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Lower bounds on the minus domination and k-subdomination numbers☆

Liying Kang[a,*], Hong Qiao[b], Erfang Shan[a], Dingzhu Du[c]

[a] *Department of Mathematics, Shanghai University, Shanghai 200436, China*
[b] *Department of Manufacturing Engineering and Engineering Management, City University of Hong Kong, Hong Kong, China*
[c] *Department of Computer Science and Engineering, University of Minnesota, Minneapolis, MN 55455, USA*

Abstract

A three-valued function f defined on the vertex set of a graph $G=(V,E)$, $f:V\to\{-1,0,1\}$ is a *minus dominating function* if the sum of its function values over any closed neighborhood is at least one. That is, for every $v\in V$, $f(N[v])\geqslant 1$, where $N[v]$ consists of v and all vertices adjacent to v. The weight of a minus function is $f(V)=\sum_{v\in V} f(v)$. The minus domination number of a graph G, denoted by $\gamma^-(G)$, equals the minimum weight of a minus dominating function of G. In this paper, sharp lower bounds on minus domination of a bipartite graph are given. Thus, we prove a conjecture proposed by Dunbar et al. (Discrete Math. 199 (1999) 35), and we give a lower bound on $\gamma_{ks}(G)$ of a graph G.

Keywords: Domination number; Minus domination; k-subdomination

1. Introduction

For a graph $G=(V,E)$ with vertex set V and edge set E, the *open neighborhood* of $v\in V$ is $N(v)=\{u\in V\mid uv\in E\}$ and the *closed neighborhood* of v is $N[v]=\{v\}\cup N(v)$. For a set S of vertices, we define the open neighborhood $N(S)=\bigcup_{v\in S} N(v)$, and the closed neighborhood $N[S]=N(S)\cup S$. A *dominating set* S for a graph $G=(V,E)$ is a subset of the vertex set V such that every vertex $v\in V$ is either in S or adjacent to a vertex in S. The *domination number* of G, $\gamma(G)$, equals the minimum cardinality of a dominating set.

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* Corresponding author.
E-mail address: kangliying39@hotmail.com (L. Kang).

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For a real function f defined on vertices of a graph G and $S\subseteq V$, write $f(S)=\sum_{v\in S} f(v)$ and $f[v]=f(N[v])$. A *minus dominating function* of G is defined in [3] as a function $f:V\to\{-1,0,1\}$ such that $f[v]\geqslant 1$ for each $v\in V$. A *signed dominating function* of G is defined in [4] as $f:V\to\{-1,1\}$ satisfying $f[v]\geqslant 1$ for all $v\in V$. A minus (signed) dominating function f is minimal if every minus (signed) dominating function g satisfying $g(v)\leqslant f(v)$ for every $v\in V$, is equal to f. It is easy to see that a minus dominating function is minimal if and only if for every vertex $v\in V$ with $f(v)\geqslant 0$, there exists a vertex $u\in N[v]$ with $f[u]=1$ and a signed function is minimal if and only if every vertex v of weight 1, there exists some $u\in N[v]$ such that $f[u]=1$ or 2. The *minus domination number* for a graph G is $\gamma^-(G)=\min\{f(V)\mid f$ is a minimal minus dominating function$\}$. Likewise, the *signed domination number* for a graph G is $\gamma_s(G)=\min\{f(V)\mid f$ is a minimal signed dominating function$\}$.

A *majority dominating function* of G is defined in [1] as $f:V\to\{-1,1\}$ such that $f[v]\geqslant 1$ for at least half the vertices of G, and the minimum weight of such a function is the *majority domination number*.

For a positive integer k, a *k-subdominating function* (kSF) of G is a function $f:V\to\{-1,1\}$ such that $f[v]=\sum_{u\in N(v)} f(u)\geqslant 1$ for at least k vertices of G. The *aggregate* $ag(f)$ of such a function is defined by $ag(f)=\sum_{v\in V} f(v)$ and the *k-subdomination number* $\gamma_{ks}(G)$ by $\gamma_{ks}=\min\{ag(f)$: f is a kSF of $G\}$. In the special cases $k=|V|$ and $k=\lceil |V|/2\rceil$, γ_{ks} is respectively the signed domination number $\gamma_s(G)$ and the majority domination number $\gamma_{maj}(G)$.

