On the Zeros of a Minimal Realization*

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ABSTRACT

In an earlier work, the authors have introduced a coordinate-free, module-theoretic definition of zeros for the transfer function $G(s)$ of a linear multivariable system (A, B, C). The first contribution of this paper is the construction of an explicit $k[z]$-module isomorphism from that zero module, $Z(G)$, to V^*/R^*, where V^* is the supremal (A, B)-invariant subspace contained in $\ker C$ and R^* is the supremal (A, B)-controllable subspace contained in $\ker C$, and where (A, B, C) constitutes a minimal realization of $G(s)$. The isomorphism is developed from an exact commutative diagram of k-vector spaces. The second contribution is the introduction of a zero-signal generator and the establishment of a relation between this generator and the classic notion of blocked signal transmissions.

1. INTRODUCTION

Even since the beginnings of control theory, and particularly since the rise of "state-space" formalism, there has been a constant interplay between the "internal description" of a linear control system, written here in the discrete-

time case

$$x(t+1) = Ax(t) + Bu(t),$$

$$y(t) = Cx(t),$$

and the corresponding "external" or "input-output" description

$$G(z) = C(zI - A)^{-1}B.$$

The context here involves a field k of scalars, three finite-dimensional vector spaces U (inputs), X (states), Y (outputs), and three k-linear maps $B: U \to X$, $A: X \to X$, and $C: X \to Y$. The resulting map $G(z)$ is $k(z)$-linear, for $k(z)$ the field of rational functions with coefficients in k, on the extended[1] $k(z)$-vector space $U(z)$ to the extended $k(z)$-vector space $Y(z)$.

The *linear system* $(X, U, Y; A, B, C)$ is called a minimal realization of $G(z)$ if the dimension of X is minimal among spaces yielding A, B, C with $G(z) = C(zI - A)^{-1}B$. Minimal realizations are uniquely determined by $G(z)$; and, in particular, the $k[z]$-module structure on X defined for polynomials $p(z)$ in $k[z]$ by

$$p(z)x = p(A)x$$

is uniquely determined up to $k[z]$-module isomorphism. This module, denoted $X(G)$, will be called the *pole module* of $G(z)$, because the eigenvalues of A, which correspond to the poles of $G(z)$, are the classical poles of the system.

The use of *zeros* of a single-input, single-output system for design purposes is of the earliest origins. An interesting account can be examined in Truxal [13], who makes references to Guillemin. More recently, the zeros of a multivariable transfer function were defined by Rosenbrock [11]. Other definitions have been given: [3] compares various definitions, and [10] contains a survey of the subject. Two approaches which are particularly important for the present paper are the Wonham-Morse construction in terms of (A, B)-invariant subspace theory [14], and the Desoer-Schulman paper [1] on "blocked transmissions." Some of the technical aspects of this work are inspired also by [2, 4].

More recently, a coordinate-free, module-theoretic definition of zeros was given in [15], where it was shown that the *zero module* $Z(G)$ could be

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[1]See Section 2 for references and a review of notation.

computed in a natural way either from a Smith-Macmillan form, as in Rosenbrock's work, or from matrix factorization. Furthermore, if $G(z)$ is either monic or epic, then the zero module is closely related to, and in some cases coincides with, the pole module of an inverse system.

The first contribution of this paper is the construction of an explicit $k[z]$-module isomorphism

$$p: Z(G) \rightarrow V^*/R^*,$$

where $Z(G)$ is the zero module of $G(z)$ as defined in [15] and V^*/R^* is the module implicit in the Wonham-Morse theory [14, p. 111]. This isomorphism fits into an elaborate framework, presented in terms of an exact commutative diagram; and the main result is Theorem I, in Section 3.

This result is not surprising, being already contained, in some sense, in [13]. Alternative algebraic approaches are available in the literature, particularly [2,4,5,6,9]. A result equivalent to Theorem I can be obtained by combining results in [4,5]; and an alternative approach to part of Theorem I appears in [6]. More detailed discussion is provided in Section 3.

The second contribution relates the zero module to the "blocked transmission" intuition of [1]. Suppose $u(z)$ is an input to the system, which can be split up into polynomial and strictly proper parts in the manner

$$u(z) = u_{\text{poly}} + u_{\text{sp}}.$$

The polynomial input u_{poly} sets up a state x in X, and the state $-x$ "blocks the transmission" of the input u_{poly}. This means that, if u_{poly} is inserted into the system during negative time, ending at $t = 0$, and the "initial state" at time $t = 0$ is $-x$, then no output is emitted. A much more interesting question concerns which strictly proper inputs u_{sp} can be blocked by some state x in X. Suppose, for a moment, that $G(z): U(z) \rightarrow Y(z)$ is monic. Then the zero module $Z(G)$ can be considered as the state module for a state-output system $(Z(G), \tilde{H}_1)$, where $\tilde{H}_1: Z(G) \rightarrow U(z)$ maps "zeros" into strictly proper inputs. The pair $(Z(G), \tilde{H}_1)$ is called a "zero-signal generator"; and, if a strictly proper input u_{sp} is blocked by a state x, then $u(z) = \tilde{H}_1\omega$ for ω in $Z(G)$, and x lies in V^*.

