On Characterization of Best Approximation 
with Certain Constraints

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Communicated by Rong-Qing Jia

Received January 30, 1991; accepted in revised form December 3, 1996

The paper improves the characterization theorem of a best uniform approximation by a set of generalized polynomials having restricted ranges of derivatives obtained in an earlier paper and gives a characterization of a best approximation with certain constraints in the $L_p$ norm $(1 < p < +\infty)$. These results are applicable to many standard approximations with constraints.

1. INTRODUCTION

Assume $X \subset [a, b]$ is a compact set containing at least $n+1$ points, $\Phi_n = \text{span}\{\varphi_1, \ldots, \varphi_n\}$ is an $n$-dimensional subspace of $L_p[a, b]$ with $1 \leq p \leq +\infty$, and for a fixed nonnegative integer $k$, the $k$th derivatives $\varphi_1^{(k)}, \ldots, \varphi_n^{(k)}$ are continuous. For $s = 0, 1, \ldots, k$, assume that $\{\varphi_1^{(s)}, \ldots, \varphi_n^{(s)}\}$ has a maximal linearly independent subset which is an extended Chebyshev system of order $r_s$ on $[a, b]$ (see the definition in [10, Chap. 1, Sect. 2]), and write

$$K_s = \{ q \in \Phi_n : l_s(x) \leq q(x) \leq u_s(x), x \in [a, b] \},$$

where $l_s$ and $u_s$ are extended real valued functions such that $-\infty \leq l_s(x) \leq u_s(x) \leq +\infty$. Let

$$K_S = \bigcap_{s=0}^{k} K_s.$$

With respect to uniform approximation (i.e., $p = +\infty$) by $K_0$, which is the set of generalized polynomials having restricted ranges, Taylor [2] (1969) got a characterization theorem of a best approximation under the hypothesis $l_0 < u_0$. The investigation by Shih [3] (1980) allows $l_s(x_i) = u_s(x_i)$ at a set of nodes $\{x_i\}$, but some strong conditions are required. Getting rid of Shih’s strong conditions, the author [4] (1992) and Zhong [5] (1993) independently gave the characterization theorems in forms of
convex hulls and alternation in the general case of \( l_0(x) \leq u_0(x) \), which contains the special cases of approximation with interpolatory constraints, one-sided approximation, and copositive approximation. As we pointed out in [4], all the characterization theorems in [6], [7], and [8] are special cases of the case in [4]. However, the later result of Zhong [9] (1993) is not a special case of [4] because in order to apply it to the copositive case, \( \{ \varphi_1, \ldots, \varphi_n \} \) must be a Chebyshev system of order 2 while it is only required to be a Chebyshev system by [9].

Recently, we [1] got a characterization of a best uniform approximation by \( K_s \), which has many special cases such as monotone approximation, coconvex approximation, multiple comonotone approximation, approximation with Hermite-Birkhoff interpolatory side conditions, and approximation by algebraic polynomials having bounded coefficients (if \( 0 \leq a, b \)), etc.

In this paper, we first improve the result of [1] and then give a characterization theorem of a best \( L_p \) (\( 1 \leq p < +\infty \)) approximation by the product of \( K_s \) and a so-called “local convex cone.”

2. MAIN RESULTS

To introduce the main results of this paper, we need some notation. For a fixed \( q_0 \in K_s \), let

\[
d(q_0^s(x), l_s) = \inf_{\xi \in (a, b)} \sqrt{f(x - \xi)^2 + [l_s(\xi) - q_0^s(x)]^2},
\]

and define \( d(q_0^s(x), u_s) \) similarly. Write the set of all the nodes of \( K_s \) as

\[
X_s^* = \{ x \in [a, b] : d(q_0^s(x), l_s) = d(q_0^s(x), u_s) = 0 \}.
\]

If \( x \in [a, b) \), by the use of

\[
\lim_{\xi \to x^+} \frac{u_s(\xi) - q_0^s(\xi)}{[\xi - x]^{t-1}} = 0
\]

we define an integer-valued function \( t_{s, 1, 1}(x) \) as follows:

\[
t_{s, 1, 1}(x) =
\begin{cases}
0, & \text{if } x \notin X_s^* \text{ and } (1) \text{ does not hold for any positive integer } t, \\
1, & \text{if } x \in X_s^* \text{ and } (1) \text{ does not hold for any positive integer } t, \\
r, & \text{if there exists a positive integer } r < r_s \text{ such that } (1) \text{ holds for } t = r \text{ but not for } t = r + 1, \\
r_s + 1, & \text{if } (1) \text{ holds for } t = r_s \text{ but not for any positive integer } t, \\
+\infty, & \text{if } (1) \text{ holds for any positive integer } t.
\end{cases}
\]
Similarly, using
\[ \lim_{\xi \to x^+} \frac{q_n^\varepsilon(x) - l(\xi)}{|\xi - x|^{r-1}} = 0 \]  
we define \( t_{n,1,+}(x) \). And substituting \( x - 0 \) for \( x + 0 \) in (1) and (2), we define \( t_{n,-1,+}(x) \) and \( t_{n,-1,-}(x) \) respectively for \( x \in (a, b] \).

Given \( x \in [a, b] \), write

\[ t_\pm = \max \{ \min \{ t_{n,1,+}(x), t_{n,1,-}(x) \}, \min \{ t_{n,-1,+}(x), t_{n,-1,-}(x) \} \} \]
\[ \omega = (-1)^s, \]
and define

\[ t_s(x) = \begin{cases} 
    t_\pm + 1, & \text{if there exists a } v \text{ such that } t_{n,1,+}(x), t_{n,-1,-}(x) > t_\pm, \\
    t_\pm, & \text{otherwise,}
\end{cases} \]

\[ T_s = \max_{x \in [a, b]} \{ t_s(x) \}. \]

Similar to the explanation for \( t(x) \) at the end of Section 3 of [4], where \( t(x) \) coincides with \( t_s(x) \) here, we see that under the condition of (4) below \( t_s(x) \) is just the minimum of the orders of the zero \( x \) of \( q_1 - q_2 \) for all choices of \( q_1, q_2 \in K_s \). So in fact \( t_s(x) \) and \( T_s \) are independent of the choices of \( q_0 \), and hence we call \( t_s(x) \) the order of quasi-touch of \( I_x \) and \( u_s \) at \( x \), and \( T_s \) the order of quasi-touch of \( I_x \) and \( u_s \) on \([a, b] \).

In what follows we always assume that \( q_0 \in K_s \) unless otherwise stated, and for each \( s = 0, \ldots, k \),

\[ \{ q^{(s)} : q \in K_s \} \neq \emptyset \]  
and

\[ \{ \sigma^{(s)} : x \in X^*_s \} \]

where \( X^*_s \) will be defined later.