Since the problems of determining the signed domination number and minus domination number are NP-complete, many works on bounds for $\gamma^-(G)$ and $\gamma_s(G)$ were studied in [2,5–9,11]. In [3], the following conjecture was given.

Conjecture 1 (Dunbar et al. [3]). *If G is a bipartite graph of order n, then $\gamma^-(G)\geqslant 4(\sqrt{n+1}-1)-n$.*

2. Lower bounds on minus domination of a bipartite graph

Theorem 1. *If $G=(X,Y)$ is a bipartite graph of order n, then $\gamma^-(G)\geqslant 4(\sqrt{n+1}-1)-n$.*

Proof. Let f be a minus dominating function of G satisfying $f(V)=\gamma^-(G)$ and

$$M=\{v\in V\mid f(v)=-1\},$$

$$P=\{v\in V\mid f(v)=1\},$$

$$Z=\{v\in V\mid f(v)=0\}.$$

$M_X=M\cap X$, $M_Y=M\cap Y$, $P_X=P\cap X$, $P_Y=P\cap Y$, $Z_X=Z\cap X$, $Z_Y=Z\cap Y$, $m_x=|M_X|$, $m_y=|M_Y|$, $p_x=|P_X|$, $p_y=|P_Y|$, $q_x=|Z_X|$, $q_y=|Z_Y|$.

Since $f[v] \geqslant 1$ for every $v \in V$, we have $|N(v) \cap P_X| \geqslant 2$ for every $v \in M_Y$. So

$$e(P_X, M_Y) \geqslant 2m_y. \tag{1}$$

For every $v \in P_X$, $|N(v) \cap M_Y| \leqslant |N(v) \cap P_Y|$. Then

$$\begin{aligned} e(P_X, M_Y) &= \sum_{v \in P_X} |N(v) \cap M_Y| \\ &\leqslant \sum_{v \in P_X} |N(v) \cap P_Y| \\ &\leqslant p_x p_y. \end{aligned} \tag{2}$$

By (1) and (2) we have

$$2m_y \leqslant p_x p_y.$$

Similarly,

$$2m_x \leqslant p_x p_y,$$

then

$$m_x + m_y \leqslant p_x p_y. \tag{3}$$

Since

$$n = q_x + q_y + m_x + m_y + p_x + p_y$$

and

$$2\sqrt{p_x p_y} \leqslant p_x + p_y,$$

we have

$$2\sqrt{p_x p_y} + m_x + m_y + q_x + q_y \leqslant n. \tag{4}$$

Using (3) and (4) we have

$$2\sqrt{m_x + m_y} + m_x + m_y + q_x + q_y \leqslant n \tag{5}$$

and

$$2\sqrt{m_x + m_y} + m_x + m_y \leqslant n. \tag{6}$$

By the definition, the inequalities can be deduced as follows:

$$\begin{aligned} \gamma^-(G) &= f(V(G)) \\ &= p_x + p_y - (m_x + m_y) \\ &\geqslant 2\sqrt{p_x p_y} - (m_x + m_y) \\ &\geqslant 2\sqrt{m_x + m_y} - (m_x + m_y). \end{aligned} \tag{7}$$

For notation convenience, we define the following

$$a = \sqrt{m_x + m_y},$$

$$h(y) = y^2 + 2y \ (y \geqslant 1),$$

$$g(y) = 2y - y^2 \ (y \geqslant 1).$$

Since $dh/dy = 2y + 2 \geqslant 2$, $dg/dy = 2 - 2y \leqslant 0$, so $h(y)$ is a monotonous increasing function and $g(y)$ is a monotonous decreasing function. By (6) we have $h(a) = a^2 + 2a \leqslant n$. And when $y = -1 + \sqrt{1+n}$,

$$\begin{aligned} h(y) &= (-1 + \sqrt{1+n})^2 + 2(-1 + \sqrt{1+n}) \\ &= 1 - 2\sqrt{1+n} + 1 + n - 2 + 2\sqrt{1+n} \\ &= n. \end{aligned}$$

So $a \leqslant -1 + \sqrt{1+n}$.