If $G(z)$ is not monic, then the corresponding results are slightly more technical; but one can still say that states in V^* correspond to "zero signals". See Theorem II, Section 4 for details.

We view the contributions of this paper as largely methodological. We have defined a very explicit "zero module" and related it to standard and useful ideas in control theory. The proofs are for the most part new, although

a number of the results are equivalent (and sometimes the proofs are very similar) to work in papers mentioned above. We have attempted to bring the machinery of commutative algebra to bear on questions of this type, and we hope that we have succeeded to some extent.

2. POLES AND ZEROS OF TRANSFER FUNCTIONS

This section fixes notation and recalls without proof the basic facts of realization theory and the definition of the zero module of a transfer function. Treatments of realization theory similar to the one given here can be found in [7, 8, 12, 17]. The zero module was defined in [15].

Let k be a field, and let U and Y be vector spaces over k of dimensions m and p. Let $k(z)$ be the field of rational functions, and write

$$U(z) = U \otimes_k k(z), \qquad Y(z) = Y \otimes_k k(z).$$

A *transfer function* is a $k(z)$-linear map

$$G(z) : U(z) \to Y(z).$$

The choice of k-bases for U and Y gives $k(z)$-bases for $U(z)$ and $Y(z)$, and a $p \times m$ matrix representation for $G(z)$. In this paper $G(z)$ will be taken *strictly* proper: that is, the matrix representation of $G(z)$ contains only strictly proper fractions. It should be noted that this notion is basis independent.

Let $k[z]$ be the ring of polynomials, and introduce the notation

$$\Omega U = U \otimes_k k[z],$$

so that ΩU is a free $k[z]$-module of rank m. In fact, it's true that Ω is a "left adjoint functor." Concretely, for any $k[z]$-module X and any k-linear map $B : U \to X$, there exists a unique $k[z]$-linear map $\tilde{B} : \Omega U \to X$ which restricts to B on $U \subset \Omega U$. More specifically, define

$$\tilde{B}(u \otimes z^i) = z^i Bu.$$

A "dual" or right-adjoint construction may be given by $\Gamma Y = Y(z)/\Omega Y$. The module ΓY is a divisible, torsion $k[z]$-module endowed with a k-linear map $\phi : \Gamma Y \to Y$. To define this, for $y(z)$ in $Y(z)$, write
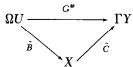
$$y(z) = y_{-n} z^n + \cdots + y_{-1} z + y_0 + y_1 z^{-1} + y_2 z^{-2} + \cdots$$

as a formal Laurent series, and define $\phi_1(y(z)) = y_1$, the coefficient of z^{-1}. Since ϕ_1 kills ΩY, it induces the required map $\phi: \Gamma Y \to Y$. Now let $C: X \to Y$ be any k-linear map out of a $k[z]$-module X. There exists a unique $k[z]$-linear map $\tilde{C}: X \to \Gamma Y$ such that $\phi \circ \tilde{C} = C$, given by the action

$$\tilde{C}x = Cxz^{-1} + C(zx)z^{-2} + \cdots .$$

The *restricted* or *Kalman* transfer function corresponding to $G(z): U(z) \to Y(z)$ is the $k[z]$-module map $G^{\#}: \Omega U \to \Gamma Y$ defined by $G^{\#} = \pi \circ G(z) \circ i$, where $i: \Omega U \to U(z)$ is the inclusion map and $\pi: Y(z) \to \Gamma Y$ is the natural projection.

