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A Generalized Invariant Imbedding Equation II

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I. INTRODUCTION

In a previous paper [6], the technique of invariant imbedding was extended to a class of two-point boundary value problems for a first order differential equation in a Banach Space X. If $x(t)$ denotes the solution to the equation then the boundary conditions were of the form $P_1x(0) + P_2x(T) = v$, $v \in X$ and P_1, P_2 bounded linear. In addition, we required that $P_1 + P_2$ have a bounded inverse. This condition is typical of those boundary value problems that arise in Control Theory [2] and Transport Theory [3]. However, it does not cover such important problems as Dirichlet conditions for second order scalar equations.

In this paper, we relax the condition that $P_1 + P_2$ be invertible. The main difficulty in this case is the derivation of appropriate initial conditions for the invariant imbedding equation. In fact, the solutions in general become unbounded as $T \to 0$. In order to retain the method as a computational device, we supplement the invariant imbedding equation with the asymptotic behavior of the solution as $T \to 0$. In the linear case, one can effect this analysis completely; but only partially for the non-linear situation. This analysis constitutes the bulk of the paper.

II. NOTATION

The notation for Calculus in Banach Spaces is that of [4]. X will denote a Banach Space. If $F(x, y)$ is a function from $X \times Y \to Z$, $F_x(x, y)$ will denote the partial Frechet differential. The Landau O and o notation follows that in [7]. $L(X)$ will denote the space of bounded linear operators on X.

III. An Existence Theorem

We consider the family of two-point boundary value problems:

$$\frac{dx(t)}{dt} = Ax(t) + f(t, x(t)), \qquad x(t) \in X, \tag{3.1}$$

$$P_1x(0) + P_2x(T) = v, \qquad v \in X, \quad 0 < T \leqslant \bar{T}, \tag{3.2}$$
$$P_1, P_2, A \in L(X).$$

Definition. The triple of operators $\{P_1, P_2, A\}$ will be called a boundary compatible set if for $0 < T \leqslant \bar{T}$ the operator $P_1 + P_2e^{TA}$ has a bounded inverse.

The first thing that we do is to establish a suitable integral representation of (3.1) and (3.2).

Lemma 3.1. *Let $\{P_1, P_2, A\}$ be a boundary compatible set. Let $f(t)$ be a continuous function on* $[0, T]$. *Then the linear problem*

$$\frac{dx(t)}{dt} = Ax(t) + f(t), \qquad x(t) \in X, \tag{3.3}$$

$$P_1x(0) + P_2x(T) = v, \qquad v \in X, \tag{3.4}$$

has a unique solution given by

$$x(t) = e^{tA}H_T(A)\, v + \int_0^T G(t, \tau) f(\tau)\, d\tau, \tag{3.5}$$

where

$$H_T(A) = (P_1 + P_2e^{TA})^{-1}, \tag{3.6}$$

and

$$G(t, \tau) = \begin{cases} e^{tA}[I - H_T(A)\, P_2e^{TA}]\, e^{-\tau A}, & 0 \leqslant \tau < t. \\ -\, e^{tA}[H_T(A)\, P_2e^{TA}]\, e^{-\tau A}, & t \leqslant \tau \leqslant T. \end{cases} \tag{3.7}$$

Proof. Using the variation of constants formula [7], the general solution to (3.3) is given by

$$x(t) = e^{tA}\xi + \int_0^t e^{(t-\tau)A}f(\tau)\, d\tau, \qquad \xi \in X. \tag{3.8}$$

Using this in (3.4) gives

$$P_1x(0) + P_2x(T) = (P_1 + P_2e^{TA})\, \xi + \int_0^T P_2e^{(T-\tau)A}f(\tau)\, d\tau. \tag{3.9}$$

Using the fact that $P_1 + P_2 e^{TA}$ is invertible, we can solve uniquely for ξ getting

$$\xi = H_T(A)\, v - \int_0^T H_T(A)\, P_2 e^{(T-\tau)A} f(\tau)\, d\tau. \tag{3.10}$$

Substituting (3.10) in (3.8) and rearranging gives (3.5). Q.E.D.

