Discrete Mathematics 310 (2010) 3437–3445

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Partitions of graphs into cographs

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A R T I C L E   I N F O

Article history:
Received 16 October 2008
Received in revised form 13 July 2010
Accepted 13 July 2010
Available online 5 October 2010

Keywords:
Graph coloring
Cograph

A B S T R A C T

Cographs form the minimal family of graphs containing K_1 that is closed with respect to complementation and disjoint union. We discuss vertex partitions of graphs into the smallest number of cographs. We introduce a new parameter, calling the minimum order of such a partition the *c-chromatic number* of the graph. We begin by axiomatizing several well-known graphical parameters as motivation for this function. We present several bounds on c-chromatic number in terms of well-known expressions. We show that if a graph is triangle-free, then its chromatic number is bounded between the c-chromatic number and twice this number. We show that both bounds are sharp for graphs with arbitrarily high girth. This provides an alternative proof to a result by Broere and Mynhardt; namely, there exist triangle-free graphs with arbitrarily large c-chromatic numbers. We show that any planar graph with girth at least 11 has a c-chromatic number at most two. We close with several remarks on computational complexity. In particular, we show that computing the c-chromatic number is NP-complete for planar graphs.

1. Introduction (an axiomatization of colorings)

In the field of graph coloring, there are a plethora of variants of the notion of coloring. Always, the vertex set is partitioned so that the subgraph induced by a single color class is uniform or simplified in some way. The family of *cographs* is the minimum family of graphs containing an isolated vertex that is closed with respect to disjoint union and complementation. Thus, they form the simplest family of graphs closed with respect to disjoint union and complementation. We define the *c-chromatic number* of a graph G to be the minimum number of colors needed in a vertex coloring so that each color class induces a cograph.

We begin by axiomatizing some frequently studied types of colorings. We then extend this in a natural way to c-colorings. For undefined terms and concepts the reader is referred to [4,20]. We shall only concern ourselves with simple graphs. By c, c', c_1, c_2, etc. we shall denote positive constants. Given U, a set of vertices in a graph G, we shall denote by $\langle U\rangle_G$ the subgraph induced by U. When the host graph G is clearly understood, this will be denoted by $\langle U\rangle$. By $G+H$ we denote the disjoint union of graphs G and H. A function f is *partition invariant* if for every graph G, $f(G)$ is the least t such that $V(G)$ has a partition of order t such that f has value one on the subgraph induced by each class. A *graphical function* is a real-valued function defined on graphs. Chromatic number of graphs and most of its variations (see [1,6,8,9,15,26]) can be expressed as particular real-valued and mostly integer-valued partition invariant graphical functions. Many of these functions may be expressed by the following scheme, which resembles the definition of the dimension in other fields of mathematics: first, we postulate that some graphs G are easy and as such satisfy $f(G)=1$. Second, we postulate some simple operations (such as union or complement) under which the function is invariant. Third, we postulate the function to be partition invariant. This

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doi:10.1016/j.disc.2010.07.011

approach is taken in this paper. For example, chromatic number is partition invariant. Furthermore $1 = \chi(\overline{K_n})$, for all n. Note that, χ is the maximum function satisfying these two conditions. We note a few other examples. Consider the following.

Remark 1. If f is a graphical function satisfying each of the following for all graphs G and H:

a. $f(K_1) = 1$
b. $f(G + H) = \max\{f(G), f(H)\}$
c. f is partition invariant

then $f(G) \leq \chi(G)$, for all graphs G.

Proof. Let us suppose f is a graphical function satisfying these three conditions. From a and b we see that $f(G) = 1$ for any empty graph G. Hence the class of graphs where $f(G) = 1$ includes the set of empty graphs. So, by c, the minimum order of a partition of $V(G)$ where each part induces a graph on which f is one must be at most the minimum order of a partition where each part induces an empty graph. □

As the chromatic number satisfies each condition in this remark, we can axiomatize χ as the maximum function satisfying the three conditions a, b and c.