A *realization* of $G^{\#}$ [or of $G(z)$] is a commutative diagram of $k[z]$-modules

$$\begin{array}{ccc}
\Omega U & \xrightarrow{\quad G^{\#} \quad} & \Gamma Y \\
 & \tilde{B} \searrow \quad \nearrow \hat{C} & \\
 & X &
\end{array}$$

where X is a finitely generated torsion $k[z]$-module, hence finite-dimensional as a vector space over k. This realization is called *reachable* if $\tilde{B}$ is epic, *observable* if $\tilde{C}$ is monic, and *canonical* or *minimal* if it is both reachable and observable. A realization with X of smallest possible dimension is automatically canonical, and its dimension is called the *Macmillan degree* of $G(z)$. Canonical realizations are unique up to an appropriate isomorphism, and in particular the state module X of canonical realization is unique up to $k[z]$-module isomorphism. This uniquely determined module will be called the *pole module* of $G(z)$. Explicitly, $X \cong \Omega U/\ker G^{\#}$, or

$$X \cong \frac{\Omega U}{G^{-1}(\Omega Y) \cap \Omega U}.$$

Associated with the realization diagram above is a triple (A, B, C) of k-linear transformations. The map $A: X \to X$, called the *dynamics*, is given by the action of z on X. The maps $B: U \to X$ and $C: X \to Y$ are given by $\tilde{B} \mid U$ and $\phi \circ \tilde{C}$, respectively. $\Sigma = (X, U, Y; A, B, C)$ is called a *linear dynamical system*. If $G(z)$ is strictly proper, the commutativity of the realization diagram is equivalent to the equation $G(z) = C(zI - A)^{-1}B$.

The *zero module* of the transfer function $G(z): U(z) \to Y(z)$ was defined in [15] by

$$Z(G) = \frac{G^{-1}(\Omega Y) + \Omega U}{\ker G + \Omega U}.$$

It is a finitely generated torsion $k[z]$-module whose invariant factors corre-
spond to the *transmission zeros*, which have been treated extensively in the
engineering literature. See [15] for more information on the zero module, and
for example [1-6, 9-11] for references to the multivariable zero literature.

3.   ZEROS AND (A, B)-INVARIANT SUBSPACES

Suppose given a linear system $\Sigma = (X, U, Y; A, B, C)$. The subspace $W \subset X$
is called (A, B)-*invariant* if $A(W) \subset W + B(U)$, or, equivalently, $(A + BF)$
$(W) \subset W$ for some k-linear map $F: X \rightarrow U$ (see [14, pp. 87–88]). In addition,
W is (A, B)-*controllable* if the system $(W, B^{-1}(W \cap B(U)), Y; (A + BF) \mid$
$W, B \mid B^{-1}(W \cap B(U)), C)$ is controllable. This notion is independent of the
choice of F [14, pp. 103–105].

Consider $\ker C$ in X. A supremal (A, B)-invariant subspace V^* and a
supremal (A, B)-controllable subspace R^* exist within $\ker C$, and $R^* \subset V^*$
[14, pp. 90, 108]. The factor space $T = V^*/R^*$ becomes a $k[z]$-module as
follows. Choose any F such that $(A + BF)(V^*) \subset V^*$. Then also $(A +
BF)(R^*) \subset R^*$, so that an induced version of $A + BF$ acts on T, and in fact
this action is independent of the choice of F [14, p. 111]. The spectrum of the
map induced by $A + BF$ on T has been identified with the *transmission zeros*
of the transfer function $G(z) = C(zI - A)^{-1}B$; see [14, pp. 112–113] and the
references cited there. More recently, [4] and [6] have discussed these
relationships from a polynomial-module point of view. In this section, we
carry this algebraicization a little further, and establish an explicit $k[z]$-mod-
ule isomorphism between $Z(G)$ and T if $G(z)$ is a strictly proper transfer
function and Σ is its minimal realization.

Suppose given the strictly proper transfer function $G(z): U(z) \rightarrow Y(z)$.
The pole module of $G(z)$ can be described as

$$X = \frac{\Omega U}{G^{-1}(\Omega Y) \cap \Omega U}.$$

The action of z on X gives the dynamics of the map A. The map $B: U \rightarrow X$
is defined by

$$B(u) = u \bmod G^{-1}(\Omega Y) \cap \Omega U.$$

It is easy to see that

$$\ker B = \ker G(z) \cap U.$$

See also the corollary to Theorem I in the next section. The map $\tilde{C}: X \to \Gamma Y$ is induced from $G^{\#}: \Omega U \to \Gamma Y$, and C is $\phi \circ \tilde{C}$, where ϕ was defined in Section 2. Define also two $k[z]$-modules

$$Z_1 = \frac{G^{-1}(\Omega Y)}{G^{-1}(\Omega Y) \cap \Omega U}$$

and

$$Z_2 = \frac{\ker G(z)}{\ker G(z) \cap \Omega U}.$$

The inclusion $\ker G(z) \subset G^{-1}(\Omega Y)$ induces a monic $k[z]$-module map $i: Z_2 \to Z_1$. Furthermore, the identity map on $G^{-1}(\Omega Y)$ induces an epic map $e: Z_1 \to Z(G)$; and it is easy to verify that the sequence

$$0 \to Z_2 \to Z_1 \to Z(G) \to 0$$

of $k[z]$-modules is exact. Consider next the k-linear map

$$p': G^{-1}(\Omega Y) \to \Omega U$$

defined by the action

$$p'(u(z)) = u_{\text{poly}},$$

where u_{poly} is the polynomial part of the rational function vector $u(z)$.