Let

\[ X^*_s = \{ x \in [a, b] \setminus X^*_s : d(q_0^{(s)}(x), I_s) \text{ or } d(q_0^{(s)}(x), u_s) = 0 \}, \]
\[ \sigma_s(x) = \begin{cases} 
    1, & \text{if } x \in X^*_s \text{ and } d(q_0^{(s)}(x), I_s) = 0, \\
    -1, & \text{if } x \in X^*_s \text{ and } d(q_0^{(s)}(x), u_s) = 0;
\end{cases} \]
\[ X^*_s = \{ x \in X^*_s : \text{there exist } \mu \text{ and } v \text{ such that } t_{\mu,v}(x) > t_s(x) \}, \]
\[ \sigma_s(x) = -v(1 - 1)^{[n-1](x)}, \text{ if } x \in X^*_s \text{ and } t_{\mu,v}(x) > t_s(x); \]
and
\[ \hat{x} = (\varphi_1(x), ..., \varphi_n(x)), \]
\[ \hat{x}^{(s + t)} = (\varphi_1^{(s + t)}(x), ..., \varphi_n^{(s + t)}(x)), \]
\[ N_x = \{ \pm \hat{x}^{(s + t)} : t = 0, 1, ..., t_j(x) - 1, x \in X_j \} \]
\[ \cup \{ -\sigma_j(x) \hat{x}^{(s + t)(x)} : x \in X_j \cup X_j' \}. \]

Moreover, for \( f \in C(X) \) or \( f \in L_p[a, b] \) with \( 1 \leq p < +\infty \), we write respectively
\[ K_{q_0}^\infty = \{ q \in \Phi_n : \| f - q \|_\infty < \| f - q_0 \|_\infty \} \]
or
\[ K_{q_0}^p = \{ q \in \Phi_n : \| f - q \|_p < \| f - q_0 \|_p \}. \]

And if \( f \in C(X) \), we write
\[ X = \{ x \in X : |f(x) - q_0(x)| = \| f - q_0 \|_\infty \} \]
and
\[ N_q = \{ -\text{sgn} [f(x) - q_0(x)] \hat{x} : x \in X \}. \]

By letting \( q_1 = \sum_{j=1}^m a_j \varphi_j \) and \( q_2 = \sum_{j=1}^m b_j \varphi_j \) be any elements of \( \Phi_n \), we define their inner product by \( (q_1, q_2) = \sum_{j=1}^m a_j b_j \). For any subset \( A \) of the space \( \Phi_n \), we define
\[ A^\circ = \{ h \in \Phi_n : (q, h) \leq 0, \forall q \in A \}. \]

Let
\[ \text{cc}(A) = \{ q : q = \sum_{j=1}^m \lambda_j q_j, q_j \in A, \lambda_j \geq 0, m \text{ is an arbitrary positive integer} \} \]
if \( A \neq \emptyset \), and \( \text{cc}(A) = \{ 0 \} \) if \( A = \emptyset \). By \( \text{cc}(A) \) we denote the closure of \( \text{cc}(A) \). And the relative interior of \( A \) in \( \Phi_n \), which we denote by \( \text{ri}(A) \), is defined as follows:
\[ \text{ri}(A) = \{ q \in \text{aff}(A) : \exists \delta > 0, O(q, \delta) \cap \text{aff}(A) \subset A \}, \]
where
\[
\text{aff}(A) := \{\lambda_1 q_1 + \cdots + \lambda_m q_m \mid q_i \in A, \lambda_1 + \cdots + \lambda_m = 1\}
\]
and \(O(q, \delta)\) is the \(\delta\)-neighborhood of \(q\).

Now we can restate the main result of [1] as follows:

**Theorem A.** Assume that \(f \in C(X) \setminus K_{\mathcal{S}}, K_{\mathcal{S}}^\infty \neq \emptyset\). If
\[
\bigcap_{s=0}^{k} \text{ri}(K_s) \neq \emptyset,
\]
then \(q_0\) is a best uniform approximation to \(f\) from \(K_{\mathcal{S}}\) if and only if there exists a vector \(h \in \mathcal{C}(N_{q_0}) \setminus \{0\}\) such that
\[
-h \in \mathcal{C} \left( \bigcup_{s=0}^{k} N_s \right).
\]

Given a subscript set \(A\), and for each \(\lambda \in A\) a real number \(d_\lambda\) and a vector \(h_\lambda \in \Phi_n \setminus \{0\}\), we say that
\[
K_\lambda := \{q \in \Phi_n : (q, h_\lambda) \leq d_\lambda, \lambda \in A\}
\]
is a local convex cone at \(q_0 \in K := \bigcap_{s=0}^{k} \text{ri}(K_s) \neq \emptyset\) if there exists a \(\delta > 0\) such that the \(\delta\)-neighborhood of \(q_0\) in \(\Phi_n\) \(O(q_0, \delta)\) satisfies
\[
O(q_0, \delta) = \{q \in \Phi_n : (q, h_\lambda) \leq d_\lambda, \lambda \in A \setminus A'\},
\]
where
\[
A' = \{\lambda \in A : (q_0, h_\lambda) = d_\lambda\}.
\]

Now, the first result of this paper is as follows:

**Theorem 1.** Assume that \(K_\lambda\) is a local convex cone at \(q_0 \in K := K_d \cap K_{\mathcal{S}}, f \in C(X) \setminus K, K_{\mathcal{S}}^\infty \neq \emptyset\). If
\[
\text{ri}(K_d) \cap \left[ \bigcap_{s=0}^{k} \text{ri}(K_s) \right] \neq \emptyset,
\]
then \(q_0\) is a best uniform approximation to \(f\) from \(K\) if and only if there exists a vector \(h \in \mathcal{C}(N_{q_0}) \setminus \{0\}\) such that
\[
-h \in \mathcal{C} \left( \{h : \lambda \in A'\} \cup \left( \bigcup_{s=0}^{k} N_s \right) \right).
\]
And if in addition $A'$ is a finite set, then (6) can be substituted by

$$-h \in \mathbb{C}\left( \{ h_j : \lambda \in A' \} \cup \left( \bigcup_{s=0}^{k} N_s \right) \right).$$

Theorem 1 improves Theorem A in two respects. First, it allows us to add some linear constraints (i.e., $(q, h_\lambda) \leq d_\nu$) to the coefficients of $q$ in $K$. For example, the set of generalized polynomials with bounded coefficients 
\[ \{ q = \sum_{i=1}^{n} a_i \varphi_i : a_i \leq \alpha_i, \beta_i, \ i = 1, ..., n \} \] is a special case of $K_{A'}. Second, when $A'$ is a finite set, $\mathbb{C}(\bullet)$ in (6) can be rewritten as $\mathbb{C}(\bullet)$, which is more precise in formulation and more valuable in applications.