The *cochromatic number*, $z(G)$, of a graph G is the minimum order of a partition of $V(G)$ where each part induces a complete or empty graph. For a discussion of this parameter, see [10,9,12,13,15].

Remark 2. The function z is the maximum graphical function f satisfying each of the following:

a. $f(K_n) = 1$, for all n
b. $f(G) = f(\overline{G})$ for each graph G
c. f is partition invariant.

Remark 3. Let F be a nontrivial graph. The *F-chromatic number* of G, as defined in [3], is the minimum number of sets needed to partition $V(G)$ in such a way that no part has F as an induced subgraph. Such partitions are also known as *F-free colorings*. For example, the K_2-chromatic number is simply the traditional chromatic number. The F-chromatic number can be axiomatized as the maximum graphical function f satisfying each of the following:

a. $f(G) = 1$, for any graph G that contains no induced copy of F.
b. f is partition invariant.

Remark 4. The *subchromatic number*, discussed in [2,8] and other places, is the minimum size of a partition of $V(G)$ where no part contains three vertices that induce a path. The subchromatic number of G is denoted $\chi_s(G)$ and is sometimes referred to as the *P_3-chromatic number*. In such a partition, each part induces a union of complete graphs. We further note that χ_s is the maximum function f satisfying

a. $f(K_n) = 1$
b. $f(G + H) = \max\{f(G), f(H)\}$
c. f is partition invariant.

Remark 5. The 1-*defective coloring number*, $\chi_1(G)$, is the minimum size of a partition of $V(G)$ such that the subgraph induced by each part has maximum degree at most 1. Interesting remarks on 1-defective colorings can be found in [7,6]. We note that χ_1 is the maximum function f such that

a. $f(K_1) = f(K_2) = 1$
b. $f(G + H) = \max\{f(G), f(H)\}$
c. f is partition invariant.

Taking inspiration from these examples, we produce a "hybrid" definition which we will further explore.

Remark 6. Let c be the maximum graphical function f satisfying the following for all graphs G and H:

a. $f(K_1) = 1$
b. $f(G + H) = \max\{f(G), f(H)\}$
c. $f(G) = f(\overline{G})$
d. f is partition invariant.

We shall refer to $c(G)$ as the *c-chromatic number* of G and a partition of $V(G)$ where each part induces a graph H with $c(H) = 1$ as a *c-coloring*. This parameter is briefly studied in [21] with a different motivation.

From a, b and c in Example 2 we know if G is a cograph then $c(G) = 1$. Since c is partition invariant, $c(G)$ is the minimum order of all partitions of $V(G)$ where each part induces a cograph.

In [28] and other places cographs are characterized as those graphs which do not contain an induced copy of P_4. Hence, the c-chromatic number is the minimum size of a partition of $V(G)$ such that the subgraph induced by each part is P_4-free. Thus, c is also known as the *P_4-chromatic number*.

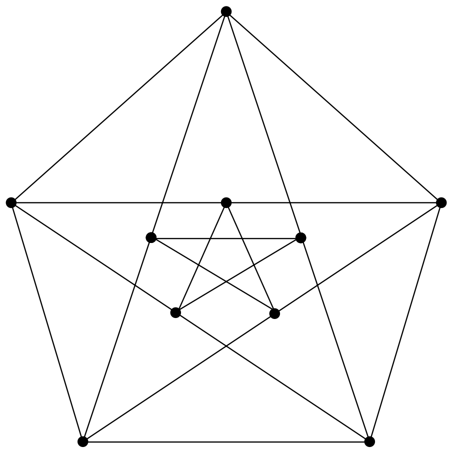

Fig. 1. A graph with $c(J) = 3$.

One can find cographs discussed in a variety of places (e.g. [5,12,26]). They are a well structured and relatively simple class. Indeed, difficult problems such as maximum clique, maximum independent set, chromatic number, minimum clique cover, and Hamiltonicity are polynomial when restricted to cographs. One can determine in linear time if a graph is a cograph. Further, one can also determine in linear time if two given cographs are isomorphic.