THEOREM 3.1. *Let P_1, P_2, A be boundary compatible. Assume $\exists\ c_1 > 0$ such that*

$$\| H_T(A)\| \leqslant c_1 \left(1 + \frac{1}{T}\right), \qquad 0 < T \leqslant \bar{T}.$$

Let $f(t, x)$ be continuous from $[0, \bar{T}] \times X \to X$ and be uniformly Lipshitz in its second variable with Lipshitz constant K. Let $c_2 = c_1 \| P_2 \|$. If $2c_2 K < 1$ $\exists 0 < s \leqslant \bar{T}$ such that for $0 < T \leqslant s$ (3.1) and (3.2) have unique solutions.

Proof. By Lemma 3.1, (3.1), (3.2) has a unique solution iff the integral equation

$$x(t) = e^{tA} H_T(A)\, v + \int_0^T G(t, \tau)\, f(\tau, x(\tau))\, d\tau, \tag{3.11}$$

has a unique continuous solution.

We let $C^\circ[0, T]$ be the Banach Space of continuous functions from $[0, T] \to X$ with the norm of $x(t)$ given by

$$\| x(t)\|_\infty = \sup_{t\in[0,T]} \| x(t)\| .$$

To prove the theorem it suffices to show that the map $\phi_T : C^\circ[0, T] \to C^\circ[0, T]$ given by

$$\phi_T(x) = e^{tA} H_T(A)\, v + \int_0^T G(t, \tau)\, f(\tau, x(\tau))\, d\tau \tag{3.12}$$

is contracting and then apply the contraction mapping principle. Therefore we consider

$$\phi_T(x) - \phi_T(y) = \int_0^T G(t, \tau)\, [f(\tau, x(\tau)) - f(\tau, y(\tau))]\, d\tau. \tag{3.13}$$

Using the Lipshitz property of f we get that

$$\| \phi_T(x) - \phi_T(y)\|_\infty \leqslant K \| x - y \|_\infty \int_0^T \| G(t, \tau)\|\, d\tau. \tag{3.14}$$

We now estimate $\int_0^T \| G(t, \tau)\| \, d\tau$. From the assumption on $H_T(A)$ we get that

$$\| I - H_T(A) P_2 e^{TA} \| \leqslant 1 + c_2 e^{T\|A\|} + \frac{c_2}{T} e^{T\|A\|}. \tag{3.15}$$

Using (3.15) and the expression for $G(t, \tau)$ given in Lemma 3.1, we get that

$$\begin{aligned} \int_0^T \| G(t, \tau)\| \, d\tau &\leqslant \left\{1 + c_2 e^{T\|A\|} + \frac{c_2}{T} e^{T\|A\|}\right\} \int_0^t e^{(t+\tau)\|A\|} \, d\tau \\ &\quad + \left\{c_2 e^{T\|A\|} + \frac{c_2}{T} e^{T\|A\|}\right\} \int_t^T e^{(t+\tau)\|A\|} \, d\tau \qquad (3.16) \\ &\leqslant \frac{e^{T\|A\|}}{\| A \|} \{e^{T\|A\|} - 1\} \left\{1 + 2c_2 e^{T\|A\|} + \frac{2c_2}{T} e^{T\|A\|}\right\} \\ &= h(T). \qquad (3.17) \end{aligned}$$

We note that $h(T)$ is a continuous function and that

$$\lim_{T \to 0} h(T) = 2c_2 . \tag{3.18}$$

$$\therefore \quad \| \phi(x) - \phi(y)\|_\infty \leqslant K h(T) \| x - y \|_\infty . \tag{3.19}$$

Since $2c_2 K < 1$ by assumption, it follows that $\exists 0 < s \leqslant \bar{T}$ such that for $0 < T \leqslant s$, $Kh(T) < 1$. Therefore, for this range of T, each ϕ_T is contracting and by the contraction mapping principal, ϕ_T has a unique fixed point.

Q.E.D.

COROLLARY 3.1. *If $f(t, x)$ is differentiable in x and*

$$\sup_{\substack{t \in [0, \bar{T}] \\ x \in X}} \| f_x(t, x)\| \leqslant K,$$

then the theorem holds.

Proof. By the mean-value theorem, it follows that

$$\sup_{t \in [0, \bar{T}]} \| f(t, x) - f(t, y)\| \leqslant K \| x - y \| . \tag{3.20}$$

$\therefore f(t, x)$ is uniformly Lipshitz on $[0, \bar{T}]$ with Lipshitz constant $\leqslant K$.

