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γ-Radonifying operators and UMD-valued Littlewood–Paley–Stein functions in the Hermite setting on BMO and Hardy spaces ☆

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Abstract

In this paper we study Littlewood–Paley–Stein functions associated with the Poisson semigroup for the Hermite operator on functions with values in a UMD Banach space $\mathbb{B}$. If we denote by H the Hilbert space $L^2((0,\infty), dt/t)$, $\gamma(H,\mathbb{B})$ represents the space of γ-radonifying operators from H into $\mathbb{B}$. We prove that the Hermite square function defines bounded operators from $BMO_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})$ (respectively, $H^1_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})$) into $BMO_{\mathcal{L}}(\mathbb{R}^n,\gamma(H,\mathbb{B}))$ (respectively, $H^1_{\mathcal{L}}(\mathbb{R}^n,\gamma(H,\mathbb{B}))$), where $BMO_{\mathcal{L}}$ and $H^1_{\mathcal{L}}$ denote BMO and Hardy spaces in the Hermite setting. Also, we obtain equivalent norms in $BMO_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})$ and $H^1_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})$ by using Littlewood–Paley–Stein functions. As a consequence of our results, we establish new characterizations of the UMD Banach spaces.

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1. Introduction

The Littlewood–Paley–Stein g-function associated with the classical Poisson semigroup $\{P_t\}_{t>0}$ is given by

$$g\big(\{P_t\}_{t>0}\big)(f)(x) = \left(\int_0^\infty \left|t\partial_t P_t f(x)\right|^2 \frac{dt}{t}\right)^{1/2}, \quad x \in \mathbb{R}^n.$$

It is well-known that this g-function defines an equivalent norm in $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Indeed, for every $1 < p < \infty$ there exists $C_p > 0$ such that

$$\frac{1}{C_p}\|f\|_{L^p(\mathbb{R}^n)} \leqslant \big\|g\big(\{P_t\}_{t>0}\big)(f)\big\|_{L^p(\mathbb{R}^n)} \leqslant C_p\|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p\big(\mathbb{R}^n\big). \tag{1}$$

Equivalence (1) is useful, for instance, to study L^p-boundedness properties of certain type of spectral multipliers.

In [31] g-functions associated with diffusion semigroups $\{T_t\}_{t>0}$ on the measure space (Ω, μ) were considered. In this general case (1) takes the following form, for every $1 < p < \infty$,

$$\frac{1}{C_p}\big\|f - E_0(f)\big\|_{L^p(\Omega,\mu)} \leqslant \big\|g\big(\{T_t\}_{t>0}\big)(f)\big\|_{L^p(\Omega,\mu)} \leqslant C_p\|f\|_{L^p(\Omega,\mu)}, \quad f \in L^p(\Omega, \mu),$$

where $C_p > 0$. Here E_0 is the projection onto the fixed point space of $\{T_t\}_{t>0}$.

Suppose that $\mathbb{B}$ is a Banach space. For every $1 < p < \infty$, we denote by $L^p(\mathbb{R}^n, \mathbb{B})$ the p-Bochner–Lebesgue space. The natural way of extending the definition of $g(\{P_t\}_{t>0})$ to $L^p(\mathbb{R}^n, \mathbb{B})$, $1 < p < \infty$, is the following

$$g_{\mathbb{B}}\big(\{P_t\}_{t>0}\big)(f)(x) = \left(\int_0^\infty \big\|t\partial_t P_t f(x)\big\|_{\mathbb{B}}^2 \frac{dt}{t}\right)^{1/2}, \quad f \in L^p\big(\mathbb{R}^n, \mathbb{B}\big),\ 1 < p < \infty.$$

Kwapień in [25] proved that $\mathbb{B}$ is isomorphic to a Hilbert space if and only if

$$\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})} \sim \big\|g_{\mathbb{B}}\big(\{P_t\}_{t>0}\big)(f)\big\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p\big(\mathbb{R}^n, \mathbb{B}\big), \tag{2}$$

for some (or equivalently, for any) $1 < p < \infty$.

Xu [41] considered generalized g-functions defined by

$$g_{\mathbb{B},q}\big(\{P_t\}_{t>0}\big)(f)(x) = \left(\int_0^\infty \big\|t\partial_t P_t(f)(x)\big\|_{\mathbb{B}}^q \frac{dt}{t}\right)^{1/q}, \quad f \in L^p\big(\mathbb{R}^n, \mathbb{B}\big),\ 1 < p < \infty,$$

where $1 < q < \infty$. He characterized those Banach spaces $\mathbb{B}$ for which one of the following inequalities holds

- $\|g_{\mathbb{B},q}(\{P_t\}_{t>0})(f)\|_{L^p(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})}$, $f \in L^p(\mathbb{R}^n,\mathbb{B})$, $1 < p < \infty$,
- $\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})} \leqslant C\|g_{\mathbb{B},q}(\{P_t\}_{t>0})(f)\|_{L^p(\mathbb{R}^n)}$, $f \in L^p(\mathbb{R}^n,\mathbb{B})$, $1 < p < \infty$.

The validity of these inequalities is characterized by the q-martingale type or cotype of the Banach space $\mathbb{B}$.

Xu's results were extended to diffusion semigroups by Martínez, Torrea and Xu [27].

In order to get new equivalent norms in $L^p(\mathbb{R}^n,\mathbb{B})$ for a wider class of Banach spaces, Hytönen [22] and Kaiser and Weis [23,24] have introduced new definitions of g-functions for Banach valued functions.

In this paper we are motivated by the ideas developed by Kaiser and Weis [23,24]. They defined g-functions for Banach valued functions by using γ-radonifying operators.

The main definitions and properties about γ-radonifying operators can be found in [40]. We now recall those aspects of the theory of γ-radonifying operators that will be useful in the sequel. We consider the Hilbert space $H = L^2((0,\infty), dt/t)$. Suppose that $(e_k)_{k=1}^\infty$ is an orthonormal basis in H and $(\gamma_k)_{k=1}^\infty$ is a sequence of independent standard Gaussian random variables on a probability space $(\Omega,\mathbb{P})$. A bounded operator T from H into $\mathbb{B}$ is a γ-radonifying operator, shortly $T \in \gamma(H,\mathbb{B})$, when $\sum_{k=1}^\infty \gamma_k T e_k$ converges in $L^2(\Omega,\mathbb{B})$. We define the norm $\|T\|_{\gamma(H,\mathbb{B})}$ by

$$\|T\|_{\gamma(H,\mathbb{B})} = \left(\mathbb{E}\left\|\sum_{k=1}^\infty \gamma_k T e_k\right\|_{\mathbb{B}}^2\right)^{1/2}.$$

This definition does not depend on the orthonormal basis $(e_k)_{k=1}^\infty$ of H. $\gamma(H,\mathbb{B})$ is a Banach space which is continuously contained in the space $L(H,\mathbb{B})$ of bounded operators from H into $\mathbb{B}$.

If $f : (0,\infty) \longrightarrow \mathbb{B}$ is a measurable function such that for every $S \in \mathbb{B}^*$, the dual space of $\mathbb{B}$, $S \circ f \in H$, there exists $T_f \in L(H,\mathbb{B})$ for which

$$\langle S, T_f(h)\rangle_{\mathbb{B}^*,\mathbb{B}} = \int_0^\infty \langle S, f(t)\rangle_{\mathbb{B}^*,\mathbb{B}} h(t)\frac{dt}{t}, \quad h \in H \text{ and } S \in \mathbb{B}^*,$$

where $\langle\cdot,\cdot\rangle_{\mathbb{B}^*,\mathbb{B}}$ denotes the duality pairing in $(\mathbb{B}^*,\mathbb{B})$. When $T_f \in \gamma(H,\mathbb{B})$ we say that $f \in \gamma(H,\mathbb{B})$ and we write $\|f\|_{\gamma(H,\mathbb{B})}$ to refer us to $\|T_f\|_{\gamma(H,\mathbb{B})}$.

The Hilbert transform $\mathcal{H}(f)$ of $f \in L^p(\mathbb{R})$, $1 \leqslant p < \infty$, is defined by

$$\mathcal{H}(f)(x) = \frac{1}{\pi}\lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon}\frac{f(y)}{x-y}, \quad \text{a.e. } x \in \mathbb{R}.$$

The Hilbert transform $\mathcal{H}$ is defined on $L^p(\mathbb{R}) \otimes \mathbb{B}$, $1 \leqslant p < \infty$, in a natural way. We say that $\mathbb{B}$ is a UMD Banach space when for some (equivalent, for every) $1 < p < \infty$

the Hilbert transformation can be extended from $L^p(\mathbb{R},\mathbb{B})$ as a bounded operator from $L^p(\mathbb{R},\mathbb{B})$ into itself. There exist many other characterizations of the UMD Banach spaces (see, for instance, [1,9,10,17,18,22,24]). Every Hilbert space is a UMD space and $\gamma(H,\mathbb{B})$ is UMD provided that $\mathbb{B}$ is UMD.