The second result of the paper is a similar characterization theorem of a best approximation in the $L_p$ norm ($1 \leq p < +\infty$):

**Theorem 2.** Assume that $K_{A'}$ is a local convex cone at $q_0 \in K = K_{A'} \cap K_S$, $f \in L_p(K)$, $1 \leq p < +\infty$, $K_{q_0}^p \neq \emptyset$, and (5) holds. If $\text{mes}(f - q_0) = 0$ when $p = 1$, where $\text{mes}(f - q_0)$ is the measure of the set

\[ Z(f - q_0) = \{ x \in [a, b] : f(x) - q_0(x) = 0 \}, \]

then $q_0$ is a best $L_p$ approximation to $f$ from $K$ if and only if

$$\begin{align*}
(c_1, ..., c_n) &\in \mathbb{C}\left( \{ h_j : \lambda \in A' \} \cup \left( \bigcup_{s=0}^{k} N_s \right) \right),
\end{align*}$$

where

$$c_i = \int_a^b \varphi_i |f - q_0|^{p-1} \text{sgn}(f - q_0) \, dx, \quad i = 1, ..., n.$$ 

And if in addition $A'$ is a finite set, then (7) can be substituted by

$$\begin{align*}
(c_1, ..., c_n) &\in \mathbb{C}\left( \{ h_j : \lambda \in A' \} \cup \left( \bigcup_{s=0}^{k} N_s \right) \right).
\end{align*}$$

3. PROOF OF THEOREM 1

If we apply Theorem (6.9.7) in [11] to the case being discussed here, then the theorem can be rewritten as
**Lemma A.** Assume that $K \subset \Phi_n$ is a closed convex set, $q_0 \in K$. If $f \in C(\mathcal{X}) \setminus K$ and $K^c_m \neq \emptyset$ (or $f \in L_p[a, b] \setminus K$, $1 \leq p < +\infty$, and $K^p_{q_0} \neq \emptyset$), then $q_0$ is a best approximation to $f$ from $K$ in uniform norm (or $L_p$ norm) if and only if there exists a vector $h \in (K^c_m - q_0) \setminus \{0\}$ (or $(K^p_{q_0} - q_0) \setminus \{0\}$) such that $-h \in (K - q_0)^c$.

Now we restate Proposition (6.9.2) in [11] and Lemmas 3 and 4 in [1] as follows:

**Lemma B.** If $A \subset \Phi_n$, then

$$A^c \circ = \overline{\mathbb{C}}(A).$$

And if $A$ is a convex compact set not containing the origin, then

$$A^c \circ = \mathbb{C}(A).$$

**Lemma C.** For $s = 0, ..., k$, we have

$$(K_s - q_0)^c = \overline{\mathbb{C}}(N_s).$$

**Lemma D.** If $f \in C(\mathcal{X})$, $q_0 \in \Phi_n$, and $K^c_m \neq \emptyset$, then

$$(K^c_m - q_0)^c = \mathbb{C}(N_{q_0}).$$

**Lemma 1.** Assume $C_i$, $i = 0, 1, ..., m$, are closed convex subsets of $\Phi_n$, $0 \in \bigcap_{i=0}^m C_i$ and $\bigcap_{i=0}^m \operatorname{ri}(C_i) \neq \emptyset$, then

$$\left( \bigcap_{i=0}^m C_i \right)^c = \mathbb{C} \left( \bigcup_{i=0}^m C_i \right).$$

**Proof.** Since $(C_0)^c = \mathbb{C}(C_0)^c$, we can assume inductively

$$\left( \bigcap_{i=0}^{l-1} C_i \right)^c = \mathbb{C} \left( \bigcup_{i=0}^{l-1} C_i \right).$$

We will now prove

$$\left( \bigcap_{i=0}^l C_i \right)^c = \mathbb{C} \left( \bigcup_{i=0}^l C_i \right).$$
Take $g_0 \in \cap_{i=0}^{l} \text{ri}(C_i)$. For $j = 0, \ldots, l$, by $C_j^\circ = \text{cc}(\bigcup_{i=0}^{j} C_i)$, the definition of $(\star)^\circ$, and Lemma B we get
\[
\left( \text{cc} \left( \bigcup_{i=0}^{j} C_i \right) \right)^\circ \subseteq C_j^\circ = \text{cc}(C_j).
\]
So for any $g \in (\text{cc}(\bigcup_{i=0}^{l} C_i))^\circ$, by the convexity of $\text{cc}(C_j)$ we see that for any $\lambda \in (0, 1)$
\[
g_\lambda := \lambda g + (1 - \lambda) g_0 \in \text{cc}(C_j), \quad j = 0, 1, \ldots, l.
\]
Since $0 \in \cap_{i=0}^{l} C_i$, there exists an $\varepsilon > 0$ such that $sg_x \in \cap_{i=0}^{l} C_i$. So $g_\lambda \in \text{cc}(\bigcap_{i=0}^{l} C_i)$ and hence $g \in (\text{cc}(\bigcap_{i=0}^{l} C_i))^\circ$. So
\[
\left( \text{cc} \left( \bigcup_{i=0}^{l} C_i \right) \right)^\circ \subseteq \text{cc} \left( \bigcap_{i=0}^{l} C_i \right).
\]
On the other hand, for any $g \in \text{cc}(\bigcap_{i=0}^{l} C_i)$, based on Lemma B we have $g \in \text{cc}(C_j) = C_j^\circ$, $j = 0, 1, \ldots, l$. So by the definition of $(\star)^\circ$ we get $g \in (\text{cc}(\bigcup_{i=0}^{l} C_i))^\circ$. Then
\[
\left( \text{cc} \left( \bigcup_{i=0}^{l} C_i \right) \right)^\circ = \text{cc} \left( \bigcap_{i=0}^{l} C_i \right).
\]
Combined with Lemma B we get
\[
\left( \bigcap_{i=0}^{l} C_i \right)^\circ = \left( \text{cc} \left( \bigcap_{i=0}^{l} C_i \right) \right)^\circ = \left( \text{cc} \left( \bigcap_{i=0}^{l} C_i \right) \right)^{\circ \circ} = \text{cc} \left( \bigcup_{i=0}^{l} C_i \right).
\]
Now to complete the proof it is sufficient to show
\[
\text{cc} \left( \bigcup_{i=0}^{l} C_i \right) = \text{cc} \left( \bigcap_{i=0}^{l} C_i \right).
\]
Write $\Psi = \text{span}(\bigcap_{i=0}^{l-1} C_i)$. For any $g \in \text{cc}(\bigcup_{i=0}^{l} C_i)$, there exist $h_j \in \text{cc}(\bigcup_{i=0}^{j} C_i)$, $j = 1, 2, \ldots$ such that $h_j \rightarrow g$ (j → ∞).
Let
\[
h_j = h_{1j} + h_{2j} + h_{3j} + h_{4j},
\]
where
\[ h_{1j} + h_{2j} \in \text{cc}\left( \bigcup_{i=0}^{l-1} C_i \right) = \left( \bigcap_{i=0}^{l-1} C_i \right)^{\circ}, \] (8)
\[ h_{3j} + h_{4j} \in \text{cc}(C_j) = C_j^{\circ}, \]
\[ \left\{ h_{1j}, h_{3j} \in \mathcal{V} + \text{span} C_j, \right\} \]
\[ \left\{ h_{2j}, h_{4j} \perp \mathcal{V} + \text{span} C_j, \right\} \] (9)