The definitions of X and Z_1 above can be applied to show that p' induces a k-linear map $p_1: Z_1 \to X$. Let $p_2: Z_2 \to X$ be given by $p_2 = p_1 \circ i$. The first goal of this section is to establish that the diagram of Figure 1 is a commutative exact diagram of k-vector spaces. Notice that Figure 1 shows R^* and V^* for codomains of p_2 and p_1, respectively, in place of X. This is part of the assertion to be proved; and we leave the p_i notation unchanged. In fact, commutativity of the diagram follows almost by definition; and exactness is routine except for three crucial points: we need to show that the image of p_1 is exactly V^*, that the image of p_2 is exactly R^*, and that p is monic.

We are indebted to the referees for raising several important questions about Figure 1. In particular they ask about module structures on V^* and R^* individually (not just on V^*/R^*), and about the significance of $\ker p_1$ and

$\ker p_2$. We do not have a useful response at present, but hope to return to these questions in a future work. For now, we can only emphasize that Figure 1 is not *a priori* a module diagram: p_1 and p_2 are just k-vector space maps. As far as we can tell, R^* has no reasonable module structure at all; and V^* has infinitely many different structures (according to Wonham, but visible here as well). Figure 1 does not seem to illuminate this particular question.

The proof of exactness involves some rather complicated computations in the state space X, and some notational conventions will be helpful. According to Section 2, we denote by $\tilde{B}u(z)$ the image in X of a polynomial vector in ΩU. If $x = \tilde{B}u(z)$ is a state, then

$$Ax = \tilde{B}zu(z).$$

For u in $U \subset \Omega U$, we write Bu in the manner $\tilde{B}u$; and for x equal to $\tilde{B}u(z)$ in X, Cx is the coefficient of z^{-1} in $G(z)u(z)$. Thus, for $u(z)$ in ΩU,

$$G(z)u(z) = y_{\text{poly}} + C \circ \tilde{B}u(z) \cdot z^{-1} + C \circ \tilde{B}zu(z) \cdot z^{-2} + \cdots,$$

where y_{poly} is an element of ΩY. From this, it is clear that $x = \tilde{B}u(z)$ lies in $\ker C$ exactly when $G(z)u(z)$ has no z^{-1} term.

The first task is to show that $p_1(Z_1) = V^*$, the supremal (A, B)-invariant subspace in $\ker C$; and the first step is to show that $p_1(Z_1) \subset \ker C$. Suppose ζ in Z_1 has representative $u(z)$ in $G^{-1}(\Omega Y)$, and write $u(z) = u_{\text{poly}} + u_{\text{sp}}$, as in our previous discussion, so that

$$G(z)u_{\text{poly}} = G(z)u(z) - G(z)u_{\text{sp}}.$$

FIG. 1.

Now by assumption $G(z)u(z)$ is a polynomial in ΩY, and $G(z)u_{sp}$ has no z^{-1} term, since $G(z)$ and u_{sp} are both strictly proper. Therefore $C \circ \tilde{B}u_{poly} = Cp_1(\zeta) = 0$, as required.

Furthermore, $p_1(Z_1)$ is (A, B)-invariant, because if

$$x = p_1(\zeta) = \tilde{B}u_{poly},$$

with

$$u_{poly} + u_1 z^{-1} + u_2 z^{-2} + \cdots \in G^{-1}(\Omega Y),$$

then

$$Ax = \tilde{B}zu_{poly}$$

$$= \tilde{B}(zu_{poly} + u_1) - \tilde{B}u_1.$$

Now $\tilde{B}u_1$ lies in $B(U)$; and

$$\tilde{B}(zu_{poly} + u_1) \in p_1(Z_1),$$

because

$$zu_{poly} + u_1 + u_2 z^{-1} + \cdots \in G^{-1}(\Omega Y),$$

so that Ax lies in $p_1(Z_1) + B(U)$. Accordingly, $p_1(Z_1) \subset V^*$.