2. Bounds

By definition we have the following:

Remark 7. For any graph G, $c(G) \leq \min\{\chi(G), z(G), \chi_s(G), \chi_1(G)\}$.

Given graphs G and H, the join of G and H, denoted $G \oplus H$, is the graph with vertex set $V(G) \cup V(H)$; and edge set consisting of the edges of G, the edges of H, and all pairs consisting of a vertex of G and a vertex of H. Since the join of two cographs is a cograph, we have the following:

Remark 8. For any graphs G and H,

$$c(G \oplus H) = \max\{c(G), c(H)\}.$$

Given a graph G, let G_1, G_2, and G_3 be disjoint copies of G. Form a graph G' by taking the union of $G_1 \oplus G_2$, $G_2 \oplus G_3$, and $G_3 \oplus K_1$. We note the following:

Remark 9. If G is a graph then $c(G') = 1 + c(G)$.

This provides a method for building graphs with arbitrarily large c-chromatic numbers. We note that $c(P_4) = 2$. Thus, $c((P_4)') = 3$. Let us repeatedly apply the operation $'$ and each time increase the c-chromatic number by 1. To build a graph with c-chromatic number k using this method, we will need at most $(3^k - 1)/2$ vertices. However, we shall see this is not best possible.

Consider the graph J in Fig. 1. We claim that $c(J) = 3$. Otherwise, consider a c-coloring of J with the colors 1 and 2. We cannot color four of the five outer vertices of this illustration with the same color. So, without loss of generality we can assume the color 1 appears three times and color 2 appears twice. Now either the three vertices colored 1 induce a path or they induce an independent vertex and an edge. In either case, these colorings cannot be extended to a 2-coloring of J that does not contain a monochromatic induced P_4. In this example J has 10 vertices. We do not know of a graph on nine vertices with a c-chromatic number equal to 3. Later, we shall establish the correct order of growth of the maximum c-chromatic number of a graph with n vertices. Note also that J is 4-regular. As will follow from Remark 12 below, every 4-regular graph has a c-chromatic number of at most 3.

Bounds on the cochromatic number in terms of order, size and genus are developed in [10,13]. These yield the following.

Remark 10. If G has order n, size e, and genus g, then

$$c(G) \leq c_1 \frac{n}{\log n}$$

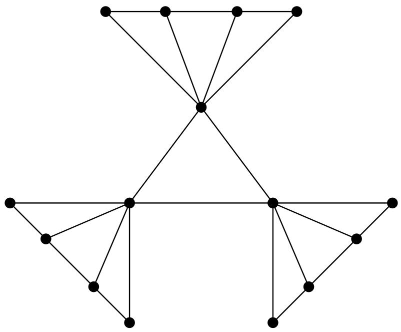

Fig. 2. Outerplanar graph with c-chromatic number equal to 3.

$$c(G) \le c_2 \frac{\sqrt{e}}{\log e}$$
$$c(G) \le c_3 \frac{\sqrt{g}}{\log g}.$$

It is shown in [9] that there exists a constant c with the property that if G is a random graph of order n with edge probability $1/2$, then almost surely every subset of G having at least $c\log(n)$ vertices contains vertices that induce P_4. Thus, each of the preceding bounds is best possible.

If H is an induced subgraph of G, then the restriction to H of any c-coloring of G is also a c-coloring. Hence, this parameter is monotone in the following sense.

Remark 11. If H is an induced subgraph of G then $c(H) \le c(G)$.

We close this section with another bound. The proof technique is found in a variety of places (e.g. [18]).

Remark 12. If G has maximum degree Δ, then $c(G) \le \lceil \frac{1+\Delta}{2} \rceil$.