UMD Banach spaces are a suitable setting to establish Banach valued Fourier multiplier theorems [15,20]. Convolution operators are closely connected with Fourier multipliers. Suppose that $\psi \in L^2(\mathbb{R}^n)$. We consider $\psi_t(x) = \frac{1}{t^n}\psi(x/t)$, $x \in \mathbb{R}^n$ and $t > 0$. The wavelet transform W_ψ associated with ψ is defined by

$$W_\psi(f)(x,t) = (f * \psi_t)(x), \quad x \in \mathbb{R}^n \text{ and } t > 0,$$

where $f \in \mathcal{S}(\mathbb{R}^n,\mathbb{B})$, the $\mathbb{B}$-valued Schwartz space.

In [24, Theorem 4.2] Kaiser and Weis gave sufficient conditions for ψ in order to

$$\|W_\psi f\|_{E(\mathbb{R}^n,\gamma(H,\mathbb{B}))} \sim \|f\|_{E(\mathbb{R}^n,\mathbb{B})}, \tag{3}$$

for every $f \in E(\mathbb{R}^n,\mathbb{B})$, where $\mathbb{B}$ is a UMD Banach space and E represents L^p, $1 < p < \infty$, H^1 or BMO. Here, as usual, H^1 and BMO denote the Hardy spaces and the space of bounded mean oscillation functions, respectively.

If $P(x) = \Gamma((n+1)/2)/\pi^{(n+1)/2}(1+|x|^2)^{-(n+1)/2}$, $x \in \mathbb{R}^n$, then $P_t(x) = \frac{1}{t^n}P(\frac{x}{t})$, $x \in \mathbb{R}^n$ and $t > 0$, is the classical Poisson kernel. By taking $\psi(x) = \partial_t P_t(x)_{|t=1}$, $x \in \mathbb{R}^n$, we have that

$$W_\psi(f)(x,t) = t\partial_t P_t(f)(x), \quad x \in \mathbb{R}^n \text{ and } t > 0.$$

Moreover, $\gamma(H,\mathbb{C}) = H$ and $\gamma(H,\mathbb{H}) = L^2((0,\infty), dt/t; \mathbb{H})$, provided that $\mathbb{H}$ is a Hilbert space [40, p. 3]. Then, when $E = L^p$, $1 < p < \infty$, (3) can be seen as a Banach valued extension of (1) and (2).

Also, in [24, Remark 4.6] UMD Banach spaces are characterized by using wavelet transforms.

Harmonic analysis associated with the harmonic oscillator (also called Hermite) operator $L = -\Delta + |x|^2$ on $\mathbb{R}^n$ has been developed in last years by several authors (see [1,5,33,35,36,38,39], amongst others). Littlewood–Paley g-functions in the Hermite setting were analyzed in [35] for scalar functions and in [6] for Banach valued functions. Motivated by the ideas developed by Kaiser and Weis [24], the authors in [2, Theorem 1] established new equivalent norms for the Bochner–Lebesgue space $L^p(\mathbb{R}^n,\mathbb{B})$ by using Littlewood–Paley functions associated with Poisson semigroups for the Hermite operator and γ-radonifying operators, provided that $\mathbb{B}$ is a UMD space. Our objectives in this paper are the following ones:

(a) To obtain equivalent norms for the $\mathbb{B}$-valued Hardy space $H^1_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})$ and $BMO_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})$ associated to the Hermite operator, when $\mathbb{B}$ is a UMD Banach space, and
(b) To characterize the UMD Banach spaces in terms of $H^1_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})$ and $BMO_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})$, by using Littlewood–Paley functions for the Poisson semigroup in the Hermite context and γ-radonifying operators.

We recall some definitions and properties about the Hermite setting. For every $k \in \mathbb{N}$ the k-th Hermite function is $h_k(x) = (\sqrt{\pi}\, 2^k k!)^{-1/2} H_k(x) e^{-x^2/2}$, $x \in \mathbb{R}$, where H_k represents the k-th Hermite polynomial [26, p. 60]. If $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ the k-th multidimensional Hermite function h_k is defined by

$$h_k(x) = \prod_{j=1}^{n} h_{k_j}(x_j), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

and we have that

$$Lh_k = \big(2|k| + n\big) h_k,$$

where $|k| = k_1 + \cdots + k_n$. The system $\{h_k\}_{k \in \mathbb{N}^n}$ is a complete orthonormal system for $L^2(\mathbb{R}^n)$. We define, the operator $\mathcal{L}$ as follows

$$\mathcal{L} f = \sum_{k \in \mathbb{N}^n} \big(2|k| + n\big) \langle f, h_k \rangle h_k, \quad f \in D(\mathcal{L}),$$

where the domain $D(\mathcal{L})$ is constituted by all those $f \in L^2(\mathbb{R}^n)$ such that $\sum_{k \in \mathbb{N}^n} (2|k| + n)^2 |\langle f, h_k \rangle|^2 < \infty$. Here $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $L^2(\mathbb{R}^n)$. It is clear that if $\phi \in C_c^\infty(\mathbb{R}^n)$, the space of smooth functions with compact support in $\mathbb{R}^n$, then $L\phi = \mathcal{L}\phi$.

For every $t > 0$ we consider the operator $W_t^{\mathcal{L}}$ defined by

$$W_t^{\mathcal{L}}(f) = \sum_{k \in \mathbb{N}^n} e^{-t(2|k|+n)} \langle f, h_k \rangle h_k, \quad f \in L^2\big(\mathbb{R}^n\big).$$

The family $\{W_t^{\mathcal{L}}\}_{t>0}$ is a semigroup of operators generated by $-\mathcal{L}$ in $L^2(\mathbb{R}^n)$ which is usually called the heat semigroup associated to $\mathcal{L}$. By taking into account the Mehler's formula [38, (1.1.36)] we can write, for every $f \in L^2(\mathbb{R}^n)$,

$$W_t^{\mathcal{L}}(f)(x) = \int_{\mathbb{R}^n} W_t^{\mathcal{L}}(x, y) f(y)\, dy, \quad x \in \mathbb{R}^n \text{ and } t > 0,$$

where, for every $x, y \in \mathbb{R}^n$ and $t > 0$,

$$W_t^{\mathcal{L}}(x, y) = \left(\frac{e^{-2t}}{\pi(1 - e^{-4t})} \right)^{n/2} \exp\left(-\frac{1}{4} \left(\frac{1 + e^{-2t}}{1 - e^{-2t}} |x - y|^2 + \frac{1 - e^{-2t}}{1 + e^{-2t}} |x + y|^2 \right) \right).$$

The Poisson semigroup $\{P_t^{\mathcal{L}}\}_{t>0}$ associated to $\mathcal{L}$, that is, the semigroup of operators generated by $-\sqrt{\mathcal{L}}$, can be written by using the subordination formula by

$$P_t^{\mathcal{L}}(f) = \frac{t}{\sqrt{4\pi}} \int_0^\infty s^{-3/2} e^{-t^2/(4s)} W_s^{\mathcal{L}}(f)\, ds, \quad f \in L^2\big(\mathbb{R}^n\big) \text{ and } t > 0. \tag{4}$$

The families $\{W_t^{\mathcal{L}}\}_{t>0}$ and $\{P_t^{\mathcal{L}}\}_{t>0}$ are also C_0-semigroups in $L^p(\mathbb{R}^n)$, for every $1 < p < \infty$ (see [31]), but they are not Markovian.

In [35] Stempak and Torrea studied the Littlewood–Paley g-functions in the Hermite setting. They proved that the g-function defined by

$$g\big(\big\{P_t^{\mathcal{L}}\big\}_{t>0}\big)(f)(x) = \left(\int_0^\infty \big|t\partial_t P_t^{\mathcal{L}} f(x)\big|^2 \frac{dt}{t}\right)^{1/2}, \quad x \in \mathbb{R}^n,$$

is bounded from $L^p(\mathbb{R}^n)$ into itself, when $1 < p < \infty$ [35, Theorem 3.2]. Also, we have that

$$\|f\|_{L^p(\mathbb{R}^n)} \sim \big\|g\big(\big\{P_t^{\mathcal{L}}\big\}_{t>0}\big)\big\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p\big(\mathbb{R}^n\big), \tag{5}$$

[4, Proposition 2.3].