By (7) we obtain

$$\begin{aligned} \gamma^-(G) &\geqslant g(a) \\ &\geqslant g(-1 + \sqrt{1+n}) \\ &= 2(-1 + \sqrt{1+n}) - (-1 + \sqrt{1+n})^2 \\ &= 2(-1 + \sqrt{1+n}) - (1 - 2\sqrt{1+n} + 1 + n) \\ &= 4(\sqrt{n+1} - 1) - n. \end{aligned}$$

We now show that this bound is best possible by the following graphs G construct by Dunbar et al. [3]. Let $s \geqslant 4$ be an even integer, and let H be isomorphic to $s/2$ disjoint copies of $K_{2,s}$. Let H_1 and H_2 be two disjoint copies of H. Further, let X_i and Y_i be the sets of vertices of H_i of degree 2 and s, respectively, for $i = 1, 2$. Now let G be the graph obtained from $H_1 \cup H_2$ by joining every vertex of Y_1 to every vertex of Y_2. Then G is a bipartite graph of order $n = s(s+2)$ with partite sets $X_1 \cup Y_2$ and $X_2 \cup Y_1$. Let f be the function on G defined as follows: let $f(v) = -1$ if $v \in X_1 \cup X_2$, and let $f(v) = 1$ if $v \in Y_1 \cup Y_2$. Then it is easy to verify that f is a minus dominating function on G with $\gamma^-(G) = f(V(G)) = 2s - s^2 = 4(\sqrt{n+1} - 1) - n$. □

Theorem 2. *If $G = (X, Y)$ is a bipartite graph of order n, then*

$$\gamma^-(G) \geqslant \left\lceil n - \left(\frac{\varepsilon}{\delta} + \frac{\varepsilon}{1 + \max(\delta_X, \delta_Y)} \right) \right\rceil,$$

where $\delta_X = \min\{d(v) \mid v \in X\}$, $\delta_Y = \min\{d(v) \mid v \in Y\}$, *and the bound is sharp.*

Proof. Let $f, M_X, M_Y, P_X, P_Y, Z_X, Z_Y, m_x, m_y, q_x, q_y$, p_x and p_y be defined as in the proof of Theorem 1. For any $x \in V$, let t_x denotes the number of vertices of weight 0 in $N(x)$. Then we have

$$|N(x) \cap M| \leqslant \begin{cases} \dfrac{d(x) - t_x}{2} & \text{if } x \in P \\ \dfrac{d(x) - 1 - t_x}{2} & \text{if } x \in Z \\ \dfrac{d(x) - t_x}{2} - 1 & \text{if } x \in M. \end{cases}$$

So

$$\begin{aligned} \sum_{y \in M_Y} d(y) &= \sum_{x \in P_X} |N(x) \cap M_Y| + \sum_{x \in Z_X} |N(x) \cap M_Y| + \sum_{x \in M_X} |N(x) \cap M_Y| \\ &\leqslant \sum_{x \in P_X} \frac{d(x) - t_x}{2} + \sum_{x \in Z_X} \frac{d(x) - 1 - t_x}{2} + \sum_{x \in M_X} \left(\frac{d(x) - t_x}{2} - 1 \right) \\ &= \sum_{x \in X} \left(\frac{d(x)}{2} - \frac{t_x}{2} \right) - \frac{1}{2} q_x - m_x. \end{aligned} \tag{8}$$

Obviously,

$$m_y \delta_Y \leqslant \sum_{y \in M_Y} d(y), \tag{9}$$

$$\sum_{x \in X} \frac{t_x}{2} = \frac{1}{2} \sum_{y \in Z_Y} d(y) \geqslant \frac{1}{2} \delta_Y q_y. \tag{10}$$

Combining (8)–(10) we obtain

$$(q_y + 2m_y)\delta_Y + (q_x + 2m_x) \leqslant \varepsilon. \tag{11}$$

Similarly, we have

$$(q_x + 2m_x)\delta_X + (q_y + 2m_y) \leqslant \varepsilon. \tag{12}$$

If $q_x + 2m_x \leqslant q_y + 2m_y$, by (11) and (12) we have

$$q_x + 2m_x \leqslant \frac{\varepsilon}{1 + \max(\delta_X, \delta_Y)},$$

$$q_y + 2m_y \leqslant \frac{\varepsilon}{\delta_Y}.$$

So,

$$\begin{aligned}\gamma^-(G) &= n-(q_x+2m_x+q_y+2m_y)\\ &\geqslant \left\lceil n-\left(\frac{\varepsilon}{\delta_Y}+\frac{\varepsilon}{1+\max(\delta_X,\delta_Y)}\right)\right\rceil\\ &\geqslant \left\lceil n-\left(\frac{\varepsilon}{\delta}+\frac{\varepsilon}{1+\max(\delta_X,\delta_Y)}\right)\right\rceil.\end{aligned}$$

