcal{U}_n^+$ and the set $\{U_n^+\colon n \in \mathbb{N}\}$ is a γ-cover of $\mathcal{A}$. Then $\{U_n\colon n \in \mathbb{N}\}$ is a γ-cover of A which shows that A satisfies $\mathsf{S}_1((\Gamma_k)_{A,X}, \Gamma_{A,X})$. Indeed, if $x \in A$, then x belongs to some $K_x \in \mathcal{A}$. But $K_x \in U_n^+$ for all but finitely many n, and so $x \in U_n$ for all but finitely many n. □

Corollary 5.7. *If a space X is $\mathsf{S}_1(\Gamma_k, \Gamma)$-bounded, then $(\mathbb{K}(X), \mathsf{V})$ is $\mathsf{S}_1(\Gamma, \Gamma)$-bounded.*

Problem 5.8. *Is the converse in Corollary* 5.7 *true*?

Analogously we can show the next theorem.

Theorem 5.9. *If $\mathcal{A} \subset (\mathbb{K}(X), \mathsf{V})$ satisfies $\mathsf{S}_1(\Gamma_{\mathcal{A},\mathbb{K}(X)}, \mathcal{O}_{\mathcal{A},\mathbb{K}(X)})$, then $A = \bigcup \mathcal{A}$ satisfies $\mathsf{S}_1((\Gamma_k)_{A,X}, \mathcal{O}_{A,X})$.*

Proof. Let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence of γ_k-covers of A by sets open in X. Suppose $\mathcal{U}_n = \{U_{n,m}\colon m \in \mathbb{N}\}$. Let $K \in \mathcal{A}$. For each $n \in \mathbb{N}$ there is $m_K \in \mathbb{N}$ such that $K \subset U_{n,m}$ for all $m \geqslant m_K$, which just means that for each n the set $\mathcal{U}_n^+$ is a γ-cover for $\mathcal{A}$ by sets open in $(\mathbb{K}(X), \mathsf{V})$. By assumption one can choose a sequence $(U_n^+\colon n \in \mathbb{N})$ such that for each n, $U_n^+ \in \mathcal{U}_n^+$ and the set $\mathcal{U}^+ = \{U_n^+\colon n \in \mathbb{N}\}$ is an open cover of $\mathcal{A}$. Then $\{U_n\colon n \in \mathbb{N}\}$ is an open cover of A. Indeed, let $x \in A$. Then x belongs to some $K_x \in \mathcal{A}$, and K_x is in a set $U_{n_x}^+ \in \mathcal{U}^+$. So, $x \in U_{n_x}$. □

In a similar way, using Lemma 3.1, one obtains:

Theorem 5.10. *If $\mathcal{A} \subset (\mathbb{K}(X), \mathsf{V})$ satisfies $\mathsf{S}_1(\Omega_{\mathcal{A},\mathbb{K}(X)}, \Gamma_{\mathcal{A},\mathbb{K}(X)})$, then $A = \bigcup \mathcal{A}$ satisfies $\mathsf{S}_1(\mathcal{K}_{A,X}, \Gamma_{A,X})$.*

Proof. Let $(\mathcal{U}_n\colon n \in \mathbb{N})$ be a sequence of elements of $\mathcal{K}_{A,X}$. Working as in the proof of (b) in Lemma 3.1, we conclude that $(\mathcal{U}_n^+\colon n \in \mathbb{N})$ is a sequence of ω-covers of $\mathcal{A}$ (by $(\mathbb{K}(X), \mathsf{V})$-open sets). For each $n \in \mathbb{N}$ choose an element $U_n^+ \in \mathcal{U}_n^+$ so that $\{U_n^+\colon n \in \mathbb{N}\}$ is a γ-cover of $\mathcal{A}$. Then $\{U_n\colon n \in \mathbb{N}\}$ is a γ-cover of A (see the proof of Theorem 5.6). □

We end this section with a general question.

Problem 5.11. *What about the converses in theorems in this section claiming implications in one direction*?

6. Closing remarks: Abstract boundedness

In 1949, S.T. Hu [8] defined and studied an abstract boundedness in any topological space. A modified version of his notion was applied as a useful tool in investigation of different hyperspace topologies, including the Hausdorff–Bourbaki, Attouch–Wets, bounded (proximal) Vietoris and Wijsman topology (see [20, p. 53], [5]). For example, in [5] it was shown that the boundedness generated by closed balls of a metric space is a powerful device for investigation of the Wijsman hyperspace topology and offers deeper and simple proofs without epsilonetics.