From [6, Theorems 1 and 2] and [25] we deduce that by defining, for every $1 < p < \infty$,

$$g_{\mathbb{B}}\big(\big\{P_t^{\mathcal{L}}\big\}_{t>0}\big)(f)(x) = \left(\int_0^\infty \big\|t\partial_t P_t^{\mathcal{L}} f(x)\big\|_{\mathbb{B}}^2 \frac{dt}{t}\right)^{1/2}, \quad f \in L^p\big(\mathbb{R}^n, \mathbb{B}\big),$$

then, for some (equivalently, for every) $1 < p < \infty$,

$$\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})} \sim \big\|g_{\mathbb{B}}\big(\big\{P_t^{\mathcal{L}}\big\}_{t>0}\big)(f)\big\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p\big(\mathbb{R}^n, \mathbb{B}\big),$$

if, and only if, $\mathbb{B}$ is isomorphic to a Hilbert space.

We consider the operator $\mathcal{G}_{\mathcal{L},\mathbb{B}}$ defined by

$$\mathcal{G}_{\mathcal{L},\mathbb{B}}(f)(x,t) = t\partial_t P_t^{\mathcal{L}}(f)(x), \quad x \in \mathbb{R}^n \text{ and } t > 0,$$

for every $f \in L^p(\mathbb{R}^n, \mathbb{B})$, $1 \leqslant p < \infty$.

In [2] the authors proved that, for every $1 < p < \infty$,

$$\|f\|_{L^p(\mathbb{R}^n,\mathbb{B})} \sim \big\|\mathcal{G}_{\mathcal{L},\mathbb{B}}(f)\big\|_{L^p(\mathbb{R}^n,\gamma(H,\mathbb{B}))}, \tag{6}$$

provided that $\mathbb{B}$ is a UMD Banach space. Since $\gamma(H, \mathbb{C}) = H$, (6) can be seen as a Banach valued extension of (5).

Our first objective is to establish (6) when the space L^p is replaced by the Hardy space H^1 and the BMO space associated with the Hermite operator.

Dziubański and Zienkiewicz [14] investigated the Hardy space $H^1_{\mathcal{S}_V}(\mathbb{R}^n)$ in the Schrödinger context, where $\mathcal{S}_V = -\Delta + V$ and V is a suitable positive potential. The Hermite operator is a special case of the Schrödinger operator. In [13] the dual space of $H^1_{\mathcal{S}_V}(\mathbb{R}^n)$ is characterized as the space $BMO_{\mathcal{S}_V}(\mathbb{R}^n)$ that is contained in the classical $BMO(\mathbb{R}^n)$ of bounded mean oscillation function in $\mathbb{R}^n$. The results in [13] and [14] hold when the dimension n is greater than 2, but when $V(x) = |x|^2$, $x \in \mathbb{R}^n$, that is, when

$\mathcal{S}_V = \mathcal{L}$ the results in [13] and [14] about Hardy and BMO spaces hold for every dimension $n \geqslant 1$.

We say that a function $f \in L^1(\mathbb{R}^n, \mathbb{B})$ is in $H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$ when

$$\sup_{t>0} \left\| W_t^{\mathcal{L}}(f) \right\|_{\mathbb{B}} \in L^1(\mathbb{R}^n).$$

As usual we consider on $H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$ the norm $\| \cdot \|_{H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})}$ defined by

$$\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})} = \left\| \sup_{t>0} \left\| W_t^{\mathcal{L}}(f) \right\|_{\mathbb{B}} \right\|_{L^1(\mathbb{R}^n)}, \quad f \in H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B}).$$

The dual space of $H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$ is the space $BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B}^*)$ defined as follows, provided that $\mathbb{B}$ satisfies the Radon–Nikodým property (see [7]). Note that every UMD space is reflexive [28, Proposition 2, p. 205] and therefore verifies the Radon–Nikodým property [11, Corollary 13, p. 76]. A function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{B})$ is in $BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$ if there exists $C > 0$ such that

(i) for every $a \in \mathbb{R}^n$ and $0 < r < \rho(a)$

$$\frac{1}{|B(a,r)|} \int_{B(a,r)} \left\| f(z) - f_{B(a,r)} \right\|_B dz \leqslant C,$$

where $f_{B(a,r)} = \frac{1}{|B(a,r)|} \int_{B(a,r)} f(z)\, dz$, and

(ii) for every $a \in \mathbb{R}^n$ and $r \geqslant \rho(a)$,

$$\frac{1}{|B(a,r)|} \int_{B(a,r)} \left\| f(z) \right\|_{\mathbb{B}} dz \leqslant C.$$

Here ρ is given by

$$\rho(x) = \begin{cases} \frac{1}{1+|x|}, & |x| \geqslant 1, \\ \frac{1}{2}, & |x| < 1. \end{cases}$$

When $\mathbb{B} = \mathbb{C}$ we simply write $H^1_{\mathcal{L}}(\mathbb{R}^n)$ and $BMO_{\mathcal{L}}(\mathbb{R}^n)$, instead of $H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{C})$ and $BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{C})$, respectively.

In [3] it was established a T1 type theorem that gives sufficient conditions in order that an operator is bounded between $BMO_{\mathcal{L}}$ spaces.

Suppose that $\mathbb{B}_1$ and $\mathbb{B}_2$ are Banach spaces and T is a linear operator bounded from $L^2(\mathbb{R}^n, \mathbb{B}_1)$ into $L^2(\mathbb{R}^n, \mathbb{B}_2)$ such that

$$T(f)(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy, \quad x \notin \operatorname{supp}(f), \; f \in L^\infty_c(\mathbb{R}^n, \mathbb{B}_1),$$

where $K(x, y)$ is a bounded operator from $\mathbb{B}_1$ into $\mathbb{B}_2$, for every $x, y \in \mathbb{R}^n$, $x \neq y$, and the integral is understood in the $\mathbb{B}_2$-Bochner sense.

As in [3] we say that T is a $(\mathbb{B}_1, \mathbb{B}_2)$-Hermite–Calderón–Zygmund operator when the following two conditions are satisfied:

(i) $\|K(x, y)\|_{L(\mathbb{B}_1,\mathbb{B}_2)} \leqslant C \frac{e^{-c(|x-y|^2+|x||x-y|)}}{|x-y|^n}$, $x, y \in \mathbb{R}^n$, $x \neq y$,

(ii) $\|K(x, y) - K(x, z)\|_{L(\mathbb{B}_1,\mathbb{B}_2)} + \|K(y, x) - K(z, x)\|_{L(\mathbb{B}_1,\mathbb{B}_2)} \leqslant C \frac{|y-z|}{|x-y|^{n+1}}$, $|x - y| > 2|y - z|$,

where $C, c > 0$ and $L(\mathbb{B}_1, \mathbb{B}_2)$ denotes the space of bounded operators from $\mathbb{B}_1$ into $\mathbb{B}_2$.

If T is a Hermite–Calderón–Zygmund operator, we define the operator $\mathbb{T}$ on $BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B}_1)$ as follows: for every $f \in BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B}_1)$,

$$\mathbb{T}(f)(x) = T(f\chi_B)(x) + \int_{\mathbb{R}^n \setminus B} K(x, y) f(y)\, dy,$$

$$\text{a.e. } x \in B = B(x_0, r_0),\ x_0 \in \mathbb{R}^n \text{ and } r_0 > 0.$$

This definition is consistent in the sense that it does not depend on x_0 or r_0. Note that if $f \in BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B}_1)$, $B = B(x_0, r_0)$, and $B^* = B(x_0, 2r_0)$ where $x_0 \in \mathbb{R}^n$ and $r_0 > 0$, then

$$\mathbb{T}(f)(x) = T\big((f - f_B)\chi_{B^*}\big)(x) + \int_{\mathbb{R}^n \setminus B^*} K(x, y)\big(f(y) - f_B\big)\, dy + \mathbb{T}(f_B)(x),$$

a.e $x \in B^*$.

Note that if $f \in L^\infty_c(\mathbb{R}^n, \mathbb{B}_1)$ then $\mathbb{T}(f) = T(f)$. In Theorems 1.2 and 1.3 below we establish the boundedness of certain Banach valued Hermite–Calderón–Zygmund operators between $BMO_{\mathcal{L}}$ spaces. When we say that an operator T is bounded between $BMO_{\mathcal{L}}$ spaces we always are speaking of the corresponding operator $\mathbb{T}$, although we continue writing T. In order to show the boundedness of our operators in Banach valued $BMO_{\mathcal{L}}$ spaces we will use a Banach valued version of [3, Theorem 1.1] (see [3, Remark 1.1]).