If $q_y+2m_y<q_x+2m_x$, by (11) and (12) we have

$$q_y+2m_y<\frac{\varepsilon}{1+\max(\delta_X,\delta_Y)},$$

$$q_x+2m_x\leqslant\frac{\varepsilon}{\delta_X}.$$

So,

$$\begin{aligned}\gamma^-(G) &= n-(q_x+2m_x+q_y+2m_y)\\ &\geqslant \left\lceil n-\left(\frac{\varepsilon}{\delta_X}+\frac{\varepsilon}{1+\max(\delta_X,\delta_Y)}\right)\right\rceil\\ &\geqslant \left\lceil n-\left(\frac{\varepsilon}{\delta}+\frac{\varepsilon}{1+\max(\delta_X,\delta_Y)}\right)\right\rceil.\end{aligned}$$

In fact, this bound is sharp, it is easy to check that $\gamma^-(K_{1,k})=1=\lceil n-\varepsilon/\delta+\varepsilon/(1+\max(\delta_X,\delta_Y))\rceil$.

3. A lower bound on *k*-subdomination number of a graph

The concept of k-subdomination was introduced by Cockayne and Mynhardt [1]. In [1], Cockayne and Mynhardt established a sharp lower bound on γ_{ks} for trees. Moreover, they also gave a sharp lower bound on γ_{ks} for trees if $k\leqslant n/2$ and proposed a conjecture.

Theorem 3 (Cockayne and Mynhardt [1]). *For any n-vertex tree T and integer* $k\in\{1,2,\ldots,n\}$, $\gamma_{ks}\leqslant 2(k+1)-n$.

Conjecture 2 (Cockayne and Mynhardt [1]). *For any n-vertex tree and any k with* $\frac{1}{2}n<k\leqslant n$, $\gamma_{ks}\leqslant 2k-n$.

In [10], the conjecture was proved and a upper bound for a connected graph was given.

Theorem 4 (Kang et al. [10]). *For any connected graph of order n and any k with $1/2n<k\leqslant n$, then*

$$\gamma_{ks} \leqslant 2\left\lceil \frac{k}{n-k+1} \right\rceil (n-k+1)-n.$$

In this section we give a lower bound for a graph G.

Theorem 5. *For any graph G of order n and size ε,*

$$\gamma_{ks} \geqslant n-\frac{2\varepsilon+(n-k)(\Delta+2)}{\delta+1}.$$

Proof. Let f be a k-subdominating function on G with $f(V)=\gamma_{ks}(G)$. Let P and M be the sets of vertices in G that are assigned the values 1 and -1, respectively. Then $|P|+|M|=n$ and $\gamma_{ks}(G)=|P|-|M|=n-2|M|$. Furthermore, we let

$$P_1=\{v\in P \mid f[v]\geqslant 1\},$$

$$P_2=P-P_1,$$

$$M_1=\{v\in M \mid f[v]\geqslant 1\},$$

$$M_2=M-M_1.$$

Clearly, $|P_1|+|M_1|\geqslant k$. Since each vertex v of P_1 is adjacent to at most $(1/2)d(v)$ vertices of M, each vertex v of M_1 is adjacent to at most $d(v)/2-1$ vertices of M. We have

$$\begin{aligned}
\delta|M| &\leqslant \sum_{v\in M} d(v)=\sum_{v\in V}|M\cap N(v)| \\
&\leqslant \sum_{v\in P_1}\frac{d(v)}{2}+\sum_{v\in M_1}\left(\frac{d(v)}{2}-1\right)+\sum_{v\in P_2\cup M_2} d(v) \\
&= \frac{1}{2}\sum_{v\in V} d(v)-|M_1|+\frac{1}{2}\sum_{v\in P_2\cup M_2} d(v) \\
&\leqslant \varepsilon-|M|+\frac{1}{2}\sum_{v\in P_2\cup M_2}(d(v)+2) \\
&\leqslant \varepsilon-|M|+(|P_2|+|M_2|)\frac{\Delta+2}{2}.
\end{aligned}$$