Q.E.D.

We now write the solution to (3.1), (3.2) as $x(t, T, v)$, explicitly showing its dependence on the parameters T and v.

THEOREM 3.2. *Let $f(t, x)$ be a C^2 function on $[0, \bar{T}] \times X$. Let*

$$\operatorname{Max}\{\sup_{\substack{t\in[0,\bar{T}]\\ x\in X}} \|f_x(t, x)\|, \sup_{\substack{t\in[0,\bar{T}]\\ x\in X}} \|f_{xx}(t, x)\|, \sup_{\substack{t\in[0,\bar{T}]\\ x\in X}} \|f_{xt}(t, x)\|\} \leqslant L.$$

Also assume that $f_t(t, x)$, $f_{tt}(t, x)$ are uniformly Lipshitz in x on $[0, \bar{T}] \times X$. Then $x(t, T, v)$ is C^2 on $(0, s] \times (0, s] \times X$.

Proof. The proof is essentially the same as that given in Theorem 4 of (6). See (6) for details.

IV. THE INVARIANT IMBEDDING EQUATION

THEOREM 4.1. *Let $R(T, v) = x(T, T, V)$. Then $R(T, v)$ satisfies the partial differential equation*

$$\begin{aligned} &R_T(T, v) + R_v(T, v)\, P_2\{AR(T, v) + f(T, R(T, v))\} \\ &\quad = AR(T, v) + f(T, R(T, v)), \qquad 0 < T \leqslant s, \quad v \in X. \end{aligned} \tag{4.1}$$

Proof. The proof is identical to that of Theorem 1 of [6]. It only needs to be noted that the derivation of (4.1) is independent of the nature of the boundary conditions and relies only on the differentiability given in Theorem 3.2 above. See [6] for details.

V. INITIAL CONDITIONS

In order to make (4.1) useful for numerical purposes, it must be supplemented by appropriate initial conditions. It can be seen in general from (3.11) that $R(T, v)$ will be unbounded as $T \to 0$ since it is assumed that $P_1 + P_2 e^{TA}$ is not invertible for $T = 0$. Consequently, we must supplement (4.1) by the asymptotic behavior of $R(T, v)$ as $T \to 0$ in order to utilize the invariant imbedding technique for the numerical solution of two-point boundary value problems when $P_1 + P_2$ is not invertible.

DEFINITION. Let $f(T, v)$ be a function defined from $(0, s] \times X$ to Y; X and Y are arbitrary Banach Spaces. We will say that $f(T, v)$ is $\tilde{O}(g(T))$ as $T \to 0$ if for a fixed $v \in X$ $\exists M > 0$ such that $\|f(T, v)\| \leqslant Mg(T)$, $T \in (0, s]$; where $g(T)$ is a function defined on $(0, s]$.

LEMMA 5.1. *Under the conditions of Theorem 3.1,*

$$R(T, v) = \tilde{O}\left(\frac{1}{T}\right).$$

Proof. We first note that because of the Lipshitz condition that $f(t, x)$ satisfies

$$\| f(t, x)\| \leqslant K_1 + K \| x \| , \qquad 0 \leqslant t \leqslant \bar{T}. \tag{5.1}$$

(K is the Lipshitz constant of $f(t, x)$.) Therefore, using (3.11) we get that

$$\begin{aligned} \| x(t, T, v)\|_\infty &\leqslant c_1 e^{T\|A\|} \left(1 + \frac{1}{T}\right) \| v \| + \int_0^T \| G(t, \tau)\| \, \| f(\tau, x(\tau, T, v))\| \, d\tau \\ &\leqslant c_1 e^{T\|A\|} \left(1 + \frac{1}{T}\right) \| v \| + Kh(T) \| x(t, T, v)\|_\infty + K_1 h(T). \end{aligned} \tag{5.2}$$

Since $Kh(T) < 1$ for $0 < T \leqslant s$, we get by transposing that

$$\| x(t, T, v)\|_\infty \leqslant \frac{c_1 e^{T\|A\|} \left(1 + \frac{1}{T}\right) \| v \| + K_1 h(T)}{1 - Kh(T)} .$$