Theorem 1.1. *Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be Banach spaces. Suppose that T is a $(\mathbb{B}_1, \mathbb{B}_2)$ Hermite–Calderón–Zygmund operator. Then, the operator T is bounded from $BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B}_1)$ into $BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B}_2)$ provided that there exists $C > 0$ such that*:

(i) *for every $b \in \mathbb{B}_1$ and $x \in \mathbb{R}^n$,*

$$\frac{1}{|B(x, \rho(x))|} \int_{B(x,\rho(x))} \big\|T(b)(y)\big\|_{\mathbb{B}_2}\, dy \leqslant C\|b\|_{\mathbb{B}_1},$$

(ii) *for every* $b \in \mathbb{B}_1$, $x \in \mathbb{R}^n$ *and* $0 < s \leqslant \rho(x)$,

$$\left(1 + \log\left(\frac{\rho(x)}{s}\right)\right)\frac{1}{|B(x,s)|}\int_{B(x,s)} \left\|T(b)(y) - \big(T(b)\big)_{B(x,s)}\right\|_{\mathbb{B}_2} dy \leqslant C\|b\|_{\mathbb{B}_1},$$

where $(T(b))_{B(x,s)} = \frac{1}{|B(x,s)|}\int_{B(x,s)} T(b)(y)\,dy$.

This result can be proved in the same way as [3, Theorem 1.1]. In some special cases the conditions (i) and (ii) reduce to simpler forms. For instance, if $T(b) = \widetilde{T}(1)b$, $b \in \mathbb{B}_1$, where $\widetilde{T}$ is a $(\mathbb{C}, L(\mathbb{B}_1, \mathbb{B}_2))$ operator (where $(\mathbb{C}, L(\mathbb{B}_1, \mathbb{B}_2))$ has the obvious meaning) then properties (i) and (ii) are satisfied provided that $\widetilde{T}(1) \in L^\infty(\mathbb{R}^n, L(\mathbb{B}_1, \mathbb{B}_2))$ and $\nabla\widetilde{T}(1) \in L^\infty(\mathbb{R}^n, L(\mathbb{B}_1, \mathbb{B}_2))$.

We denote by $\{P_t^{\mathcal{L}+\alpha}\}_{t>0}$ the Poisson semigroup associated with the operator $\mathcal{L} + \alpha$, when $\alpha > -n$. We can write

$$P_t^{\mathcal{L}+\alpha}(f) = \frac{t}{\sqrt{4\pi}}\int_0^\infty s^{-3/2}e^{-t^2/(4s)}e^{-\alpha s}W_s^{\mathcal{L}}(f)\,ds.$$

The operator $\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}$ is defined by

$$\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f)(x,t) = t\partial_t P_t^{\mathcal{L}+\alpha}(f)(x), \quad x \in \mathbb{R}^n \text{ and } t > 0.$$

Our first result is the following one.

Theorem 1.2. *Let* $\mathbb{B}$ *be a UMD Banach space and* $\alpha > -n$. *Then, if* E *represents* $H^1_{\mathcal{L}}$ *or* $BMO_{\mathcal{L}}$ *we have that*

$$\|f\|_{E(\mathbb{R}^n,\mathbb{B})} \sim \left\|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f)\right\|_{E(\mathbb{R}^n,\gamma(H,\mathbb{B}))}, \quad f \in E\big(\mathbb{R}^n, \mathbb{B}\big).$$

In order to establish our characterization for the UMD Banach spaces we introduce the operators $T^{\mathcal{L}}_{j,\pm}$, $j = 1, \ldots, n$, defined as follows:

$$T^{\mathcal{L}}_{j,\pm}(f)(x,t) = t(\partial_{x_j} \pm x_j)P_t^{\mathcal{L}}(f)(x), \quad x \in \mathbb{R}^n \text{ and } t > 0.$$

In [2, Theorem 2] it was established that if $\mathbb{B}$ is a UMD Banach space then the operators $T^{\mathcal{L}}_{j,\pm}$ are bounded from $L^p(\mathbb{R}^n, \mathbb{B})$ into $L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))$, for every $1 < p < \infty$ and $j = 1, \ldots, n$, provided that $n \geqslant 3$ in the case of $T^{\mathcal{L}}_{j,-}$.

The behavior of the operators $T^{\mathcal{L}}_{j,\pm}$ on the spaces $H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$ and $BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$ is now stated.

Theorem 1.3. *Let* $\mathbb{B}$ *be a UMD Banach space and* $j = 1, \ldots, n$. *By* E *we represent the space* $H^1_{\mathcal{L}}$ *or* $BMO_{\mathcal{L}}$. *Then, the operators* $T^{\mathcal{L}}_{j,\pm}$ *are bounded from* $E(\mathbb{R}^n, \mathbb{B})$ *into* $E(\mathbb{R}^n, \gamma(H, \mathbb{B}))$, *provided that* $n \geqslant 3$ *in the case of* $T^{\mathcal{L}}_{j,-}$.

UMD Banach spaces are characterized as follows.

Theorem 1.4. *Let $\mathbb{B}$ be a Banach space. Then, the following assertions are equivalent.*

(i) *$\mathbb{B}$ is UMD.*
(ii) *For some (equivalently, for every) $j = 1, \dots, n$, there exists $C > 0$ such that, for every $f \in H^1_{\mathcal{L}}(\mathbb{R}^n) \otimes \mathbb{B}$,*

$$\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})} \leqslant C \big\|\mathcal{G}_{\mathcal{L}+2,\mathbb{B}}(f)\big\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,\gamma(H,\mathbb{B}))},$$

and

$$\big\|T^{\mathcal{L}}_{j,+}(f)\big\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,\gamma(H,\mathbb{B}))} \leqslant C\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})}.$$

(iii) *For some (equivalently, for every) $j = 1, \dots, n$, there exists $C > 0$ such that, for every $f \in BMO_{\mathcal{L}}(\mathbb{R}^n) \otimes \mathbb{B}$,*

$$\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})} \leqslant C \big\|\mathcal{G}_{\mathcal{L}+2,\mathbb{B}}(f)\big\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,\gamma(H,\mathbb{B}))}$$

and

$$\big\|T^{\mathcal{L}}_{j,+}(f)\big\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,\gamma(H,\mathbb{B}))} \leqslant C\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})}.$$

In (ii) and (iii) the operators $\mathcal{G}_{\mathcal{L}+2,\mathbb{B}}$ and $T^{\mathcal{L}}_{j,+}$, $j = 1, \dots, n$, can be replaced by $\mathcal{G}_{\mathcal{L}-2,\mathbb{B}}$ and $T^{\mathcal{L}}_{j,-}$, $j = 1, \dots, n$, respectively, provided that $n \geqslant 3$.

In the following sections we present proofs of Theorems 1.2, 1.3 and 1.4. In Appendix A we show that the Riesz transforms in the Hermite setting can be extended as bounded operators from $BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$ into itself and from $H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$ into itself. These boundedness properties will be needed when proving Theorem 1.4. Moreover, they have interest in themself and complete the results established in [3] and in [14].

Throughout this paper by C and c we always denote positive constants that can change on each occurrence.

2. Proof of Theorem 1.2

We distinguish four parts in the proof of Theorem 1.2.

2.1. We are going to show that the operator $\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}$ is bounded from $BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$ into $BMO_{\mathcal{L}}(\mathbb{R}^n, \gamma(H, \mathbb{B}))$. In order to see this we will use Theorem 1.1. According to [2, Theorem 1] the operator $\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}$ is bounded from $L^2(\mathbb{R}^n, \mathbb{B})$ into $L^2(\mathbb{R}^n, \gamma(H, \mathbb{B}))$, because $\mathbb{B}$ is UMD.

Suppose that $f \in BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$. Then, f is a $\mathbb{B}$-valued function with bounded mean oscillation and hence $\int_{\mathbb{R}^n} \|f(x)\|_{\mathbb{B}}/(1+|x|)^{n+1}\,dx < \infty$. The kernel $P_t^{\mathcal{L}+\alpha}(x,y)$ of the operator $P_t^{\mathcal{L}+\alpha}$ can be written as

$$P_t^{\mathcal{L}+\alpha}(x,y) = \frac{t}{\sqrt{4\pi}} \int_0^\infty s^{-3/2} e^{-t^2/(4s)-\alpha s} W_s^{\mathcal{L}}(x,y)\,ds, \quad x, y \in \mathbb{R}^n \text{ and } t > 0.$$

We have that

$$t\partial_t P_t^{\mathcal{L}+\alpha}(x,y) = \frac{t}{\sqrt{4\pi}} \int_0^\infty s^{-3/2}\left(1 - \frac{t^2}{2s}\right) e^{-t^2/(4s)-\alpha s} W_s^{\mathcal{L}}(x,y)\,ds,$$
$$x, y \in \mathbb{R}^n \text{ and } t > 0.$$

By [3, (4.4) and (4.5)] we have that, for every $x, y \in \mathbb{R}^n$ and $s > 0$,

$$\begin{aligned} W_s^{\mathcal{L}}(x,y) &\leqslant C \frac{e^{-ns}}{(1-e^{-4s})^{n/2}} \exp\left(-c\left(\frac{|x-y|^2}{1-e^{-2s}} + \left(1-e^{-2s}\right)|x+y|^2 \right.\right. \\ &\qquad \left.\left. + \left(|x|+|y|\right)|x-y|\right)\right) \\ &\leqslant C e^{-c(|x-y|^2+(|x|+|y|)|x-y|)} \frac{e^{-ns-c\frac{|x-y|^2}{s}-c(1-e^{-2s})|x+y|^2}}{(1-e^{-4s})^{n/2}}. \end{aligned} \tag{7}$$