From the boundedness of \( \{h_i\} \) we see that \( \{h_{2j} + h_{4j}\} \) is bounded. So there exists a subsequence of \( \{h_{2j} + h_{4j}\} \) (we still denote it by \( \{h_{2j} + h_{4j}\} \) for convenience) and a \( g_2 \perp \mathcal{V} + \text{span} C_j \) such that when \( j \to \infty \)
\[ h_{2j} + h_{4j} \to g_2 \in C_j^{\circ}. \] (10)

Since \( (h_{2j}, g) = 0 \) for any \( g \in \mathcal{V} \), by (8) we have
\[ h_{1j} = (h_{1j} + h_{2j}) - h_{2j} \in \left( \bigcap_{i=0}^{l-1} C_i \right)^{\circ}. \] (11)

Similarly
\[ h_{3j} \in C_j^{\circ}. \] (12)

Assume that \( \{|h_{1j}|\} \) is unbounded, then \( \{h_{1j}/|h_{1j}|\} \) has a subsequence which converges to an \( h \neq 0 \). And by the boundedness of \( \{h_{1j} + h_{3j}\} \) we see that \( \{h_{3j}/|h_{3j}|\} \) converges to \(-h\). Thus by (9), (11), and (12)
\[ \left\{ \begin{array}{l}
    h \in (\mathcal{V} + \text{span} C_j) \cap \left( \bigcap_{i=0}^{l-1} C_i \right)^{\circ}, \\
    -h \in C_j^{\circ}.
  \end{array} \right. \] (13)

For \( g_0 \in \bigcap_{i=0}^{m} \text{rel}(C_i) \) and any \( g \in \mathcal{V} \) there exists an \( \epsilon > 0 \) such that \( g_0 \pm \epsilon g \in \bigcap_{i=0}^{l-1} C_i \). So \( (g_0, \pm h) \leq 0 \). Since (13) implies \( (g_0, \pm h) \leq 0 \), hence \( (g_0, h) = 0 \), we have \( (g, h) = 0 \). Similarly, \( (\tilde{g}, h) = 0 \) for any \( \tilde{g} \in \text{span} C_j \). Then \( h \perp (\mathcal{V} + \text{span} C_j) \) which contradicts (13). Now we see that \( \{|h_{1j}|\} \) is bounded and hence \( \{|h_{3j}|\} \) is bounded too. So by (11) and (12) there exist \( g_1 \) and \( g_3 \) such that
\[ h_{1j} \to g_1 \in \left( \bigcap_{i=0}^{l-1} C_i \right)^{\circ}, \quad h_{3j} \to g_3 \in C_j^{\circ}. \]
(taking subsequences if necessary) when \( j \to \infty \). Thus by (10) and the inductive assumption we have

\[
g = g_1 + g_2 + g_3 \in \mathcal{C}(\bigcup_{i=0}^{j} C_j^i).
\]

**Lemma 2.** For each \( s = 0, 1, ..., k \), if

\[
\delta_s = \inf \{ |x_1 - x_2| : x_1, x_2 \in X^*_s, x_1 \neq x_2 \},
\]

then for any \( x \in X^*_s \) there exists a positive \( \delta_0 < \delta_s \) such that

\[
[x, x + \delta_0] \cap X^*_s = \emptyset \quad \text{or} \quad \sigma_s(\xi) = \sigma_s(x), \quad \xi \in (x, x + \delta_0] \cap X^*_s, \tag{14}
\]

and

\[
[x - \delta_0, x) \cap X^*_s = \emptyset \quad \text{or} \quad \sigma_s(\xi) = (-1)^{(s+1)} \sigma_s(x), \quad \xi \in [x - \delta_0, x) \cap X^*_s. \tag{15}
\]

**Proof.** Because for any \( q \in K \), we have \( q^{(s)}(x) = q^{(s)}_0(x) \), \( x \in X^*_s \), by (3) and the definition of the extended Chebyshev system we conclude that \( X^*_s \) is a finite set and hence \( \delta_s > 0 \).

Assume \( (x, x + \delta] \cap X^*_s \neq \emptyset \) for any positive \( \delta < \delta_s \).

If for any positive \( \delta < \delta_s \) there exists \( \xi, \eta \in (x, x + \delta] \cap X^*_s \) such that \( \sigma_s(\xi) = 1, \sigma_s(\eta) = -1 \), then there exist two sequences \( \{\xi_i\} \) and \( \{\eta_i\} \) such that \( \xi_i, \eta_i \to x + 0 \) \( (i \to \infty) \) and

\[
\begin{align*}
(dq^{(s)}_0(\xi_i), l_i) &= 0, \\
(dq^{(s)}_0(\eta_i), u_i) &= 0, \\
& \quad i = 1, 2, ...
\end{align*}
\]