To prove the opposite inclusion, let $F: X \to U$ be any k-linear feedback such that V^* is $(A + BF)$-invariant. Then

$$(A + BF)\tilde{B}u(z) = A\tilde{B}u(z) + BF\tilde{B}u(z)$$

$$= \tilde{B}zu(z) + \tilde{B}F\tilde{B}u(z)$$

$$= \tilde{B}\{zu(z) + F\tilde{B}u(z)\}$$

$$= \tilde{B}\{(z + F\tilde{B})u(z)\}.$$

To complete the argument, we introduce a computational lemma, which is useful in itself.

COMPUTATIONAL LEMMA.   *Suppose* $x = \tilde{B}u(z)$ *lies in* V^*. *For* $i \geqslant 1$, *define* u_i *in* U *by*

$$u_i = F\tilde{B}\{(z + F\tilde{B})^{i-1}u(z)\}.$$

Then

$$u(z) + \sum_{i=1}^{\infty} u_i z^{-i} \in G^{-1}(\Omega Y).$$

Proof of Lemma.   An easy induction shows that

$$(z + F\tilde{B})^t u(z) = z^t u(z) + u_1 z^{t-1} + u_2 z^{t-2} + \cdots + u_t$$

and that $\tilde{B}(z + F\tilde{B})^t u(z)$ lies in $V^* \subset \ker C$. Actually, more is true. For each t,

$$G(z)\{(z + F\tilde{B})^t u(z)\} = z^t v(z) + y_2 z^{-2} + y_3 z^{-3} + \cdots$$

for $v(z)$ in ΩY and y_i, possibly dependent upon t, in Y. This is clear for $t = 0$. Assume done for $t = k$, and note that

$$G(z)\{(z + F\tilde{B})^{k+1}u(z)\} = zG(z)(z + F\tilde{B})^k u(z) + G(z)F\tilde{B}(z + F\tilde{B})^k u(z)$$

$$= z^{k+1}v(z) + y_2 z^{-1} + y_3 z^{-2} + \cdots + G(z)u_{k+1}.$$

Here $G(z)u_{k+1}$ is a strictly proper power series, and the coefficient of z^{-1} vanishes because

$$(z + F\tilde{B})^{k+1}u(z)$$

is already known to be in $\ker C$. Thus

$$G(z)\left(z^t u(z) + u_1 z^{t-1} + \cdots + u_t\right) = z^t v(z) + y_2 z^{-2} + y_3 z^{-3} + \cdots,$$

so that a multiplication in both members by z^{-t} produces

$$G(z)\left(u(z) + u_1 z^{-1} + \cdots + u_t z^{-t}\right) = v(z) + y_2 z^{-t-2} + y_3 z^{-t-3} + \cdots.$$

Inasmuch as this equation holds for $t = 1, 2, 3, \ldots$, we can pass to the limit for the formal power series, which gives

$$G(z)\left(u(z) + \sum_{i=1}^{\infty} u_i z^{-i}\right) = v(z);$$

and this completes the proof of the lemma.   ∎

The lemma shows immediately that $V^* \subset p_1(Z_1)$, because, if $x = \tilde{B}u(z)$ lies in V^*, then $u(z) + \sum_{i=1}^{\infty} u_i z^{-i}$ gives an element ζ in Z_1 with $p_1(\zeta) = x$. Another proof that p_1 is epic can be found in [6, Theorem 7.1].

The analysis of Figure 1 continues with the study of $p_2 \colon Z_2 \to X$; and we must show that $\operatorname{im} p_2$ is R^*. Observe that

$$\operatorname{im} p_2 \subset \operatorname{im} p_1 = V^*.$$

Recall also that

$$(A + BF)\tilde{B}u(z) = \tilde{B}(z + F\tilde{B})u(z),$$

where V^* is $(A + BF)$-invariant. It is clearly necessary to establish that $\operatorname{im} p_2$ is $(A + BF)$-invariant. Select $x = \tilde{B}u(z)$ in $\operatorname{im} p_2$. Then there exist v_i in U such that

$$w_1(z) = u(z) + \sum_{i=1}^{\infty} v_i z^{-i} \in \ker G(z).$$

By the Lemma, write

$$w_2(z) = u(z) + \sum_{i=1}^{\infty} u_i z^{-i} \in G^{-1}(\Omega Y).$$

It follows that

$$G(z)\{w_1(z) - w_2(z)\} = y(z) \in \Omega Y,$$

and hence that $y(z)$ is equal to zero because the left member is strictly proper. With both $w_1(z)$ and $w_1(z) - w_2(z)$ in $\ker G(z)$, we have $w_2(z)$

there also, and consequently

$$zw_2(z) = zu(z) + u_1 + u_2 z^{-1} + \cdots$$

$$= zu(z) + F\tilde{B}u(z) + u_2 z^{-1} + \cdots$$

$$= (z + F\tilde{B})u(z) + u_2 z^{-1} + \cdots$$

as well. Consequently, $\tilde{B}(z + F\tilde{B})u(z)$ lies in $\operatorname{im} p_2$ as desired.