Hence, since $\alpha + n > 0$, for each $x, y \in \mathbb{R}^n$ and $t > 0$,

$$\begin{aligned} \left|t\partial_t P_t^{\mathcal{L}+\alpha}(x,y)\right| &\leqslant C t e^{-c(|x-y|^2+(|x|+|y|)|x-y|)} \int_0^\infty \frac{e^{-c\frac{|x-y|^2+t^2}{s}}}{s^{3/2}} \frac{e^{-(\alpha+n)s}}{(1-e^{-4s})^{n/2}}\,ds \\ &\leqslant C t e^{-c(|x-y|^2+(|x|+|y|)|x-y|)} \int_0^\infty \frac{e^{-c\frac{|x-y|^2+t^2}{s}}}{s^{(n+3)/2}}\,ds \\ &\leqslant C e^{-c(|x-y|^2+(|x|+|y|)|x-y|)} \frac{t}{(t+|x-y|)^{n+1}} \\ &\leqslant C \frac{t}{(t+|x-y|)^{n+1}}. \end{aligned} \tag{8}$$

Then, $\int_{\mathbb{R}^n} |t\partial_t P_t^{\mathcal{L}+\alpha}(x,y)|\|f(y)\|_{\mathbb{B}}\,dy < \infty$, for every $x \in \mathbb{R}^n$ and $t > 0$, and we deduce that

$$t\partial_t P_t^{\mathcal{L}+\alpha}(f)(x) = \int_{\mathbb{R}^n} t\partial_t P_t^{\mathcal{L}+\alpha}(x,y) f(y)\,dy, \quad x \in \mathbb{R}^n \text{ and } t > 0.$$

Moreover, by (8) we get that,

$$\left\| t\partial_t P_t^{\mathcal{L}+\alpha}(x,y)\right\|_H \leqslant Ce^{-c(|x-y|^2+|y||x-y|)}\left(\int_0^\infty \frac{t}{(t+|x-y|)^{2(n+1)}}\,dt\right)^{1/2}$$
$$\leqslant C\frac{e^{-c(|x-y|^2+|y||x-y|)}}{|x-y|^n}, \quad x,y\in\mathbb{R}^n,\ x\neq y. \tag{9}$$

Let $x,y\in\mathbb{R}^n$, $x\neq y$. We write $F(x,y;t)=t\partial_t P_t^{\mathcal{L}+\alpha}(x,y)$, $t>0$. Since $F(x,y;\cdot)\in H$, for every $b\in\mathbb{B}$, the function $F_b(x,y;t)=F(x,y;t)b$, $t>0$, defines an element $\widetilde{F}_b(x,y;\cdot)\in\gamma(H,\mathbb{B})$ satisfying that

$$\left\langle S,\widetilde{F}_b(x,y;\cdot)(h)\right\rangle_{\mathbb{B}^*,\mathbb{B}} = \int_0^\infty \left\langle S,F_b(x,y;t)\right\rangle_{\mathbb{B}^*,\mathbb{B}} h(t)\frac{dt}{t}$$
$$= \langle S,b\rangle_{\mathbb{B}^*,\mathbb{B}}\int_0^\infty F(x,y;t)h(t)\frac{dt}{t}, \quad S\in\mathbb{B}^* \text{ and } h\in H.$$

Then, for every $b\in\mathbb{B}$,

$$\widetilde{F}_b(x,y;\cdot)(h) = \left(\int_0^\infty F(x,y;t)h(t)\frac{dt}{t}\right)b, \quad h\in H.$$

We consider the operator $\tau(x,y)(b)=\widetilde{F}_b(x,y;\cdot)$, $b\in\mathbb{B}$. We have that

$$\left\|\tau(x,y)(b)\right\|_{\gamma(H,\mathbb{B})} = \left(\mathbb{E}\left\|\sum_{k=1}^\infty \gamma_k \widetilde{F}_b(x,y;\cdot)(e_k)\right\|_{\mathbb{B}}^2\right)^{1/2}$$
$$= \left(\mathbb{E}\left\|\sum_{k=1}^\infty \gamma_k \int_0^\infty F(x,y;t)e_k(t)\frac{dt}{t}b\right\|_{\mathbb{B}}^2\right)^{1/2}$$
$$= \left(\mathbb{E}\left|\sum_{k=1}^\infty \gamma_k \int_0^\infty F(x,y;t)e_k(t)\frac{dt}{t}\right|^2\right)^{1/2}\|b\|_{\mathbb{B}}$$
$$= \left\|F(x,y;\cdot)\right\|_H \|b\|_{\mathbb{B}}, \quad b\in\mathbb{B}. \tag{10}$$

Hence, if $L(\mathbb{B},\gamma(H,\mathbb{B}))$ denotes the space of bounded operators from $\mathbb{B}$ into $\gamma(H,\mathbb{B})$, we obtain

$$\left\|\tau(x,y)\right\|_{L(\mathbb{B},\gamma(H,\mathbb{B}))} \leqslant C\frac{e^{-c(|x-y|^2+|y||x-y|)}}{|x-y|^n}.$$

Let $j = 1, \dots, n$. We have that

$$\partial_{x_j}\big(t\partial_t P_t^{\mathcal{L}+\alpha}(x,y)\big) = \frac{t}{\sqrt{4\pi}}\int_0^\infty s^{-3/2}\left(1-\frac{t^2}{2s}\right)e^{-t^2/(4s)-\alpha s}\partial_{x_j}\big(W_s^{\mathcal{L}}(x,y)\big)\,ds$$
$$x, y \in \mathbb{R}^n \text{ and } t > 0.$$

Since

$$\partial_{x_j}\big(W_s^{\mathcal{L}}(x,y)\big) = -\frac{1}{2}\left(\frac{1+e^{-2s}}{1-e^{-2s}}(x_j-y_j) + \frac{1-e^{-2s}}{1+e^{-2s}}(x_j+y_j)\right)W_s^{\mathcal{L}}(x,y),$$
$$x, y \in \mathbb{R}^n \text{ and } s > 0,$$

we obtain that

$$\big|\partial_{x_j}\big(W_s^{\mathcal{L}}(x,y)\big)\big| \leqslant Ce^{-c(|x-y|^2+(|x|+|y|)|x-y|)}\frac{e^{-ns-c\frac{|x-y|^2}{s}}}{(1-e^{-4s})^{(n+1)/2}}, \quad x, y \in \mathbb{R}^n \text{ and } s > 0. \tag{11}$$

By proceeding as above we get

$$\big\|\partial_{x_j}\big(t\partial_t P_t^{\mathcal{L}+\alpha}(x,y)\big)\big\|_H \leqslant \frac{C}{|x-y|^{n+1}}, \quad x, y \in \mathbb{R}^n,\ x \neq y,$$

and then

$$\big\|\partial_{x_j}\tau(x,y)\big\|_{L(\mathbb{B},\gamma(H,\mathbb{B}))} \leqslant \frac{C}{|x-y|^{n+1}}, \quad x, y \in \mathbb{R}^n,\ x \neq y.$$

By taking into account symmetries we obtain the same estimates when ∂_{x_j} is replaced by ∂_{y_j}.

Next we show that if $f \in L_c^\infty(\mathbb{R}^n, \mathbb{B})$ then

$$t\partial_t P_t^{\mathcal{L}+\alpha}(f)(x) = \int_{\mathbb{R}^n}\tau(x,y)f(y)\,dy, \quad x \notin \operatorname{supp}(f), \tag{12}$$

where the integral is understood in the $\gamma(H,\mathbb{B})$-Bochner sense. Indeed, let $f \in L_c^\infty(\mathbb{R}^n, \mathbb{B})$ and $x \notin \operatorname{supp}(f)$. We have that

$$\int_{\mathbb{R}^n}\big\|\tau(x,y)f(y)\big\|_{\gamma(H,\mathbb{B})}\,dy \leqslant C\int_{\operatorname{supp}(f)}\frac{\|f(y)\|_{\mathbb{B}}}{|x-y|^n}\,dy < \infty.$$

Since $\gamma(H,\mathbb{B})$ is continuously contained in the space $L(H,\mathbb{B})$, $\tau(x,\cdot)f \in L^1(\mathbb{R}^n, L(H,\mathbb{B}))$. Then, there exists a sequence $(T_k)_{k\in\mathbb{N}}$ in $L^1(\mathbb{R}^n)\otimes L(H,\mathbb{B})$ such that

$$T_k \longrightarrow \tau(x,\cdot)f, \quad \text{as } k\to\infty, \text{ in } L^1\big(\mathbb{R}^n, L(H,\mathbb{B})\big).$$