So for any \( q \in K \) we have

\[
\begin{align*}
&q^{(s)}(\xi_i) - q^{(s)}_0(\xi_i) \geq 0, \\
&q^{(s)}(\eta_i) - q^{(s)}_0(\eta_i) \leq 0,
\end{align*}
\]

which implies that \( q^{(s)} - q^{(s)}_0 \equiv 0 \) by the definition of the extended Chebyshev system. This contradicts the hypothesis of (3). Now we see that there exists a positive \( \delta_0 < \delta_s \) such that \( \sigma_s(\xi) \equiv \text{constant} \) for any \( \xi \in (x, x + \delta_0] \cap X^*_s \). Without loss of generality, we assume that the constant equals 1. So there exists a sequence \( \{\xi_i\} \) with \( \xi_i \to x + 0 \) \( (i \to \infty) \) and \( dq^{(s)}_0(\xi_i), l_i) = 0 \). Then by the definition we get directly \( t_{s,1}(..., x) = \infty \) and \( \sigma_s(\eta) = 1 \) which implies (14). The proof of (15) is similar. \( \square \)
Lemma 3. For $0 \leq s \leq k$, $x \in X^*_r$, if there is a positive $\delta_0 < \delta$, that satisfies (14) and (15), then

$$(H - q_0)^2 = \mathcal{C}(M),$$

where

$$H = \{ q \in \Phi_n : t_q(x) \leq q^{(\ell)}(x) \leq u_q(x), x \in [x - \delta_0, x + \delta_0] \},$$

$$M = \{ \pm \hat{x}^{(s+j)} : j = 0, 1, \ldots, t_q(x) - 1 \}
\cup \{ -\sigma(x) \hat{x}^{(s+j)} : \xi \in [x - \delta_0, x + \delta_0] \cap (X^*_r \cup X^*_d) \}. $$

Proof. By $\varphi^{(s)}[x_0, x_1, \ldots, x_j]$ we denoted the difference quotient of the $j$th order of $\varphi^{(s)}$. Write

$$[x_0, x_1, \ldots, x_j]^{(s)} = (\varphi^{(s)}[x_0, x_1, \ldots, x_j], \ldots, \varphi^{(s)}[x_0, x_1, \ldots, x_j]).$$

Based on the well-known property of the difference quotient with coalescent knots we have

$$[x_0, \ldots, x_{j+1}]^{(s)} = \frac{1}{j!} \hat{x}^{(s+j)}$$

and

$$\frac{1}{x_{i+1} - x_{i}} \left( [x_0, \ldots, x_{j}]^{(s)} - \frac{1}{(j-1)!} \hat{x}^{(s+j-1)} \right) = [x_0, \ldots, x_{j}]^{(s)}.$$ (19)

Write $t_q(x)$ as $t$ for convenience. Since Lemma C implies $(H - q_0)^2 = \mathcal{C}(M)$, it is sufficient to prove that $h \in \mathcal{C}(M)$ if $h \in \mathcal{C}(M)$.

If $h = 0$, then $h \in \mathcal{C}(M)$ clearly. Otherwise, there exist $h_i \neq 0$, $i = 1, 2, \ldots$, such that $h_i \in \mathcal{C}(M)$ and

$$h_i \rightarrow h \quad (i \rightarrow \infty).$$

(i) Provided $x \in X^*_r$, let $\sigma = \sigma(x)$. Since by the definition of $t_q$ we have $t_q(\xi_0) = 0$ for any $\xi \in X^*_r$, from the Carathéodory theorem we can write

$$h_i = \sum_{j=0}^{\ell} \theta_{ij} \hat{x}^{(s+j)} + \sum_{j=1}^{m} \theta_{ij} \hat{x}^{(s+j)}.$$ (20)
where \(0 \leq m_j \leq n + 1\), \(x_j \in [x - \delta_o, x + \delta_o] \cap X^r\), and

\[
\begin{cases}
- \sigma \theta_{y_j} > 0, \\
- \delta_j(x_j) \theta_{y_j} > 0,
\end{cases} \quad j = t + 1, \ldots, t + m_j. \tag{21}
\]

Take a subsequence of \(\{h_i\}\) if necessary (still denoted by \(\{h_i\}\)) such that \(m_i = \text{a constant} m\) (clearly, \(0 \leq m \leq n + 1\)); for each \(j = t + 1, \ldots, t + m\), \(\sigma_j(x_j) (i = 1, 2, \ldots)\) is a constant; and there exists an \(x_j\) such that \(x_j \to x_j\) (\(i \to \infty\)). Then from (21), (14), and (15) we have

\[
\begin{cases}
- \sigma_j(x_j) \theta_{y_j} > 0, & \text{if } j \in J_0 := \{ j : x_j \neq x, j = t + 1, \ldots, t + m \}, \\
- \sigma \theta_{y_j} > 0, & \text{if } j \in J := \{ j : x_j = x, j = t + 1, \ldots, t + m \} \text{ and } x_j > x, \\
- (1)^j \sigma \theta_{y_j} > 0, & \text{if } j \in J := \{ j : x_j = x, j = t + 1, \ldots, t + m \} \text{ and } x_j < x.
\end{cases}
\tag{22}
\]

Let

\[
\begin{cases}
\theta'_j = \theta_{y_j}, & j \in J_0 \text{ or } j = t, \\
\theta'_j = \theta_{y_j} + \frac{1}{n} \sum_{j \in J} \theta_{y_j} (x_j - x), & l = 0, \ldots, t - 1, \\
\theta'_j = \theta_{y_j} (x_j - x), & j \in J.
\end{cases}
\tag{23}
\]

Since (19) implies

\[
\tilde{\xi}'(x) - \sum_{j=0}^{t-1} (x_j - x)^{l} \frac{1}{n} \tilde{\xi}(x) = (x_j - x) \left[ x, x_j, x \right]^{(s)} - \sum_{l=1}^{t-1} (x_j - x)^{l} \frac{1}{n} \tilde{\xi}(x) \\
= (x_j - x)^{s} \left[ x, x_j, x \right]^{(s)} - \sum_{l=2}^{t-1} (x_j - x)^{l} \frac{1}{n} \tilde{\xi}(x) \\
= \cdots \\
= (x_j - x)^{t} \left[ x, x_j, x \right]^{(s)},
\]

we can rewrite \(h_i\) as

\[
h_i = \sum_{j=0}^{t_i} \theta'_{y_j} \tilde{\xi}(x) + \sum_{j \in J} \theta_{y_j} \left[ x, x_j, x \right]^{(s)} + \sum_{j \in J_0} \theta_{y_j} \tilde{\xi}(x).
\]

Now we shall prove that the sequence \(\{A_i\}\), \(A_i := \max_{j=0, \ldots, t+m} \theta'_{y_j}\), is bounded. In fact, otherwise \(\{A_i\}\) (or its subsequence) satisfies \(A_i \to +\infty\).
(i \to \infty); \theta'_j / A_i$ has a limit $\theta_j$; and at least one of $\{\theta_j\}_{j=t}^{t+m}$ does not equal zero. Since $\lim_{i \to \infty} h_i / A_i = 0$, by (18) we see that zero equals

$$\sum_{j=0}^{t-1} \theta_j x^{(s+j)} + \left( \theta_j + \frac{1}{t} \sum_{j \in J} \theta_j \right) x^{(s+t)} + \sum_{j \in J} \theta_j x^{(t)},$$