To proceed, we use a characterization of R^* given in [14, Theorem 5.5, p. 109]: namely, $R^* = \langle A + BF | B(U) \cap V^* \rangle$, the subspace of X generated by the vectors in $B(U) \cap V^*$ and their successive images under $A + BF$. We claim first that $B(U) \cap V^* \subset p_2(Z_2)$. Suppose $x = \tilde{B}u$ lies in $B(U) \cap V^*$. Then there is a strictly proper power series v_{sp} such that $G(z)(u + v_{sp}) = y(z)$ in ΩY. Again, the left member is strictly proper, while the right member is polynomial; hence both members are zero; and $u + v_{sp}$ lies in $\ker G(z)$. Thus, $\tilde{B}u$ is in $p_2(Z_2)$. Therefore $R^* \subset p_2(Z_2)$, since $p_2(Z_2)$ is $(A + BF)$-invariant.

It remains to show that $p_2(Z_2) \subset R^*$. Suppose $x = \tilde{B}u(z)$ lies in $p_2(Z_2)$, and $u(z)$ in ΩU has degree r. We will show by induction on r that x lies in R^*. If $r = 0$, then we are done because

$$p_2(Z_2) \cap B(U) \subset V^* \cap B(U) \subset R^*.$$

Assume the induction hypothesis for $r = k$, and write

$$u(z) = a_{k+1} z^{k+1} + a_k z^k + \cdots + a_1 z + a_0$$

for $a_i \in U$. Then

$$G(z)\{u(z) + v_{sp}\} = 0$$

for some strictly proper v_{sp}. A multiplication by z^{-1} shows that

$$\tilde{B}(a_{k+1} z^k + \cdots + a_1) \in \operatorname{im} p_2,$$

and hence $\in R^*$ by induction. Accordingly,

$$(A + BF)\tilde{B}(a_{k+1} z^k + \cdots + a_1) = \tilde{B}(z + F\tilde{B})(a_{k+1} z^k + \cdots + a_1)$$

$$= \tilde{B}\{a_{k+1} z^{k+1} + \cdots + a_1 z$$

$$+ F\tilde{B}(a_{k+1} z^k + \cdots + a_1)\}$$

lies in $R^* \subset p_2(Z_2)$. Consequently,

$$
\tilde{B}u(z) - \tilde{B}\{a_{k+1}z^{k+1} + \cdots + a_1 z + F\tilde{B}(a_{k+1}z^k + \cdots + a_1)\}
$$
$$
= \tilde{B}\{u(z) - a_{k+1}z^{k+1} - \cdots - a_1 z - F\tilde{B}(a_{k+1}z^k + \cdots + a_1)\}
$$
$$
= \tilde{B}\{a_0 - F\tilde{B}(a_{k+1}z^k + \cdots + a_1)\}
$$

is in $p_2(Z_2) \cap B(U) \subset R^*$. As the sum of two elements in R^*, then, $\tilde{B}u(z)$ lies in R^* as well.

So far we have shown that the top two rows of Figure 1 are exact. Diagram chasing insures the existence of an epic k-linear homomorphism $p: Z(G) \to V^*/R^*$ as shown. It remains to show that p is monic, and finally that p is a $k[z]$-module homomorphism.

Suppose that ζ in Z_1 has representative $v(z)$ in $G^{-1}(\Omega Y)$, and that $\xi = e(\zeta)$ lies in the kernel of p. Then $p_1(\zeta)$ lies in R^*, so that there exist polynomial u_{poly} and strictly proper u_{sp} such that $\tilde{B}u_{\text{poly}} = p_1(\zeta)$ and $G(z)(u_{\text{poly}} + u_{\text{sp}}) = 0$. We claim that $v(z)$ lies in $\ker G + \Omega U$, so that it represents zero in $Z(G)$. This will show that p is monic. Write $v(z) = v_{\text{poly}} + v_{\text{sp}}$. Because $\tilde{B}v_{\text{poly}} = \tilde{B}u_{\text{poly}}$ in X, it follows that $v_{\text{poly}} - u_{\text{poly}}$ is in $G^{-1}(\Omega Y) \cap \Omega U$. Subtract $u_{\text{poly}} + u_{\text{sp}}$ from $v(z)$ to conclude that $v_{\text{sp}} - u_{\text{sp}}$ lies in $G^{-1}(\Omega Y)$; hence $G(z)(v_{\text{sp}} - u_{\text{sp}}) = 0$ because it is strictly proper. Now write

$$
v_{\text{poly}} + v_{\text{sp}} = (u_{\text{poly}} + u_{\text{sp}}) + (v_{\text{sp}} - u_{\text{sp}}) + (v_{\text{poly}} - u_{\text{poly}}).
$$

The first two terms lie in $\ker G(z)$, and the third lies in ΩU; so $v(z)$ represents zero in $Z(G)$ as required.