Hence,

$$\int_{\mathbb{R}^n} T_k(y)\,dy \longrightarrow \int_{\mathbb{R}^n} \tau(x,y)f(y)\,dy, \quad \text{as } k\to\infty, \text{ in } L(H,\mathbb{B}),$$

and also, for every $h\in H$,

$$T_k[h] \longrightarrow \tau(x,\cdot)f[h], \quad \text{as } k\to\infty, \text{ in } L^1\big(\mathbb{R}^n,\mathbb{B}\big).$$

Suppose that $T=\sum_{\ell=1}^m f_\ell\tau_\ell$, where $f_\ell\in L^1(\mathbb{R}^n)$ and $\tau_\ell\in L(H,\mathbb{B})$, $\ell=1,\dots,m\in\mathbb{N}$. We can write

$$\left(\int_{\mathbb{R}^n} T(y)\,dy\right)[h] = \sum_{\ell=1}^m \tau_\ell[h]\int_{\mathbb{R}^n} f_\ell(y)\,dy = \int_{\mathbb{R}^n} T(y)[h]\,dy, \quad h\in H.$$

Hence, we conclude that

$$\left(\int_{\mathbb{R}^n} \tau(x,y)f(y)\,dy\right)[h] = \int_{\mathbb{R}^n} \tau(x,y)f(y)[h]\,dy, \quad h\in H,$$

where the last integral is understood in the $\mathbb{B}$-Bochner sense.

For every $h\in H$, by (9) we have that

$$\begin{aligned}\int_{\mathbb{R}^n} \tau(x,y)f(y)[h]\,dy &= \int_{\operatorname{supp}(f)} \left(\int_0^\infty t\partial_t P_t^{\mathcal{L}+\alpha}(x,y)h(t)\frac{dt}{t}\right)f(y)\,dy \\ &= \int_0^\infty \left(\int_{\operatorname{supp}(f)} t\partial_t P_t^{\mathcal{L}+\alpha}(x,y)f(y)\,dy\right)h(t)\frac{dt}{t} \\ &= \int_0^\infty t\partial_t P_t^{\mathcal{L}+\alpha}(f)(x)h(t)\frac{dt}{t} = \big(t\partial_t P_t^{\mathcal{L}+\alpha}(f)(x)\big)[h].\end{aligned}$$

Thus (12) is established.

We conclude that $\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}$ is a $(\mathbb{B},\gamma(H,\mathbb{B}))$-Hermite–Calderón–Zygmund operator.

On the other hand, by [34, Proposition 3.3] we have that

$$W_t^{\mathcal{L}}(1)(x) = \frac{1}{\pi^{n/2}}\left(\frac{e^{-2t}}{1+e^{-4t}}\right)^{n/2} \exp\left(-\frac{1-e^{-4t}}{2(1+e^{-4t})}|x|^2\right), \quad x \in \mathbb{R}^n \text{ and } t > 0. \tag{13}$$

It follows that, for every $x \in \mathbb{R}^n$ and $t > 0$,

$$\begin{aligned}\partial_t W_t^{\mathcal{L}+\alpha}(1)(x) &= \partial_t\big(e^{-\alpha t} W_t^{\mathcal{L}}(1)(x)\big) \\ &= -e^{-\alpha t}\left(\alpha + n\frac{1-e^{-4t}}{1+e^{-4t}} + |x|^2 \frac{4e^{-4t}}{(1+e^{-4t})^2}\right) W_t^{\mathcal{L}}(1)(x). \end{aligned} \tag{14}$$

We can write

$$\begin{aligned}\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(1)(x,t) &= \frac{t}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \partial_t W_{t^2/(4u)}^{\mathcal{L}+\alpha}(1)(x)\,du \\ &= \frac{t^2}{\sqrt{4\pi}} \int_0^\infty \frac{e^{-u}}{u^{3/2}} \partial_z W_z^{\mathcal{L}+\alpha}(1)(x)_{|z=t^2/(4u)}\,du, \quad x \in \mathbb{R}^n \text{ and } t > 0. \end{aligned} \tag{15}$$

Minkowski's inequality leads to

$$\begin{aligned}\big\|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(1)(x,\cdot)\big\|_H &\leqslant C \int_0^\infty \frac{e^{-u}}{u^{3/2}} \big\|t^2 \partial_z W_z^{\mathcal{L}+\alpha}(1)(x)_{|z=t^2/(4u)}\big\|_H\,du \\ &\leqslant C \int_0^\infty \frac{e^{-u}}{u^{1/2}} \big\|z\partial_z W_z^{\mathcal{L}+\alpha}(1)(x)\big\|_H\,du, \quad x \in \mathbb{R}^n. \end{aligned}$$

Moreover, we have that

$$\big\|z\partial_z W_z^{\mathcal{L}+\alpha}(1)(x)\big\|_H \leqslant C\left(\int_0^1 e^{-cz|x|^2}\big(1+|x|^4\big)z\,dz + \int_1^\infty e^{-2(n+\alpha)z} z\,dz\right)^{1/2} \leqslant C,$$
$$x \in \mathbb{R}^n.$$

Hence, $\|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(1)(x,\cdot)\|_H \in L^\infty(\mathbb{R}^n)$. As above, this means that $\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(1) \in L^\infty(\mathbb{R}^n, H)$.

In a similar way we can see that, for every $j = 1,\dots,n$, $\partial_{x_j}\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(1) \in L^\infty(\mathbb{R}^n, H)$.

By using Theorem 1.1 we can show that the operator $\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}$ is bounded from $BMO_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})$ into $BMO_{\mathcal{L}}(\mathbb{R}^n,\gamma(H,\mathbb{B}))$.

2.2. We are going to prove that $\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}$ is a bounded operator from $H^1_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})$ into $H^1_{\mathcal{L}}(\mathbb{R}^n,\gamma(H,\mathbb{B}))$. In order to show this property we extend to a Banach valued setting the atomic characterization of Hardy spaces due to Dziubański and Zienkiewicz [12,14].

A strongly measurable function $a:\mathbb{R}^n\longrightarrow\mathbb{B}$ is an atom for $H^1_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})$ when there exist $x_0\in\mathbb{R}^n$ and $0<r_0\leqslant\rho(x_0)$ such that the support of a is contained in $B(x_0,r_0)$ and

(i) $\|a\|_{L^\infty(\mathbb{R}^n,\mathbb{B})}\leqslant|B(x_0,r_0)|^{-1}$,
(ii) $\int_{\mathbb{R}^n}a(x)\,dx=0$, provided that $r_0\leqslant\rho(x_0)/2$.

Proposition 2.1. *Let Y be a Banach space. Suppose that $f\in L^1(\mathbb{R}^n,Y)$. The following assertions are equivalent.*

(i) $\sup_{t>0}\|W_t^{\mathcal{L}}(f)\|_Y\in L^1(\mathbb{R}^n)$.
(ii) $\sup_{t>0}\|P_t^{\mathcal{L}}(f)\|_Y\in L^1(\mathbb{R}^n)$.
(iii) *There exist a sequence $(a_j)_{j\in\mathbb{N}}$ of atoms in $H^1_{\mathcal{L}}(\mathbb{R}^n,Y)$ and a sequence $(\lambda_j)_{j\in\mathbb{N}}$ of complex numbers such that $\sum_{j\in\mathbb{N}}|\lambda_j|<\infty$ and $f=\sum_{j\in\mathbb{N}}\lambda_ja_j$.*

Proof. Dziubański and Zienkiewicz proved in [14, Theorem 1.5] (see also [12]) that (i) $\Leftrightarrow$ (iii) for $Y=\mathbb{C}$. In order to show [14, Theorem 1.5] they use the atomic decomposition for the functions in the local Hardy space $h^1(\mathbb{R}^n)$ established by Goldberg [16, Lemma 5]. By reading carefully [32, Theorem 1, p. 91, and Theorem 2, p. 107] we can see that the classical Banach valued $H^1(\mathbb{R}^n,Y)$ can be defined by using different maximal functions and by atomic representations, that is, [32, Theorem 1, p. 91, and Theorem 2, p. 107] continue being true when we replace $H^1(\mathbb{R}^n)$ by $H^1(\mathbb{R}^n,Y)$. Then, if we define the Banach valued local Hardy space $h^1(\mathbb{R}^n,Y)$ in the natural way, $h^1(\mathbb{R}^n,Y)$ can be described by the corresponding maximal functions and by atomic decompositions (see [16, Theorem 1 and Lemma 5]). More precisely, the arguments in the proofs of [16, Theorem 1 and Lemma 5] allow us to show that if $f\in L^1(\mathbb{R}^n,Y)$ then $f\in h^1(\mathbb{R}^n,Y)$ if and only if $f=\sum_{j\in\mathbb{N}}\lambda_ja_j$, where $\lambda_j\in\mathbb{C}$, $j\in\mathbb{N}$, and $\sum_{j\in\mathbb{N}}|\lambda_j|<\infty$, and, for every $j\in\mathbb{N}$, a_j is an h^1-atom as in [16, p. 37] but taking values in Y. With these comments in mind and by proceeding as in the proof of [14, Theorem 1.5] we conclude that (i) $\Leftrightarrow$ (iii).