(24)

and (21)–(23) imply

$$\begin{cases}
-\sigma \theta_t \geq 0, \\
-\sigma \theta_j \geq 0, & j \in J, \\
-\sigma (x_j) \theta_j \geq 0, & j \in J_0.
\end{cases}$$

(25)

Because the definition of extended Chebyshev system of order $r_s$ and the hypothesis $t \leq r_s$ imply that $\{x^{(s+j)}\}_{j=0}^{t-1}$ are linearly independent, therefore at least one of $\theta_j$’s ($j = t, ..., t + m$) does not equal zero. Based on Lemma 5 of [4] (substituted $\phi_u$ by span$\{\varphi_{r_1}, ..., \varphi_{r_n}\}$), there exists a $q \in K_s$ such that

$$\begin{align*}
q^{(s+t)}(x) &= 0, & j = 0, ..., t - 1, \\
\sigma q^{(s+t)}(x) &> 0, \\
\sigma (x_j) q^{(s+t)}(x) &> 0.
\end{align*}$$

So by (24) and (25) we have

$$0 = (0, q)$$

$$= \sum_{j=0}^{t-1} \theta_j q^{(s+j)}(x) + \left[ \theta_j + \frac{1}{t} \sum_{j \in J} \theta_j \right] q^{(s+t)}(x) + \sum_{j \in J} \theta_j q^{(s+t)}(x) < 0,$$

which is a contradiction. Thus $A_i$ is bounded.

Now, if we write the limit of $\theta'_j$ as $\theta_j$, then $h = \lim_{i \to \infty} h_i$ still has the form of (24). And by (25) we have $h \in \text{cc}(M)$.

(ii) If $x \notin X_s^n$, then $[x - \delta_0, x + \delta_0] \cap X_s^n = \emptyset$. So in (20) we have $m_t = 0$ and $\theta_0 = 0$. Let $A_i = \max_{j=0}^{t-1} |\theta_j|$. Then from the linear independence of $\{x^{(s+j)}\}_{j=0}^{t-1}$ it is not difficult to see that $\{A_i\}$ is bounded. So $h = \lim_{i \to \infty} h_i \in \text{cc}(M)$. 

**Lemma 4.** For each $s = 1, ..., k$,

$$(K_s - q_0)^\circ = \text{cc}(N_s).$$

(26)
Proof. Assume that $X^*_s = \{x_1, ..., x_m\}$. By Lemma 2 there exists a positive $\delta_0 < \delta_i$ such that (14) and (15) hold for every $x \in X^*_s$. Write

$$H_0 = \{ q \in \Phi_s : l_*(x) \leq q^{(i)}(x) \leq u_j(x), \, s \in [a, b] \setminus O(X^*_s, \delta_0) \},$$

$$M_0 = \{ -\sigma_j(x) \xi_j^{(i)} : x \in X^*_s \setminus O(X^*_s, \delta_0) \}.$$

For each $i = 1, ..., m$, by $H_i$ and $M_i$ we denote respectively the sets of (16) and (17) with $x$ substituted by $x_i$. Then

$$K_s = \bigcap_{i=0}^m H_s,$$

$$N_s = \bigcup_{i=0}^m M_s,$$

$$(H_s - q_0)^\circ = \text{cc}(M_s), \quad i = 1, ..., m.$$  

Suppose

$$(H_0 - q_0)^\circ = \text{cc}(M_0). \quad (27)$$

If by Lemma 5 in [4] we take a $q \in K_s$ such that

$$\begin{cases} q^{(i+j)}(x_i) = 0, & j = 0, 1, ..., t_i(x_i) - 1, \quad i = 1, ..., m, \\ \sigma_j(x) q^{(i+j)}(\xi)_j > 0, & \xi \in X^*_s \cup X^*_n. \end{cases} \quad (28)$$

then it is clear that

$$\frac{1}{m} (q - q_0) \in \bigcap_{i=0}^m \text{ri}(H_s),$$

and by Lemma 1 we have

$$(K_s - q_0)^\circ = \left[ \bigcap_{i=0}^m (H_s - q_0) \right]^\circ = \text{cc} \left( \bigcup_{i=0}^m (H_s - q_0)^\circ \right) = \text{cc}(N_s).$$

Now it is sufficient to prove (27). In fact, if $0 \notin \text{co}(M_0)$, which denotes the convex hull of $M_0$, then from Lemma B we have

$$\text{cc} (\text{co}(M_0)) = \text{cc}(\text{co}(M_0)).$$

So by Lemma C with $K_s$ replaced by $H_0$ we get (27). On the other hand, it is impossible that $0 \in \text{co}(M_0)$ because otherwise we have

$$\sum_{j=0}^r \lambda_j \sigma_j(\xi_j) \xi_j^{(n)}(\xi) = 0, \quad \lambda_j < 0, \quad \xi_j \in X^*_s \setminus O(X^*_s, \delta_0).$$
and hence for the $q$ satisfying (28)

$$\sum_{j=0}^r \lambda_j \sigma_j(\xi_j) q^{(\xi_j)} = (q, 0) = 0,$$

which contradicts the second inequality of (28).

**Lemma 5.** If $K_A \subset \Phi_n$ is a local convex cone at $q_0 \in K_A$, then

$$(K_A - q_0)^\circ = \overline{\mathbb{C}}(\{h_j : \lambda \in A'\}).$$

**Proof.** Since $[\mathbb{C}(A)]^\circ = A^\circ$, by Lemma B it is sufficient to prove that

$$\mathbb{C}(K_A - q_0) = [\mathbb{C}(\{h_j : \lambda \in A'\})]^\circ.$$

Write

$$H_z = \{q \in \Phi_n : (q, h_j) \leq d_j\}.$$

Assume $q \in \mathbb{C}(K_A - q_0)$. For any $\lambda \in A'$, it is clear that $q \in \mathbb{C}(H_z - q_0)$ and $(q + q_0, h_z) \leq d_z$. So $(q, h_z) \leq 0$, $\lambda \in A'$, and hence

$$q \in [\mathbb{C}(\{h_j : \lambda \in A'\})]^\circ.$$

On the other hand, suppose $q \notin \mathbb{C}(K_A - q_0)$. By the definition of a local convex cone there exists a $\delta > 0$ such that

$$\delta q \in H_z - q_0, \quad \lambda \notin A'.$$

If

$$\delta q \in \mathbb{C}(H_z - q_0), \quad \lambda \in A',$$

then $\delta q \in K_A - q_0$ and $q \in \mathbb{C}(K_A - q_0)$, which contradicts the hypothesis. So there exists at least one $\lambda_0 \in A'$ such that $\delta q \notin \mathbb{C}(H_{\lambda_0} - q_0)$. So

$$(\delta q, h_{\lambda_0}) > 0,$$

which implies

$$q \notin [\mathbb{C}(\{h_j : \lambda \in A'\})]^\circ.$$

The Proof of Theorem 1. By Lemmas 1, 4, and 5 we have
\[(K - q_0)^c = \mathbb{C}(\{h_j : \lambda \in A' \} \cup \bigcup_{s=0}^{k} N_s)\].