Proof. Set $k = \lceil \frac{1+\Delta}{2} \rceil$. Let us color the vertices of G with k colors so that the number of monochromatic edges is minimized. Suppose this coloring contains a monochromatic path of length 3. Let u be one of the middle vertices of such a path. Some color is used at most once in the neighborhood of u. Changing u to this color decreases the number of monochromatic edges. Thus, there are no monochromatic paths of length 3 and this must be a c-coloring of G. $\square$

3. Planarity

If G is planar, then $\chi_1(G) \le 4$. This is shown in [6] without use of the four-color theorem. From [3] we know that for each natural number k there is a planar graph H having the property that every 3-coloring of H has an induced monochromatic path of length at least k. Thus, we know that there exists a planar graph G with $c(G) = 4$; hence the bound is sharp. Likewise, if G is outerplanar then $\chi(G) \le 3$ and thus $c(G) \le 3$. In Fig. 2 we give an example of an outerplanar graph with c-chromatic number equal to 3. To see this, suppose we c-color the vertices of the graph using only two colors. In doing so, the triangle at the center must have two vertices that are given the same color. These two vertices are both joined to 4-paths, and both of these paths must be 2-colored.

Grötsch's Theorem states that all triangle-free planar graphs have proper 3-colorings. Thus, if G is planar with girth at least 4 then $c(G) \le 3$. We shall see that by increasing the girth of a planar graph, we can guarantee an even lower c-chromatic number. Our argument is similar to one given in [22]. A 3-*ear* in a graph G is a path of length exactly 3 such that its internal vertices have degree 2 in G.

Lemma 1. *If G is a planar graph with girth at least 11 and no isolated vertices, then there is a partition $V_1, V_2, \ldots, V_m$ of $V(G)$ such that for all k with $1 \le k \le m-1$ either V_k consists of internal vertices of a 3-ear in $\langle V_k \cup V_{k+1} \cup \ldots \cup V_m \rangle$ or V_k is a single vertex with degree 1 in $\langle V_k \cup V_{k+1} \cup \cdots \cup V_m \rangle$.*

Proof. Suppose G is a plane graph with girth at least 11 that contains no isolated vertices. If G contains a vertex of degree 1 we put it in V_1, delete it from G, and repeat this procedure for the remaining graph. So, suppose G has no vertices of degree 1. Let us form the multigraph H by deleting all vertices of degree 2 in G and replacing each with an edge; that is, we do the reverse operation of subdividing an edge with a vertex. We note that H must have a face of length at most 5, for otherwise the dual of H would be a planar graph with minimum degree 6, a contradiction. So, consider a face of H of length at most 5. As G has girth at least 11, the corresponding cycle in G must contain at least 2 consecutive vertices of degree 2 which induce a path. Take two such internal vertices and form V_1. We can remove these vertices from G and repeat this process. In doing so, we create the desired partition. $\square$

If we subdivide each edge in the dodecahedron with a single vertex we produce a planar graph with girth 10 that has neither a pendant vertex nor 3-ear. Thus, the preceding is best possible.

Theorem 2. *If G is a planar graph with girth at least 11, then $c(G) \leq 2$.*

Proof. Suppose the statement is false. Let G be a counterexample of minimum order. We note that G has no isolated vertices, is planar, and has girth at least 11. Let $V_1, V_2, \ldots, V_m$ be a decomposition of $V(G)$ as described in the previous lemma. If V_1 consists of a pendant vertex, then we remove it from G and c-color what remains with two colors. We then color the vertex in V_1 with the color not on the vertex it is adjacent to and deduce $c(G) \leq 2$, a contradiction. So suppose V_1 contains two vertices, say u and v. Take a c-coloring of G that uses colors 1 and 2. Without loss of generality, suppose u is adjacent to a vertex colored 1. If v is also adjacent to a vertex of color 1, then we can color both u and v with 2. If v is adjacent to a vertex colored 2, then we color u with 2 and v with 1. In both cases a contradiction is reached. Hence the desired conclusion holds. □

We do not know if this bound is best possible. Perhaps it is even true that $c(G) \leq 2$ for every planar triangle-free graph G. The preceding proof can also be used to show that if G is planar with girth at least 11 then $\chi_1(G) \leq 2$. In the following section, we shall provide further remarks on girth.