By the subordination representation (4) of $P_t^{\mathcal{L}}$, $t>0$, we deduce that (i) $\Rightarrow$ (ii).

To finish the proof we are going to see that (ii) $\Rightarrow$ (iii). In order to show this we can proceed as in the proof of [14, Theorem 1.5]. We present a sketch of the proof. Firstly, by (4) and (7) and proceeding as in (8) we deduce that

$$P_t^{\mathcal{L}}(x,y)\leqslant Ce^{-c(|x-y|^2+|x||x-y|)}\frac{t}{(t+|x-y|)^{n+1}},\qquad x,y\in\mathbb{R}^n\text{ and }t>0.\tag{16}$$

Hence, for every $\ell\in\mathbb{N}$, there exists $C>0$ such that

$$P_t^{\mathcal{L}}(x,y)\leqslant C\left(1+\frac{|x-y|}{\rho(x)}\right)^{-\ell}|x-y|^{-n},\qquad x,y\in\mathbb{R}^n\text{ and }t>0.\tag{17}$$

Moreover, for every $M > 0$ we can find $C > 0$ for which

$$\left|P_t^{\mathcal{L}}(x,y) - P_t(x-y)\right| \leqslant C\left(\frac{|x-y|}{\rho(x)}\right)^{1/2}|x-y|^{-n},$$
$$x, y \in \mathbb{R}^n,\ |x-y| \leqslant M\rho(x) \text{ and } t > 0, \tag{18}$$

where P_t denotes the classical Poisson semigroup.

Indeed, let $M > 0$. According to (4) we can write

$$\left|P_t^{\mathcal{L}}(x,y) - P_t(x-y)\right| \leqslant Ct\int_0^\infty \frac{e^{-t^2/(4s)}}{s^{3/2}}\left|W_s^{\mathcal{L}}(x,y) - W_s(x-y)\right|ds,$$
$$x, y \in \mathbb{R}^n \text{ and } t > 0,$$

where $W_t(x) = e^{-|x|^2/(4t)}/(4\pi t)^{n/2}$, $x \in \mathbb{R}^n$ and $t > 0$. From (7) it follows that

$$t\int_{\rho(x)^2}^\infty \frac{e^{-t^2/(4s)}}{s^{3/2}}\left|W_s^{\mathcal{L}}(x,y) - W_s(x-y)\right|ds$$
$$\leqslant Ct\int_{\rho(x)^2}^\infty \frac{e^{-c(t^2+|x-y|^2)/s}}{s^{(n+3)/2}}\,ds \leqslant C\int_{\rho(x)^2}^\infty \frac{ds}{s^{(n+2)/2}}$$
$$\leqslant \frac{C}{\rho(x)^n} = C\left(\frac{|x-y|}{\rho(x)}\right)^n\frac{1}{|x-y|^n}, \quad x, y \in \mathbb{R}^n,\ x \neq y \text{ and } t > 0.$$

On the other hand, we have that

$$t\int_0^{\rho(x)^2} \frac{e^{-t^2/(4s)}}{s^{3/2}}\left|W_s^{\mathcal{L}}(x,y) - W_s(x-y)\right|ds$$
$$\leqslant C\left\{t\int_0^{\rho(x)^2} \frac{e^{-c(t^2+|x-y|^2)/s}}{s^{(n+3)/2}}\left|e^{-ns} - 1\right|ds\right.$$
$$+\,t\int_0^{\rho(x)^2} \frac{e^{-c(t^2+|x-y|^2)/s}}{s^{3/2}}\left|\frac{1}{(1-e^{-4s})^{n/2}} - \frac{1}{(4s)^{n/2}}\right|ds$$
$$+\,t\int_0^{\rho(x)^2} \frac{e^{-c(t^2+|x-y|^2)/s}}{s^{(n+3)/2}}\left|\exp\left(-\frac{1}{4}\frac{1-e^{-2s}}{1+e^{-2s}}|x+y|^2\right) - 1\right|ds$$

$$+t\int_0^{\rho(x)^2}\frac{e^{-t^2/(4s)}}{s^{(n+3)/2}}\left|\exp\left(-\frac{1}{4}\frac{1+e^{-2s}}{1-e^{-2s}}|x-y|^2\right)-e^{-|x-y|^2/(4s)}\right|ds\Bigg\}$$

$$=C\sum_{j=1}^4 I_j(x,y,t)\quad x,y\in\mathbb{R}^n \text{ and } t>0.$$

Since $|e^{-ns}-1|\leqslant Cs$, $s>0$, and

$$\left|\frac{1}{(1-e^{-4s})^{n/2}}-\frac{1}{(4s)^{n/2}}\right|\leqslant\frac{C}{s^{n/2-1}},\quad 0<s<1,$$

we deduce that

$$\begin{aligned}I_j(x,y,t)&\leqslant Ct\int_0^{\rho(x)^2}\frac{e^{-c(t^2+|x-y|^2)/s}}{s^{(n+1)/2}}\,ds\\&\leqslant C\int_0^1\frac{e^{-c(t^2+|x-y|^2)/s}}{s^{n/2}}\,ds\leqslant C\frac{1}{(t^2+|x-y|^2)^{(n/2-1/4)}}\int_0^1\frac{ds}{s^{1/4}}\\&\leqslant\frac{C}{|x-y|^{n-1/2}}\\&\leqslant C\left(\frac{|x-y|}{\rho(x)}\right)^{1/2}\frac{1}{|x-y|^n},\quad x,y\in\mathbb{R}^n,\ x\neq y,\ t>0 \text{ and } j=1,2.\end{aligned}$$

Also, we have that, for every $x,y\in\mathbb{R}^n$ and $s>0$,

$$\left|\exp\left(-\frac{1}{4}\frac{1+e^{-2s}}{1-e^{-2s}}|x-y|^2\right)-e^{-|x-y|^2/(4s)}\right|\leqslant Ce^{-|x-y|^2/(4s)}|x-y|^2\leqslant Cse^{-c|x-y|^2/s}.$$

Then, by proceeding as above we get

$$I_4(x,y,t)\leqslant C\left(\frac{|x-y|}{\rho(x)}\right)^{1/2}\frac{1}{|x-y|^n},\quad x,y\in\mathbb{R}^n,\ x\neq y \text{ and } t>0.$$

Finally, we analyze I_3. We have that

$$\left|\exp\left(-\frac{1}{4}\frac{1-e^{-2s}}{1+e^{-2s}}|x+y|^2\right)-1\right|\leqslant Cs|x+y|^2\leqslant C\frac{s}{\rho(x)^2},$$
$$|x-y|\leqslant M\rho(x) \text{ and } s>0.$$

Hence, it follows that

$$I_3(x,y,t) \leqslant \frac{C}{\rho(x)^2} \int_0^{\rho(x)^2} \frac{e^{-c(t^2+|x-y|^2)/s}}{s^{n/2}}\, ds$$
$$\leqslant \frac{C}{\rho(x)^2|x-y|^{n-1/2}} \int_0^{\rho(x)^2} \frac{ds}{s^{1/4}} = C\left(\frac{|x-y|}{\rho(x)}\right)^{1/2} \frac{1}{|x-y|^n},$$

provided that $|x-y| \leqslant M\rho(x)$, $x \neq y$ and $t > 0$.

By combining the above estimates we obtain (18).

Estimations (17) and (18) can be also obtained when $n \geqslant 3$ as special cases of [14, Lemma 3.0].

According to [30, p. 517, line 5]

$$\rho(x) \sim \frac{1}{M(x)} = \sup\left\{r > 0\colon\ \frac{1}{r^{n-2}} \int_{B(x,r)} |y|^2\, dy \leqslant 1\right\}.$$

Since $\rho(x) \leqslant 1/2$, there exists $m_0 \in \mathbb{Z}$ such that the set $\mathcal{B}_m = \{x \in \mathbb{R}^n\colon\ 2^{m/2} \leqslant M(x) < 2^{\frac{m+1}{2}}\}$ is empty, provided that $m < m_0$. Then, for every $m \in \mathbb{Z}$, $m \geqslant m_0$, and $k \in \mathbb{N}$ we can consider $x_{(m,k)} \in \mathbb{R}^n$ as in [14, Lemma 2.3] and choose, according to [14, Lemma 2.5], a function $\psi_{(m,k)} \in C_c^\infty(B(x_{(m,k)}, 2^{(2-m)/2}))$ such that $\|\nabla\psi_{(m,k)}\|_{L^\infty(\mathbb{R}^n)} \leqslant C2^{m/2}$ and $\sum_{(m,k)} \psi_{(m,k)} = 1$, $x \in \mathbb{R}^n$. Here $C > 0$ does not depend on (m,k). We can assume $m_0 = 0$ to make the reading easier.