And if in addition \(A'\) is a finite set, it is clear that
\[\mathbb{C}(\{h_j : \lambda \in A'\}) = \mathbb{C}(\{h_j : \lambda \in A'\})\],
and hence
\[(K - q_0)^c = \mathbb{C}(\{h_j : \lambda \in A' \} \cup \bigcup_{s=0}^{k} N_s)\].

Combining this with Lemma A and Lemma D we get the conclusion of Theorem 1.

4. PROOF OF THEOREM 2

Lemma 6. If \(f \in L_p (1 \leq p < +\infty), q_0 \in \Phi_n, K_{q_0}^p \neq \emptyset, \) and \(\text{mes } Z(f - q_0) = 0\) when \(p = 1\), then \((c_1, ..., c_n) \neq 0\) and
\[(K_{q_0}^p - q_0)^c = \{-\eta(c_1, ..., c_n) : \eta \geq 0\}\], (29)
where the \(c_i\)’s are defined below (7).

Proof. Write \(h_0 = (c_1, ..., c_n)\).

Based on the characterization theorem of a best \(L_p\) approximation by the linear subspace \(\Phi_n\) (see [12, Theorems 3.3.1 and 3.3.2]), we see that if \(h_0 = 0\) then \(q_0\) is a best approximation to \(f\) from \(\Phi_n\), which contradicts the hypothesis of \(K_{q_0}^p \neq \emptyset\). Thus \(h_0 \neq 0\).

Now, it is sufficient to prove
\[\mathbb{C}(K_{q_0}^p - q_0) = \{-h_0\}^c\] (30)
because by Lemma B it follows from (30) that
\[(\mathbb{C}(K_{q_0}^p - q_0))^c = \mathbb{C}(\{-h_0\}),\]
which implies (29).
(i) For \( q \in \mathbb{C}(K_0^p - q_0) \), we will prove \( q \in \{ -h_0 \}^c \). Assume on the contrary that \((q, -h_0) > 0\); then there must be a \( q_1 \in \mathbb{C}(K_0^p - q_0) \) such that \((q_1, -h_0) > 0\). By the definition of \( h_0 \) we get
\[
\int_a^b q_1 |f - q_0|^{p-1} \text{sgn}(f - q_0) \, dx < 0. \tag{31}
\]
It is easy to show that
\[
\|f - q_0\|_p < \|f - q_0 - \delta q_1\|_p, \quad \forall \delta > 0. \tag{32}
\]
In fact, if \( p = 1 \), by (31) we have
\[
\|f - q_0\|_1 = \int_a^b (f - q_0 - \delta q_1) \text{sgn}(f - q_0) \, dx + \delta \int_a^b q_1 \text{sgn}(f - q_0) \, dx
\]
\[
< \|f - q_0 - \delta q_1\|_1.
\]
If \( p > 1 \), then from the Hölder Inequality we have
\[
\|f - q_0\|_p = \int_a^b (f - q_0 - \delta q_1) |f - q_0|^{p-1} \text{sgn}(f - q_0) \, dx
\]
\[
+ \delta \int_a^b q_1 |f - q_0|^{p-1} \text{sgn}(f - q_0) \, dx
\]
\[
< \int_a^b |f - q_0 - \delta q_1| \|f - q_0\|_p^{p-1} \, dx
\]
\[
\leq \|f - q_0 - \delta q_1\|_p \|f - q_0\|_p^{p-1}.
\]
And hence
\[
\|f - q_0\|_p < \|f - q_0 - \delta q_1\|_p \quad (p > 1).
\]
Now we get (32) and hence \( q_1 \notin \mathbb{C}(K_0^p - q_0) \) which is a contradiction.

(ii) If \((q_1, -h_0) < 0\), then
\[
p := (q, h_0) = \int_a^b q |f - q_0|^{p-1} \text{sgn}(f - q_0) \, dx > 0. \tag{33}
\]
Since \( q \in L_p \) and \(|f - q_0|^{p-1} \in L_{p'} \) (where \((1/p) + (1/p') = 1\), \(|q| |f - q_0|^{p-1}\) is integrable on \([a, b]\). So by Lusin’s Theorem and the property of
absolute continuity of an integral there exists a closed subset $F$ of $[a, b] \setminus Z(f - q_0)$ such that both $f - q_0$ and $q$ are continuous on $F$, and the complementary set

$$E := [a, b] \setminus Z(f - q_0) - F$$

is so small that

$$\int_{E} |q| |f - q_0|^\gamma \, dx < \frac{p}{4(2\gamma - 1 + 1)}.$$  \hspace{1cm} (34)

Clearly

$$\mu := \min_{x \in F} |f(x) - q_0(x)| > 0,$$

$$M := \max_{x \in F} \max\{|f(x) - q_0(x)|, |q(x)|\} < + \infty.$$

(a) Assume that $p = 1$. Let

$$0 < \delta < \frac{\mu}{2M}$$

Then for $x \in F$ we have

$$\text{sgn}[f(x) - q_0(x) - \delta q(x)] = \text{sgn}[f(x) - q_0(x)].$$  \hspace{1cm} (35)

So by (34), (33), and the hypothesis of $\text{mes } Z(f - q_0) = 0$ we see

$$\|f - q_0 - \delta q\|_1 \leq \int_{E} |f - q_0 - \delta q| \, dx + \int_{F} (f - q_0 - \delta q) \text{sgn}(f - q_0) \, dx$$

$$\leq \int_{E} |f - q_0| \, dx + \delta \int_{E} |q| \, dx + \int_{F} |f - q_0| \, dx$$

$$- \delta \int_{F} q \text{sgn}(f - q_0) \, dx$$

$$\leq \|f - q_0\|_1 + 2\delta \int_{E} |q| \, dx - \delta \int_{E + F} q \text{sgn}(f - q_0) \, dx$$

$$\leq \|f - q_0\|_1 + \frac{\delta p}{4} - \delta p < \|f - q_0\|_1.$$
(b) Assume that $p > 1$. Let

$$ F_+ = \{ x \in F : f(x) - q_0(x) > 0 \}, $$

$$ F_- = \{ x \in F : f(x) - q_0(x) < 0 \}, $$

$$ 0 < \delta < \min \left\{ \frac{\mu}{2M^2} \frac{\rho}{(p-1)(b-a) M^2(\mu/2)^{p-2}}, \frac{\rho}{(p-1)(b-a) M^2(2M)^{p-2}} \right\}. $$