4. Girth

Suppose G is a cograph without triangles. Since G has no induced P_4, there is no induced odd cycle in G. Hence, every triangle-free cograph is bipartite. This leads us to the following.

Remark 13. If G is triangle-free, then

$$c(G) \leq \chi(G) \leq 2c(G).$$

There exist triangle-free graphs with arbitrarily high chromatic number. Hence, there exist triangle-free graphs with arbitrarily large c-chromatic number, as was established with an alternate proof in [21]. As we shall see, both bounds in Remark 13 are sharp. Moreover, we show that both of these bounds are attained for graphs of arbitrarily large girth.

Theorem 3. *For every k and all ℓ at least four, there exists a graph $G_{k,\ell}$ of girth ℓ such that $\chi(G_{k,\ell}) = c(G_{k,\ell}) = k$.*

We give a short probabilistic proof. One could modify the construction of highly chromatic graphs without short cycles given in [23] to yield a constructive proof of the same result. Our proof uses standard methods, see e.g. [24,25], which go back to Erdös. Given events A and B, by $Prob(B|A)$ we shall mean $Prob(A \cap B)/Prob(A)$. We also use the well-known bound $(1-p)^k \leq e^{-pk}$, where $0 < p < 1$.

Proof. Let ℓ and k be given with $l \geq 4$. Set $t = k - 1$. For a large positive integer n, consider pairwise disjoint sets $V_1, V_2, \ldots, V_k$, each of size n. Let G be a random graph with vertex set $V = \bigcup_{i=1}^{k} V_i$ where the edges are chosen independently from the family $\{(x, y) : x \in V_i, y \in V_j, i \neq j\}$, each with probability $p = n^{\delta-1}$, where $0 < \delta < 1/\ell$.

Let us say a pair of sets C and D is *disparate* if both have cardinality $\lceil n/t \rceil$ and there exists distinct i and j where $C \subset V_i$ and $D \subset V_j$. Given disparate sets C and D, let $X(C, D)$ count the edges in $\langle C \cup D \rangle$. Let us say a disparate pair C, D is *weak* if $X(C, D) \leq 3n$. Let us set $m = \lceil n/t \rceil^2$. The expected value of X is pm, which is bounded below by $c_1 n^{1+\delta}/t^2$, where c_1 is some positive constant. By Hoeffding's inequality (see [14]), the probability that any particular disparate pair is weak is less than $e^{-c_2 n^2/t^2}$, where c_2 is some constant. Hence, the expected number of weak pairs is bounded above by B where

$$B = \binom{nk}{\frac{2n}{t}} e^{-\frac{c_2 n^2}{t^2}} \leq \left(\frac{ket}{2}\right)^{\frac{2n}{t}} e^{-\frac{c_2 n^2}{t^2}} \leq \left(\frac{k^2 e^2 t^2}{4} e^{-\frac{c_2 n}{t}}\right)^{\frac{n}{t}}.$$

Since the last expression tends to zero, we can choose n large enough to place the expected number of weak pairs arbitrarily close to zero.

Alternatively, we can set $\gamma = 1 - (3n+1)/\mu$ and $\mu = E[X(C, D)] = n^{1+\delta}/t^2 = \omega(n)$. By Chernoff's bound,

$$Pr[X(C, D) \leq 3n] = Pr\left[X(C, D) < \left(1 - \left(1 - \frac{3n+1}{\mu}\right)\right)\mu\right] < e^{-\mu\gamma^2/2}$$
$$= e^{-\mu(1-\frac{3n+1}{\mu})^2/2} \leq e^{-c_2\mu}$$

where c_2 is a positive constant. Set $q = 2\lceil n/t \rceil$. We can bound the expected number of weak disparate pairs with

$$\binom{nk}{q} e^{-c_2\mu} \leq \left(\frac{ket}{2}\right)^q e^{-c_2\mu} = \left(\frac{ket}{2}\right)^q \left(e^{-c_2\frac{n^\delta}{2t}}\right)^q = \left(\frac{ket}{2} e^{-c_2\frac{n^\delta}{2t}}\right)^q = o(1).$$