But

$$\| R(T, v)\| = \| x(T, T, v)\| \leqslant \| x(t, T, v)\|_\infty ,$$

$$\therefore \quad R(T, v) = \tilde{0} \left(\frac{1}{T}\right) .$$

LEMMA 5.2.

$$\| x_t(t, T, v)\|_\infty = \tilde{0} \left(\frac{1}{T}\right) .$$

Proof. In the course of the proof, Lemma (5.1), we obtained

$$\| x(t, T, v)\|_\infty = \tilde{0} \left(\frac{1}{T}\right) .$$

From the differential equation (3.1) we get that

$$\begin{aligned} \| x_t(t, T, v)\|_\infty &\leqslant \| A \| \, \| x(t, T, v)\|_\infty + \| f(t, x(t, T, v)\| \\ &\leqslant (\| A \| + K) \| x(t, T, v)\|_\infty + K_1 . \end{aligned} \tag{5.3}$$

$$\therefore \quad \| x_t(t, T, v)\|_\infty = \tilde{0} \left(\frac{1}{T}\right) .$$

Q.E.D.

To make a more exact asymptotic analysis, we must impose an additional condition on $H_T(A)$. Therefore, as $T \to 0$, we assume that

$$H_T(A) = \frac{S}{T} + 0(1), \qquad S \in L(X). \tag{5.4}$$

LEMMA 5.3. *Under the previous assumption on $H_T(A)$ and under the conditions of Theorem* (3.2), *$R(T, v)$ satisfies the following relation as $T \to 0$:*

$$R(T, v) = \frac{Sv}{T} - P_2 S f(T, R(T, v)) + \tilde{0}(1). \tag{5.5}$$

Proof (All the asymptotic relations are as $T \to 0$). From Theorem (3.2) and Taylor's Theorem [7], we get that

$$\begin{aligned} f(t, x(t, T, v)) &= f(T, x(T, T, v)) \\ &\quad + f_x(\theta(T-t), x(\theta(T-t), T, v)\, x_t(\theta(T-t), T, v)\,(T-t) \\ &\quad + f_t(\theta(T-t), x(\theta(T-t), T, v)\,(T-t), \\ &= f(T, x(T, T, v)) + g(t, T, v), \qquad 0 < \theta < 1. \end{aligned} \tag{5.6}$$

Using (3.11) we see that $R(T, v)$ satisfies

$$\begin{aligned} R(T, v) = e^{TA} H_T(A)\, v + \int_0^T e^{TA}[I - H_T(A)\, P_2 e^{TA}]\, e^{-\tau A} \\ \times f(\tau, x(\tau, T, v))\, d\tau. \end{aligned} \tag{5.7}$$

We substitute (5.6) into (5.7) and use (5.4) to get

$$\begin{aligned} R(T, v) &= \frac{Sv}{T} + \tilde{0}(1) + \int_0^T e^{(T-\tau)A}\{f(T, R(T, v)) + g(\tau, T, v)\}\, d\tau \\ &\quad - \int_0^T \left\{\frac{SP_2}{T} + 0(1)\right\} e^{-\tau A}\{f(T, R(T, v)) + g(\tau, T, v)\}\, d\tau \\ &= \frac{Sv}{T} + I_1 + I_2 + \tilde{0}(1). \end{aligned} \tag{5.8}$$

Using the properties of $f_t(t, x)$, $f_x(t, x)$ given in Theorem 3.2 and Lemmas (5.1), (5.2), it follows that $I_1 = \tilde{0}(1)$. Also $I_2 = I_2' + I_2''$, where

$$I_2' = -\int_0^T \frac{SP_2}{T} e^{-\tau A}\{f(T, R(T, v) + g(\tau, T, v)\}\, d\tau,$$

and

$$I_2'' = I_2 - I_2'.$$

Again using Lemmas (5.1), (5.2), I_2'' is seen to be $\tilde{0}(1)$. Similarly,

$$I_2' = SP_2 f(T, R(T, v)) + \tilde{0}(1).$$

$$\therefore \quad R(T, v) = \frac{Sv}{T} - SP_2 f(T, R(T, v)) + \tilde{0}(1).$$

Q.E.D.

We will now examine (5.5) in the case where $f(t, x)$ is linear. In this case we are able to determine the leading term in the asymptotic expansion of $R(T, v)$ as $T \to 0$.