As $|P_2|+|M_2|\leqslant n-k$, it follows that

$$|M| \leqslant \frac{2\varepsilon+(n-k)(\Delta+2)}{2(\delta+1)}.$$

Thus,

$$\begin{aligned}\gamma_{ks} &= n - 2|M| \\ &\geqslant n - \frac{2\varepsilon + (n-k)(\Delta+2)}{\delta+1}.\end{aligned}$$

This completes the proof of Theorem 5. □

For the graphs in which each vertex has odd degree, the lower bound on γ_{ks} in Theorem 5 can be improved slightly.

Theorem 6. *For every graph G in which each vertex has odd degree,*

$$\gamma_{ks} \geqslant n - \frac{2\varepsilon + (n-k)(\Delta+2) - k}{\delta+1}.$$

Proof. Let f, P, M, P_1, P_2, M_1 and M_2 be defined as in the proof of Theorem 5.

Since every vertex of G has odd degree, it is easy to see that each vertex v of P_1 is adjacent to at most $(d(v)-1)/2$ vertices of M, each vertex v of M_1 is adjacent to at most $(d(v)-1)/2-1$ vertices of M. Hence, we have

$$\begin{aligned}\delta|M| &\leqslant \sum_{v\in M} d(v) = \sum_{v\in V} |M \cap N(v)| \\ &\leqslant \sum_{v\in P_1} \frac{d(v)-1}{2} + \sum_{v\in M_1}\left(\frac{d(v)-1}{2} - 1\right) + \sum_{v\in P_2\cup M_2} d(v) \\ &= \frac{1}{2}\sum_{v\in V} d(v) - \frac{1}{2}(|P_1|+|M_1|) - |M_1| + \frac{1}{2}\sum_{v\in P_2\cup M_2} d(v) \\ &\leqslant \varepsilon - \frac{1}{2}(|P_1|+|M_1|) - |M| + \sum_{P_2\cup M_2}\frac{d(v)+2}{2} \\ &\leqslant \varepsilon - \frac{1}{2}(|P_1|+|M_1|) - |M| + (|P_2|+|M_2|)\frac{\Delta+2}{2}.\end{aligned}$$

Since $|P_1|+|M_1|\geqslant k$, $|P_2|+|M_2|\leqslant n-k$, we have

$$(\delta+1)|M| \leqslant \varepsilon - \frac{k}{2} + \frac{(\Delta+2)(n-k)}{2}.$$

Hence,

$$|M| \leqslant \frac{2\varepsilon + (\Delta+2)(n-k) - k}{2(\delta+1)}.$$

Thus,

$$\begin{aligned}\gamma_{ks}(G) &= n - 2|M| \\ &\geqslant n - \frac{2\varepsilon + (\Delta+2)(n-k) - k}{\delta+1}.\end{aligned}$$

This completes the proof of Theorem 6. □

By Theorems 5 and 6, we easily obtain the following lower bounds on γ_{ks} for r-regular graphs.

Theorem 7. *Let G be a r-regular graph of order n, then*

$$\gamma_{ks} \geqslant \begin{cases} \dfrac{r+2}{r+1}k - n & \text{for } r \text{ even}, \\ \dfrac{r+3}{r+1}k - n & \text{for } r \text{ odd}. \end{cases}$$

In the special cases where $k=|V|$ and $k=\lceil |V|/2 \rceil$, Theorem 7 deduces to the following results.

Corollary 7 (Henning [7]). *For every r-regular graph G of order n,*

$$\gamma_{s}(G) \geqslant \begin{cases} \dfrac{2n}{r+1} & \text{for } r \text{ odd}, \\ \dfrac{n}{r+1} & \text{for } r \text{ even} \end{cases}$$

and the bounds are sharp.

Corollary 8 (Henning [7]). *For every r-regular* ($r \geqslant 2$) *graph G of order n,*

$$\gamma_{\mathrm{maj}}(G) \geqslant \begin{cases} \dfrac{1-r}{2(r+1)}n & \text{for } r \text{ odd}, \\ \dfrac{-r}{2(r+1)}n & \text{for } r \text{ even} \end{cases}$$

and the bounds are sharp.

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