Now, let $b(G)$ count the number of edges contained in all cycles of length $3, 4, \ldots, l$ in G. By linearity of expected value we have

$$E(b(G)) \leq \sum_{j=3}^{\ell} j! \binom{kn}{j} p^j \leq \ell \frac{k^\ell n^\ell n^{\delta\ell}}{n^\ell} < c_3 n^{\delta\ell} = o(n).$$

Here c_3 is a constant that depends on k and l.

Given that $E(b(G)) = o(n)$, we see that there exists a graph G'' (an instance of the random graph G) such that

- G'' contains no disparate weak pair and
- There exist $n-1$ edges $e_1, e_2, \ldots, e_{n-1}$ such that the graph G' obtained from G'' by deleting edges $e_1, e_2, \ldots, e_{n-1}$ has girth greater than ℓ.

We now prove that G' satisfies the statement of our theorem without the girth condition. Clearly, the chromatic and c-chromatic numbers of G' are no more than k. Let $A_1, A_2, \ldots, A_{k-1}$ be a partition of V. For $1 \leq i \leq k$ let $j(i)$ denote a part $A_{j(i)}$ that satisfies

$$\left|A_{j(i)} \cap V_i\right| \geq \frac{n}{k-1} = \frac{n}{t}.$$

There are two distinct indices i and i' such that $j(i) = j(i')$. However, the set $\{\{x, x'\} : x \in A_{j(i)}, x' \in A_{j(i')}\} \cap E(G')$ has size at least $2n$ in G'. Thus $A_{j(i)}$ induces a cycle (necessarily of length greater than ℓ). Accordingly, $\langle A_{j(i)} \rangle$ has an induced path of length 3. Hence $A_1, A_2, \ldots, A_{k-1}$ fails to be a coloring as well as a c-coloring and thus $\chi(G') = c(G') = k$.

To complete the proof, take the disjoint union of G' with an ℓ-cycle, creating a graph with girth ℓ and the two other desired properties. □

Thus, the lower bound in Remark 13 is tight. As we shall see, the upper bound is tight as well.

Given G, build G' by adding a pendant edge to each vertex of G. Accordingly, if G has n vertices and m edges, the corresponding G' has $2n$ vertices and $n + m$ edges.

Theorem 4. *For ℓ and k with $\ell \geq 3$ there exists a graph with girth at least ℓ and c-chromatic number k and chromatic number* $2k$.

Our proof shares similarities with [16].

Proof. We shall establish the statement by induction on k for any fixed ℓ. We note that K_2 satisfies the statement with $k = 1$. Let H be a graph with girth ℓ such that $c(H) = k$ and $\chi(H) = 2k$. Suppose H has v vertices. By [19,23], there is a v-uniform hypergraph, say K, with girth at least ℓ and chromatic number $k + 1$. Build graph H' by starting with an independent set V corresponding to the vertices of K and disjoint G' corresponding to the edges of K such that the pendant vertices of a copy of H form the vertex set of the corresponding edge in K. Thus, for each edge in K, we take the corresponding vertices in V and form an G'. The graph H' has girth at least ℓ. If we try to color it with k colors, given that the chromatic number of K is $k + 1$, there must be an edge of K having all its vertices in H' colored the same. But this would force the copy of H attached to these vertices to have at most $k - 1$ colors, a contradiction. By assigning all vertices of V the same additional color, we see H' has chromatic number $k + 1$. Repeat this procedure, with H replaced with H'. This produces a graph H'' with girth at least ℓ and chromatic number $2k + 2$. To c-color H'' we can use the same c-coloring on each copy of the original H and add a new color for all the remaining vertices. The set of vertices colored by the new color is a star forest and is therefore a cograph. This shows that the c-chromatic number goes up by at most 1. However, since $\chi(H'')/2 \leq c(H'')$, the c-chromatic number is increased by exactly 1 and our desired result holds. □