LEMMA 5.4. *Let $f(t, x) = K(t)\, x + r(t)$ where $t \mapsto K(t)$, $t \mapsto r(t)$ are C^2 maps from $[0, \bar{T}] \to L(X)$ and X respectively.* (5.9)
Then

$$x(t, T, v) = U(t, T)\, v + h(t, T) \qquad \text{where} \qquad U(t, T) \in L(X). \tag{5.10}$$

Proof. This follows immediately from the uniqueness of $x(t, T, v)$ and linearity of $f(t, x)$. See (6).

COROLLARY.

$$R(T, v) = R(T)\, v + g(T), \tag{5.11}$$

where

$$R(T) = U(T, T) \qquad \text{and} \qquad g(T) = h(T, T).$$

Proof. Let $t = T$ in (5.10).

THEOREM 5.1. *Let $f(t,x)$ satisfy the hypothesis of Lemma* (5.3). *In addition, assume that the operator $I + SP_2K(0)$ has a bounded inverse. Then*

$$R(T) = \frac{(I + SP_2K(0))^{-1}\, S}{T} + 0(1), \tag{5.12}$$

and

$$g(T) = -\,(I + SP_2K(0))^{-1}\, SP_2 r(0) + 0(T). \tag{5.13}$$

Proof. From Lemma (5.3) we get that

$$R(T, v) = \frac{Sv}{T} - SP_2[K(T)\, R(T, v) + r(T)] + \tilde{0}(1). \tag{5.14}$$

Using (5.11) in (5.14) gives

$$R(T)\, v + g(T) = \frac{Sv}{T} - SP_2[K(T)\, R(T)\, v + K(T)\, g(T) + r(T)] + \tilde{0}(1). \tag{5.15}$$

Since $R(T)$ does not depend on $r(T)$, we obtain the behavior of $R(T, v)$ by setting $r(T) = g(T) = 0$ in (5.15).

$$\therefore \quad R(T)\, v = \frac{Sv}{T} - SP_2K(T)\, R(T)\, v + \tilde{0}(1). \tag{5.16}$$

$$\therefore \quad [I + SP_2K(T)]\, R(T)\, v = \frac{Sv}{T} + \tilde{0}(1). \tag{5.17}$$

We will now analyze the operator $I + SP_2K(T)$. Since $T \mapsto K(T)$ is C^2 it follows by Taylor's Theorem that

$$\| K(T) - K(0)\| = 0(T) \qquad \text{as} \qquad T \to 0. \tag{5.18}$$

Now

$$(I + SP_2K(T)) = (I + SP_2K(0)) + SP_2[K(T) - K(0)]. \tag{5.19}$$

Since $\| K(T) - K(0)\| = 0(T)$, $\exists T'$ such that for

$$0 \leqslant T \leqslant T' \qquad \|(I + SP_2K(0))^{-1} SP_2(K(T) - K(0))\| < 1. \tag{5.20}$$

Since $(I + SP_2K(0))$ is assumed invertible, it follows from (5.20) and Banach's Lemma [7] that for $0 \leqslant T \leqslant T'$ $(I + SP_2K(T))$ has a bounded inverse. Using the Neumann series for $[I + SP_2K(T)]^{-1}$ we get that

$$[I + SP_2K(T)]^{-1} = (I + SP_2K(0))^{-1} + 0(T). \tag{5.21}$$

Using (5.21) in (5.17) gives

$$\begin{aligned} R(T)\, v &= (I + SP_2K(0))^{-1} \frac{Sv}{T} + \tilde{0}(1), \\ \therefore \quad R(T) &= (I + SP_2K(0))^{-1} \frac{S}{T} + 0(1). \end{aligned} \tag{5.22}$$

To get the behavior of $g(T)$ as $T \to 0$ we set $v = 0$ in 5.15 giving

$$(I + SP_2K(T))\, g(T) = -\, SP_2r(T) + 0(1). \tag{5.23}$$

This relation is insufficient to determine the leading term in the expansion of $g(T)$ near 0, therefore we examine the 0(1) term in (5.23) more closely. To do this, we note first of all that $h(t, T)$ satisfies the following integral equation obtained from (3.11) and Lemma 5.4:

$$h(t, T) = \int_0^T G(t, \tau)\, [K(\tau)\, h(\tau, T) + r(\tau)]\, d\tau. \tag{5.24}$$

$$\begin{aligned} \therefore \quad g(T) &= \int_0^T G(T, \tau)\, [K(\tau)\, h(\tau, T) + r(\tau)]\, d\tau \\ &= \int_0^T e^{TA}[I - H_T(A)\, P_2 e^{TA}]\, e^{-\tau A} \\ &\quad \times\ K(\tau)\, h(\tau, T) + r(\tau)]\, d\tau. \end{aligned} \tag{5.25}$$