The last gasp in this long section is to examine $p: Z(G) \to V^*/R^*$ more closely. It has been proved that p is an isomorphism of vector spaces over k, and it remains to show that p respects the $k[z]$-module structures on $Z(G)$ and V^*/R^*. But this follows from the lemma. Suppose ξ lies in $Z(G)$ and $p(\xi) = \tilde{B}u(z) \bmod R^*$ in V^*/R^* for some polynomial $u(z)$ in ΩU. Then ξ can be represented by $u(z) + \sum_{i=1}^{\infty} u_i z^{-i}$, where $u_i = F\tilde{B}\{(z + F\tilde{B})^{i-1}u(z)\}$. Therefore $p(z\xi) = \tilde{B}\{zu(z) + F\tilde{B}u(z)\} \bmod R^*$. This proves the claim, since multiplication by $z + F\tilde{B}$ corresponds to $A + BF$, the (Wonham-Morse) action on V^*/R^*.

We summarize all this work in the first main theorem of the paper.

THEOREM I.   *The diagram of Figure 1 is an exact commutative diagram of vector spaces over k. Furthermore, $p: Z(G) \to V^*/R^*$ is an isomorphism of $k[z]$-modules.*

COROLLARY [14, Exercise 4.4, p. 98]. *The transfer function $G(z)$ is left invertible (i.e., $\ker G(z) = 0$) if, and only if, $R^* = 0$ and $\ker G(z) \cap U = 0$.*

Proof. Surely $\ker G(z) = 0$ implies $Z_2 = 0$, and $R^* = 0$, since p_2 is epic. This also implies $\ker G(z) \cap U = 0$. Conversely, suppose that $R^* = 0$ and that $\ker G(z)$ contains a nonzero vector $u(z) = u_{\text{poly}} + u_{\text{sp}}$. Denote the equivalence class of $u(z)$ in $\ker G(z)/\{\ker G(z) \cap \Omega U\}$ by $[u(z)]$. Then

$$p_2[u(z)] = \tilde{B}u_{\text{poly}} = 0$$

because $R^* = 0$. But this implies that $G(z)u_{\text{poly}}$ lies in ΩY; so then it follows that

$$0 = G(z)(u_{\text{poly}} + u_{\text{sp}})$$

$$= G(z)u_{\text{poly}} + G(z)u_{\text{sp}}$$

or

$$G(z)u_{\text{poly}} = -G(z)u_{\text{sp}},$$

with left member in ΩY and right member strictly proper. From this, we have both members zero. In other words, if $R^* = 0$, the polynomial and strictly proper parts of a vector $u(z)$ in $\ker G(z)$ lie again in $\ker G(z)$. Now suppose that

$$u_{\text{sp}} = u_1 z^{-1} + u_2 z^{-1} + u_3 z^{-3} + \cdots$$

is in $\ker G(z)$. Then so is zu_{sp}, which has polynomial part u_1 in U, so that u_1 lies in $\ker G(z) \cap U$. A similar argument holds for $u_2, u_3, \ldots$. Therefore, if R^* is 0 and $\ker G(z) \cap U$ is zero, then also $\ker G(z) = 0$.

4. THE ZERO-SIGNAL GENERATOR

This section supplies an intuition about zeros somewhat different from the one in the previous section, although it will appear in the proofs that the crucial mathematics was already in place there. Roughly speaking, we show here that an input "mode" is a zero for a plant $G(z)$ if the resulting output

contains only natural "modes" of the system. Here an "input mode" is identified with a strictly proper rational vector in ΓU. "Natural modes" are outputs $y(z)$ in ΓY which lie in the image of $\tilde{C}: X \to \Gamma Y$. This view is also very closely related to the idea of blocked transmissions in Desoer and Schulman [1].