A family $\mathbb{B}$ of nonempty closed subsets of a space X is said to be an *abstract boundedness*, or simply *boundedness*, if it is closed for finite unions, closed hereditary and contains all singletons. Examples of boundedness are the families $\mathbb{F}(X)$ and $\mathbb{K}(X)$ of a topological space X, the family of all (totally) bounded subsets of a metric or uniform space. If $\mathbb{B}$ is a boundedness in a space X and $\mathcal{U}$ is an open cover of X, then $\mathcal{U}$ is said to be a $\mathbb{B}$-*cover* if each $B \in \mathbb{B}$ is contained in an element of $\mathcal{U}$ and $X \notin \mathcal{U}$. $\mathcal{U}$ is called a $\gamma_{\mathbb{B}}$-*cover* if it is infinite and each $B \in \mathbb{B}$ is not contained in at most finitely many elements of $\mathcal{U}$. For a given boundedness $\mathbb{B}$ in a space denote by $\mathcal{O}_{\mathbb{B}}$ ($\Gamma_{\mathbb{B}}$) the collection of all $\mathbb{B}$-covers ($\gamma_{\mathbb{B}}$-covers). In this section we consider only (Hausdorff) spaces in which each $\mathbb{B}$-cover (for a given $\mathbb{B}$) contains a countable $\mathbb{B}$-subcover; call such spaces $\mathbb{B}$-*Lindelöf*.

We formulate without proofs several results related to boundedness. In what follows $\mathbb{B}$ will be a fixed boundedness.

Recall that the game $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$, associated to the selection principle $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$, is played by two players, ONE and TWO, which play a round for each positive integer. In the nth round ONE chooses a set $A_n \in \mathcal{A}$, and TWO responds by choosing an element $b_n \in A_n$. TWO wins a play $(A_1, b_1; \ldots; A_n, b_n; \ldots)$ if $\{b_n\colon n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, ONE wins.

Theorem 6.1. *For a $\mathbb{B}$-Lindelöf space X the following are equivalent*:

(1) *X satisfies* $\alpha_2(\mathcal{O}_{\mathbb{B}}, \Gamma)$;
(2) *X satisfies* $\alpha_3(\mathcal{O}_{\mathbb{B}}, \Gamma)$;
(3) *X satisfies* $\alpha_4(\mathcal{O}_{\mathbb{B}}, \Gamma)$;
(4) *X satisfies* $\mathsf{S}_1(\mathcal{O}_{\mathbb{B}}, \Gamma)$;
(5) ONE *has no winning strategy in the game* $\mathsf{G}_1(\mathcal{O}_{\mathbb{B}}, \Gamma)$.

Theorem 6.2. *For a $\mathbb{B}$-Lindelöf space X the following are equivalent*:

(1) *X satisfies* $\alpha_2(\mathcal{O}_{\mathbb{B}}, \Gamma_{\mathbb{B}})$;
(2) *X satisfies* $\alpha_3(\mathcal{O}_{\mathbb{B}}, \Gamma_{\mathbb{B}})$;
(3) *X satisfies* $\alpha_4(\mathcal{O}_{\mathbb{B}}, \Gamma_{\mathbb{B}})$;
(4) *X satisfies* $\mathsf{S}_1(\mathcal{O}_{\mathbb{B}}, \Gamma_{\mathbb{B}})$;
(5) ONE *has no winning strategy in the game* $\mathsf{G}_1(\mathcal{O}_{\mathbb{B}}, \Gamma_{\mathbb{B}})$.

Theorem 6.3. *For a space X the properties $\alpha_2(\Gamma_{\mathbb{B}}, \Gamma)$, $\alpha_3(\Gamma_{\mathbb{B}}, \Gamma)$, $\alpha_4(\Gamma_{\mathbb{B}}, \Gamma)$, and $\mathsf{S}_1(\Gamma_{\mathbb{B}}, \Gamma)$ are equivalent.*

Theorem 6.4. *For a space X the properties $\alpha_2(\Gamma_{\mathbb{B}}, \Gamma_{\mathbb{B}})$, $\alpha_3(\Gamma_{\mathbb{B}}, \Gamma_{\mathbb{B}})$, $\alpha_4(\Gamma_{\mathbb{B}}, \Gamma_{\mathbb{B}})$, and $\mathsf{S}_1(\Gamma_{\mathbb{B}}, \Gamma_{\mathbb{B}})$ are equivalent.*

Consider $\mathbb{B}$ as a subspace of $(\mathsf{CL}(X), \mathsf{V}^+)$ and denote this space by $(\mathbb{B}, \mathsf{V}^+)$. Then we have the following two results.

Theorem 6.5. *A space X is $\mathbb{B}$-Lindelöf if and only if $(\mathbb{B}, \mathsf{V}^+)$ is ω-Lindelöf.*

Theorem 6.6. *For a space X the following are equivalent*:

(1) $(\mathbb{B}, \mathsf{V}^+)$ *satisfies* $\mathsf{S}_1(\mathcal{O}_{\mathbb{B}}, \Gamma)$;
(2) *X satisfies* $\mathsf{S}_1(\mathcal{O}_{\mathbb{B}}, \Gamma_{\mathbb{B}})$.

Acknowledgements

The authors thank the referee for his careful reading of the paper and useful comments and suggestions, which led to improvements on several places.

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