Then (35) holds for any $x \in F = F_+ \cup F_-$. So by the Taylor Formula we have

$$ |f - q_0 - \delta q|^p = \begin{cases} (f - q_0)^p - \delta p q(f - q_0)^{p-1} & x \in F_+, \\ (q_0 - f)^p + \delta p q(q_0 - f)^{p-1} + \frac{1}{2} \delta^2 p(p-1) q^2(f - q_0 - \Delta q)^{p-2} & x \in F_- \end{cases}, \quad (36) $$

where $\Delta = \Delta(x)$ satisfies $0 < \Delta(x) < \delta$. Considering $\delta < \mu/(2M) < 1$, by the definition of $\mu$ and $M$ we get

$$ |f - q_0 - \Delta q|^{p-2} < \left\{ \begin{array}{ll} (\mu/2)^{p-2}, & p < 2, \\ (2M)^{p-2}, & p \geq 2, \end{array} \right. \quad x \in F. $$

Then from the definition of $\delta$ it follows that

$$ \frac{1}{2} \delta(p-1) \int_E q^2 |f - q_0 - \Delta q|^{p-2} dx $$

$$ \leq \frac{1}{2} \delta(p-1)(b-a) M^2 \max \{ (\mu/2)^{p-2}, (2M)^{p-2} \} \frac{\rho}{2}. \quad (37) $$

And for $x \in E$, by the Taylor Formula we have

$$ |f - q_0 - \delta q|^p \leq \left[ |f - q_0| + \delta |q| \right]^p $$

$$ = |f - q_0|^p + \delta p |q| (|f - q_0| + \Delta |q|)^{p-1} $$

$$ \leq |f - q_0|^p + \delta p 2^{p-1} |q| (f - q_0)^{p-1} + \delta p(2\Delta)^{p-1} |q|^p, \quad (38) $$

where $\Delta = \Delta(x)$ satisfies $0 < \Delta(x) < \delta$. 

---

Best Approximation with Constraints
Now, from (38), (36), (37), (34), (33), and the definition of $\delta$ we have
\[
\|f - q_0 - \delta q\|_p^p \\
= \left( \int_{E'} + \int_{Z(f - q_0)} + \int_{F} \right) |f - q_0 - \delta q|^p dx \\
\leq \left[ \|f - q_0\|_p^p + \delta p 2^{p-1} |q| \|f - q_0\|^{p-1} \right] dx \\
+ \left[ \delta^p 2^{p-1} \int_{E'} |q|^p dx + \delta^p \int_{Z(f - q_0)} |q|^p dx \right] \\
+ \int_{F} \left[ |f - q_0|^p - \delta p q |f - q_0|^{p-1} \text{sgn}(f - q_0) \right] \frac{1}{2} \delta^p (p - 1) q^2 |f - q_0 - \delta^p q|^{p-2} dx \\
+ \int_{F} \left[ |f - q_0|^p - \delta p q |f - q_0|^{p-1} \text{sgn}(f - q_0) \right] \frac{1}{2} \delta^p (p - 1) q^2 |f - q_0 - \delta^p q|^{p-2} dx \\
\leq \|f - q_0\|_p^p + \delta p 2^{p-1} \int_{E'} |q| \|f - q_0\|^{p-1} dx \\
+ \delta p \int_{E'} q |f - q_0|^{p-1} \text{sgn}(f - q_0) dx \\
- \delta p \int_{E'} q |f - q_0|^{p-1} \text{sgn}(f - q_0) dx \\
+ \delta^p 2^{p-1} |q|_p^p + \delta p \frac{1}{2} \delta^p (p - 1) q^2 |f - q_0 - \delta^p q|^{p-2} dx \\
\leq \|f - q_0\|_p^p + \delta p (2^{p-1} + 1) \int_{E'} |q| \|f - q_0\|^{p-1} dx \\
- \delta p \int_{E'} q |f - q_0|^{p-1} \text{sgn}(f - q_0) dx + \delta p \frac{1}{2} \delta^p 2^{p-1} |q|_p^p + \delta p \frac{p}{2} \\
< \|f - q_0\|_p^p + \delta p \frac{p}{4} - \delta p + \delta p \frac{p}{4} + \delta p \frac{p}{2} \\
= \|f - q_0\|_p^p.
Based on (a) and (b), we see that if \((q_0, h_0) < 0\), then there exists a \(\delta > 0\) such that \(q_0 + \delta q \in K^r_{q_0}\), which means \(q \in \text{cc}(K^r_{q_0} - q_0)\). So if \((q_0, h_0) \leq 0\) then \(q \in \text{cc}(K^r_{q_0} - q_0)\), which is

\[
\{ -h_0 \} \circ \text{cc}(K^r_{q_0} - q_0).
\]

Combining (i) with (ii) we obtain (30), and the lemma is established. □

Note. If we omit the condition that \(\text{mes}(f - q_0) = 0\) when \(p = 1\), then (29) may be false. A counterexample is as follows: Let \([a, b] = [-1, 1]\):

\[
f(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0; \end{cases} \quad n = 2; \quad \Phi_n = \text{span}(1, x), \quad \text{and} \quad q_0(x) \equiv 0.
\]

Then \(K^1_{q_0} \neq \emptyset\) since \(\|f - q_0\|_1 = 1\), and \(\|f - ((1/2) + (x/2))\|_1 < 1\). For any \(q = a_1 + a_2 x\) with \(a_i < 0\), by drawing a diagram we can find that \(\|f - q\|_1 > 1\). So

\[
a_1 \geq 0, \quad \text{if} \quad q \in K^1_{q_0}.
\]

Now let \(q_1 = (-1, 0)\). Then for any \(q \in K^1_{q_0}\) we have \((q, q_1) \leq 0\). So

\[
q_1 \in (K^1_{q_0})^\circ = (K^r_{q_0} - q_0)^\circ.
\]

But \(q_1 \notin \{ -\eta(c_1, c_2) : \eta \geq 0 \}\) since \(c_2 = \int_{-1}^1 x \text{sgn}(f - q_0) \, dx = 1/2\).

Proof of Theorem 2. The proof is similar to that of Theorem 1 in which one uses Lemma D instead of Lemma 6. □

ACKNOWLEDGMENT

The author is grateful to the referees for their valuable corrections and suggestions, which helped in the revision of the manuscript.

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