From (5.25) we easily show that $g(T)$ is $0(1)$. Using the asymptotic expansion for $H_T(A)$ and arguing as in Lemma (5.3) we get from (5.25) that

$$g(T) = -\int_0^T \frac{SP_2}{T} e^{-\tau A}[K(T)\,g(T) + r(T)]\,d\tau + 0(T). \tag{5.26}$$

$$\therefore\quad (I + SP_2K(T))\,g(T) = -\,SP_2r(T) + 0(T) \\ = -\,SP_2r(0) + 0(T). \tag{5.27}$$

Again using the fact that $(I + SP_2K(T))^{-1} = (I + SP_2K(0))^{-1} + 0(T)$ we get that $g(T) = -\,[I + SP_2K(0)]^{-1}\,SP_2r(0) + 0(T)$. Q.E.D.

COROLLARY.

$$\operatorname*{Lim}_{T\to 0} g(T) = -\,(I + SP_2K(0))^{-1}\,SP_2r(0). \tag{5.28}$$

Note. By assuming that $H_T(A) = S/T + W + 0(T)$ and using a similar analysis to that above, it is possible to get a formula for the $0(1)$ term in the expansion for $R(T)$.

Theorem 5.1 points the way towards using invariant imbedding as a method for solving boundary value problems with $P_1 + P_2$ non invertible, at least for linear problems. Since we know the asymptotic behavior of $R(T)$ as $T \to 0$, we can integrate the Ricatti equation (6) satisfied by $R(T)$ by using as approximate initial conditions, $R(T_0) = [I + SP_2K(0)]^{-1}\,S/T_0$ for T_0 near 0. The linear equation satisfied by $g(T)$ (6) can be integrated from $T = 0$ since (5.28) establishes appropriate initial conditions.

VI. AN EXAMPLE

The motivation for the theory developed in this paper comes from the consideration of the following Dirichlet problem:

$$\frac{d^2u(t)}{dt^2} = F(t, u(t)), \qquad u(t) \text{ real}, \tag{6.1}$$

$$u(0) = v_1\,, \qquad u(1) = v_2\,. \tag{6.2}$$

If we make the usual substitution $v(t) = du(t)/dt$ then (6.1), (6.2) can be put in the form of (3.1), (3.2). We get

$$\frac{dx(t)}{dt} = Ax(t) + f(t, x(t)), \tag{6.3}$$

$$P_1x(0) + P_2x(1) = v, \tag{6.4}$$

where

$$x(t) = (u(t), v(t)), \qquad f(t, x(t)) = (0, F(t, u(t)),$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

and

$$v = (v_1, v_2). \tag{6.5}$$

The matrix

$$H_T(A) = \begin{bmatrix} 1 & 0 \\ -\frac{1}{T} & \frac{1}{T} \end{bmatrix} \tag{6.6}$$

If we use $\|(x_1, x_2)\| = |x_1| + |x_2|$, then $\| H_T(A)\| \leqslant 1 + 1/T$, $T > 0$. Also $\| P_2 \| \leqslant 1$. Using this, we get that the constant c_2 in Theorem 3.1 is $\leqslant 1$. If we let K be the Lipshitz constant of F then $2c_2K \leqslant 2K$.

Therefore, if $K < \frac{1}{2}$, Theorem 3.1 holds for (6.3), (6.4). The asymptotic analysis of (V) can be carried out if we note that

$$H_T(A) = \frac{1}{T}\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Conclusion

We have shown in this paper how to extend the invariant imbedding theory of [7] to the case where $P_1 + P_2$ is not necessarily invertible. The analysis of (V) shows that one must supplement the invariant imbedding equation with asymptotic conditions near 0. This analysis was completed for linear problems; however, coupled with a linearization technique for non-linear problems [5] the method should be applicable here too. Numerical experiments are planned.

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