5. Computational complexity

As mentioned, the problem of deciding if $c(G) = 1$ is in P. By [1], the problem of deciding if $c(G) \leq 2$ is NP-complete. By [27] $c(G) \leq 2$ whenever G is a chordal comparability graph. In this section we show computational complexity for planar graphs is not so simple. For a definition and discussion of gadgets, see [7,17].

Theorem 5. *The decision problem $c(G) \leq 2$ is NP-complete for planar graphs of maximum degree* 6.

Proof. We shall establish this by developing several gadgets that in polynomial time reduce the 3-SAT problem to the problem $c(G) \leq 2$ for planar graphs G of maximum degree six. Consider the graph shown at the top of Fig. 3.

Call this graph H. Note $c(H) = 2$. Further, if we c-color it with only 2 colors, the vertices marked by x must be given the same color. Further, the vertex adjacent to the two vertices labeled x must be given the other color if only two colors are allowed. We shall denote this graph with the representation at the bottom of Fig. 3 and refer to it as an *extender*.

The gadget at the top of Fig. 4 we shall call a *negater* and denote it with the diagram at the bottom of Fig. 4. We note that in any c-coloring of the negater with two colors, the vertices marked x and $\neg x$ must be given different colors. We note

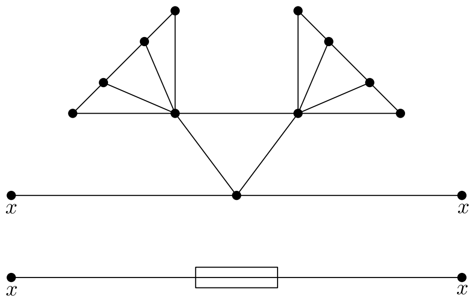

Fig. 3. Extender.

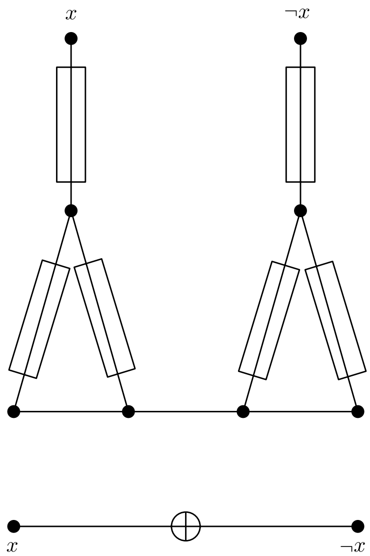

Fig. 4. Another gadget.

that both the extender and negater are planar with a maximum degree of 6. Further, there are planar embeddings where the vertices labeled x in the extender and the vertices labeled x and $\neg x$ in the negater sit on the outer region.

Consider an instance of the 3-SAT problem with variables $x_1, x_2, \ldots, x_n$ and propositions $P_1, P_2, \ldots, P_k$. First build a graph (which is probably non-planar and of very high maximum degree) on $2n$ vertices labeled $x_1, x_2, \ldots, x_n$ and $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$. For each i, extending from x_i to $\bar{x}_i$ place a negater. To this graph we now add an additional vertex, t. For each proposition $P = x \vee y \vee z$ create a path on four vertices, say t', x', y', and z'. Attach t' with an extender to t. Use negaters to attach x' to x and similarly y' to y and z' to z.