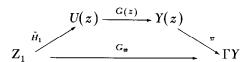The module Z_1 defined in Section 3 can be described by

$$Z_1 \cong \frac{G^{-1}(\Omega Y) + \Omega U}{\Omega U}.$$

The inclusion $G^{-1}(\Omega Y) + \Omega U \subset U(z)$ induces a monic $k[z]$-module map

$$\tilde{H}: Z_1 \to \Gamma U$$

with $\Gamma U = U(z)/U[z]$ as usual. The module Z_1 together with the map $\tilde{H}$ is called the *zero-signal generator* for $G(z)$. It should be regarded as a state-output system with state space Z_1 and observability map $\tilde{H}$. In case $\ker G(z) = 0$, so that $G(z)$ is left invertible, then $Z_1 = Z(G)$ is finite dimensional and the zero-signal generator is the output part of an essential inverse system [15]. However, $G(z)$ need not be assumed invertible here.

Let $s: \Gamma U \to U(z)$ be the k-linear map which takes an equivalence class in ΓU to its unique strictly proper representative, and let $\tilde{H}_1 = s \circ \tilde{H}: Z_1 \to U(z)$. Note that $\tilde{H}_1$ is k-linear but not $k[z]$-linear. Next define a k-linear map $G_{\#}: Z_1 \to \Gamma Y$ as $G_{\#} = \pi \circ G(z) \circ \tilde{H}_1$ in the following diagram:

The image of $G_{\#}$ will consist of outputs corresponding to the zero-signal generator. We would like to compare those with outputs arising from particular states in X under the usual output map $\tilde{C}: X \to \Gamma Y$, if $(X, U, Y; A, B, C)$ is a minimal realization of $G(z)$. Recall that V^* denotes the supremal (A, B)-invariant subspace in $\ker C$. The main facts are summarized in the second main theorem of this paper.

THEOREM II.   *The following are equivalent for an output γ in ΓY:*

(a) $\gamma = \tilde{C}x$ *for some* x *in* X, *and also* $\gamma = \pi G(z)u(z)$ *for some strictly proper input* $u(z)$ *in* $U(z)$.

(b) $\gamma = \tilde{C}x$ *for some x in X, and* $\gamma = G_{\#}\omega$ *for some ω in* Z_1.
(c) $\gamma = \tilde{C}x$ *for some x in* V^*.

Proof. We will show (a) $\rightarrow$ (b) $\rightarrow$ (c) $\rightarrow$ (a).

Assume that $u(z)$ in $U(z)$ is strictly proper, and $\pi G(z)u(z) = \tilde{C}x$. Choose a polynomial $u_0(z)$ such that $x = \tilde{B}u_0(z)$, so that $\tilde{C}x = G^{\#}u_0(z)$. This shows that $\pi G(z)(-u_0(z) + u(z)) = 0$, implying that $-u_0(z) + u(z)$ lies in $G^{-1}(\Omega Y)$ and represents an element ω in Z_1. Clearly $\pi G(z)u(z) = G_{\#}\omega = \gamma$, as required. This establishes (a) $\rightarrow$ (b).

For (b) $\rightarrow$ (c), let $x = \tilde{B}u_0(z)$ for $u_0(z)$ in ΩU. Let $v_{\text{poly}} + v_{\text{sp}}$ represent ω in Z_1 with $G_{\#}\omega = \tilde{C}x = \gamma$. This shows that $G(z)(u_0(z) - v_{\text{sp}})$ lies in ΩY, and $u_0(z) - v_{\text{sp}}$ represents an element ζ in Z_1. But $x = \tilde{B}u_0(z) = p_1(\zeta)$, where $p_1: Z_1 \rightarrow V^*$ is the crucial map of Theorem I, and so x lies in V^*.

Finally, (c) $\rightarrow$ (a) follows in much the same way. If $x \in V^*$, choose ω in Z_1 with $p_1(\omega) = x$, say ω represented by $v_{\text{poly}} + v_{\text{sp}}$. Then $x = \tilde{B}v_{\text{poly}}$ and $\tilde{C}x = G^{\#}v_{\text{poly}} = -\pi G(z)v_{\text{sp}} = \pi G(z)(-v_{\text{sp}})$. This finishes (c) $\rightarrow$ (a) and the whole proof. ∎

According to Theorem II, then, the output from a state can match the output from a strictly proper input exactly when the state lies in V^*. Furthermore, the inputs which can be used in this way are exactly the ones obtained from the zero-signal generator.

We conclude with a brief discussion of the "transmission-blocking" philosophy. Suppose $u(z) = \tilde{H}_1(\omega)$ is a "zero signal." Then $\pi G(z)u(z) = \tilde{C}x$ for some x in V^*. If the system is set to the "initial condition" $-x$ and the input $u(z)$ is fed in, then the output is identically zero. In other words, the input signal $u(z)$ can be blocked by an appropriate choice of initial condition. The converse is equally valid: a strictly proper signal that can be blocked must be a zero signal.

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