If this graph can be c-colored with two colors, then use colors 0 and 1 where without loss of generality t is labeled with 1. Such a coloring determines a truth assignment that satisfies all propositions. Further, any truth assignment that satisfies a given 3-SAT problem corresponds to a c-coloring of this graph with two colors, and given a truth assignment such a coloring can be found in polynomial time. However, such a graph may not be planar and might have vertices of degree larger than 6. We can join extenders together and use these to split vertices of large degree into collections of vertices of small degree, where each vertex in the collection must be assigned the same label in any c-coloring with two colors. So, we may assume all vertices have degree at most 6. For planarity, we refer to the gadget in Fig. 5 as an *uncrosser*. We note that in a c-coloring that uses just two labels, the vertices marked with the same letter in the uncrosser must get the same label. So, if in forming our graph we have edges which cross in some drawing, then we can replace them using an uncrosser. Thus, we may assume our constructed graph is planar. We also note the maximum degree of the uncrosser is 6.

As the number of vertices in this new graph is bounded by a polynomial function of n, we have the desired reduction. □

Let us refer to the graph in Fig. 2 as K. We note that K is outerplanar and $c(K) = 3$. Suppose we are given a planar graph G. For each vertex v in G, let us take a copy of K and attach edges joining v to each vertex in K. Call the resulting graph G'. Clearly, G' can be built from G in polynomial time. Note G' is planar. Further, $c(G') \leq 3$ if and only if $\chi(G) \leq 3$ if and only if $c(G') = 3$. By [11], testing $\chi(G) \leq 3$ is NP-complete for planar graphs. As one can tell if a graph is 2-colorable in polynomial time, we have the following.

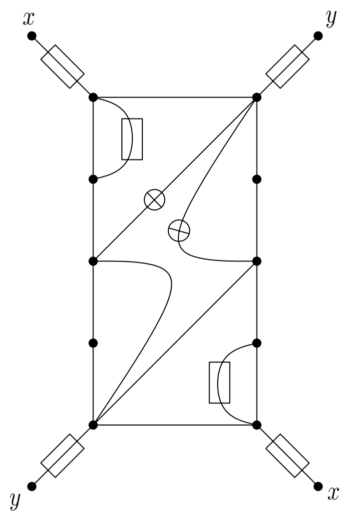

Fig. 5. Uncrosser.

Theorem 6. *The decision problem* $c(G) \leq 3$ *is NP-complete for planar graphs.*

In [28] it is shown that the decision problem $c(G) \leq 2$ is *NP*-complete for chordal graphs. This leads us to the following, which was pointed out by a referee.

Theorem 7. *For fixed* $k \geq 2$*, the decision problem* $c(G) \leq k$ *is NP-complete for chordal graphs.*

Proof. We shall establish this by induction on k, the base case being referenced above. Suppose for some k the decision problem $c(G) \leq k - 1$ is *NP*-complete for chordal graphs. Given a chordal graph G, let K be a complete graph of order $k + 1$. For each vertex v in K, make a copy of G and join it to v. Call this graph G'. Note, G' is also a chordal graph. If we c-color G' with k colors, two vertices in K, say u and v, are given the same color. If the copies of G attached to u and v use all k colors then a monochromatic induced P_4 exists in G'. So if $c(G') \leq k$ then $c(G) \leq k - 1$.

Further, if $c(G) \leq k - 1$, then we can c-color all copies of G with the same $k - 1$ colors and give the vertices of K an unused color. In this case, $c(G') \leq k$. It follows that

$$c(G) \leq k - 1 \Leftrightarrow c(G') \leq k.$$

As the transformation from G to G' is done in polynomial time, the desired conclusion is reached. □

Since $c(G) \leq 4$ for planar graphs, deciding whether $c(G) \neq 4$ is *NP*-complete for planar graphs as well. It may be the case that $c(G) = 2$ and $c(G) = 3$ are in *P* for planar triangle-free graphs. As mentioned, we do not know if there exists a planar triangle-free graph G with $c(G) = 3$.

Acknowledgements

The authors thank the referees for their excellent suggestions. Their remarks greatly improved the document. Both authors received financial support generously offered under Czech Ministry of Education Grant LN00A056.

We thank David Hartman (Prague) for the help with the final version of this article.

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