For every $m, k \in \mathbb{N}$, let us define $B_{(m,k)} = B(x_{(m,k)}, 2^{(4-m)/2})$ and $\widehat{B}_{(m,k)} = B(x_{(m,k)}, (\sqrt{n}+1)2^{(4-m)/2})$ and consider the maximal operators

$$\widetilde{\mathcal{M}}_m(f) = \sup_{0<t\leqslant 2^{-m}} \left\|P_t(f) - P_t^{\mathcal{L}}(f)\right\|_Y, \qquad \mathcal{M}_m^{\mathcal{L}}(f) = \sup_{0<t\leqslant 2^{-m}} \left\|P_t^{\mathcal{L}}(f)\right\|_Y,$$
$$\mathcal{M}_m(f) = \sup_{0<t\leqslant 2^{-m}} \left\|P_t(f)\right\|_Y,$$

and the maximal commutator operator

$$\mathcal{M}_{(m,k)}^{\mathcal{L}}(f) = \sup_{0<t\leqslant 2^{-m}} \left\|P_t^{\mathcal{L}}(\psi_{(m,k)}f) - \psi_{(m,k)}P_t^{\mathcal{L}}(f)\right\|_Y.$$

Let $m, k \in \mathbb{N}$. By using (16) we deduce that, for a certain $C > 0$ independent of m and k,

$$\sup_{y\in B_{(m,k)}} \int_{\mathbb{R}^n\setminus\widehat{B}_{(m,k)}} \sup_{0<t\leqslant 2^{-m}} \left|P_t^{\mathcal{L}}(x,y) - P_t(x,y)\right| dx \leqslant C.$$

Indeed, if $x, y \in \mathbb{R}^n$, $x \neq y$, the function $w(t) = t/(t^2+|x-y|^2)^{(n+1)/2}$, $t > 0$, is increasing in the interval $(0, |x-y|/\sqrt{n})$ and it is decreasing in the interval $(|x-y|/\sqrt{n}, \infty)$. If $x \in \mathbb{R}^n \setminus \widehat{B}_{(m,k)}$ and $y \in B_{(m,k)}$, $|x-y| \geqslant \sqrt{n}\,2^{(4-m)/2}$. Hence, from (16) it follows that

$$\sup_{y\in B_{(m,k)}} \int_{\mathbb{R}^n\setminus \widehat{B}_{(m,k)}} \sup_{0<t\leqslant 2^{-m}} \left|P_t^{\mathcal{L}}(x,y)-P_t(x,y)\right| dx$$

$$\leqslant C2^{-m} \sup_{y\in B_{(m,k)}} \int_{\mathbb{R}^n\setminus \widehat{B}_{(m,k)}} \frac{1}{(2^{-2m}+|x-y|^2)^{(n+1)/2}}\, dx$$

$$\leqslant C2^{-m} \int_{\mathbb{R}^n\setminus B(0,\sqrt{n}\,2^{(4-m)/2})} \frac{1}{(2^{-2m}+|u|^2)^{(n+1)/2}}\, du \leqslant C\frac{2^{-m}}{2^{-m}+\sqrt{n}\,2^{(4-m)/2}} \leqslant C.$$

By (18) and arguing as in [14, Lemma 3.9], we conclude that, for a certain $C>0$,

$$\left\|\widetilde{\mathcal{M}}_m(\psi_{(m,k)}f)\right\|_{L^1(\mathbb{R}^n)} \leqslant C\|\psi_{(m,k)}f\|_{L^1(\mathbb{R}^n,Y)}, \quad f\in L^1(\mathbb{R}^n,Y).$$

Also, by proceeding as in the proof of [14, Lemma 3.11] we can find $C>0$ such that

$$\sum_{(m,k)} \left\|\mathcal{M}^{\mathcal{L}}_{(m,k)}(f)\right\|_{L^1(\mathbb{R}^n)} \leqslant C\|f\|_{L^1(\mathbb{R}^n,Y)}, \quad f\in L^1(\mathbb{R}^n,Y).$$

By combining the above estimates we deduce that

$$\sum_{(m,k)} \left\|\mathcal{M}_m(\psi_{(m,k)}f)\right\|_{L^1(\mathbb{R}^n)} \leqslant C\Big(\|f\|_{L^1(\mathbb{R}^n,Y)} + \Big\|\sup_{t>0}\|P_t^{\mathcal{L}}f\|_Y\Big\|_{L^1(\mathbb{R}^n)}\Big) < \infty,$$

provided that (ii) holds.

Now the proof of (ii) $\Rightarrow$ (iii) can be finished as in [14, Section 4]. □

In the next result we complete the last proposition characterizing the Hardy space by the maximal operator associated with the semigroup $\{P_t^{\mathcal{L}+\alpha}\}_{t>0}$.

Proposition 2.2. *Let Y be a Banach space and $\alpha>-n$. Suppose that $f\in L^1(\mathbb{R}^n,Y)$. Then $f\in H^1_{\mathcal{L}}(\mathbb{R}^n,Y)$ if, and only if, $\sup_{t>0}\|P_t^{\mathcal{L}+\alpha}(f)\|_Y\in L^1(\mathbb{R}^n)$.*

Proof. We consider the operator L_α defined by

$$L_\alpha(g)=\sup_{t>0}\left\|P_t^{\mathcal{L}+\alpha}(g)-P_t^{\mathcal{L}}(g)\right\|_Y, \quad g\in L^1(\mathbb{R}^n,Y).$$

We can write

$$L_\alpha(g)(x)=\sup_{t>0}\left\|\int_{\mathbb{R}^n} L_\alpha(x,y;t)g(y)\,dy\right\|_Y, \quad x\in\mathbb{R}^n,$$

where

$$L_\alpha(x,y;t)=\frac{t}{\sqrt{4\pi}}\int_0^\infty \frac{e^{-t^2/(4u)}}{u^{3/2}}\big(e^{-\alpha u}-1\big)W_u^{\mathcal{L}}(x,y)\,du,\quad x,y\in\mathbb{R}^n \text{ and } t>0.$$

From (7) and by taking into account that $|e^{-(\alpha+n)u}-e^{-nu}|\leqslant Cue^{-cu}$, $u\in(0,\infty)$, we obtain that

$$\begin{aligned}
\big|L_\alpha(x,y;t)\big| &\leqslant Cte^{-c|x-y|^2}\int_0^\infty \frac{e^{-c(t^2+|x-y|^2)/u}}{u^{3/2}}\frac{|e^{-(\alpha+n)u}-e^{-nu}|}{(1-e^{-4u})^{n/2}}\,du\\
&\leqslant Cte^{-c|x-y|^2}\int_0^\infty \frac{e^{-c(|x-y|^2+t^2)/u}e^{-cu}}{u^{1/2}(1-e^{-4u})^{n/2}}\,du\\
&\leqslant Cte^{-c|x-y|^2}\int_0^\infty \frac{e^{-c(|x-y|^2+t^2)/u}}{u^{n/2+5/4}}\,du\\
&\leqslant C\frac{e^{-c|x-y|^2}}{|x-y|^{n-1/2}},\quad x,y\in\mathbb{R}^n,\ x\neq y \text{ and } t>0.
\end{aligned}$$

Hence, for every $g\in L^1(\mathbb{R}^n,Y)$,

$$\int_{\mathbb{R}^n}\big|L_\alpha(g)(x)\big|\,dx\leqslant C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{e^{-c|x-y|^2}}{|x-y|^{n-1/2}}\big\|g(y)\big\|_Y\,dy\,dx\leqslant C\|g\|_{L^1(\mathbb{R}^n,Y)}.$$

This shows that L_α is a bounded (sublinear) operator from $L^1(\mathbb{R}^n,Y)$ into $L^1(\mathbb{R}^n)$.

The proof of this property can be finished by using Proposition 2.1. □

As usual by $H^1(\mathbb{R}^n,\mathbb{B})$ we denote the classical $\mathbb{B}$-valued Hardy space.

Proposition 2.3. *Let Y be a UMD Banach space and $\alpha>-n$. The (sublinear) operator $T_\alpha^{\mathcal{L}}$ defined by*

$$T_\alpha^{\mathcal{L}}(f)(x)=\sup_{s>0}\big\|P_s^{\mathcal{L}+\alpha}\mathcal{G}_{\mathcal{L}+\alpha,Y}(f)(x,\cdot)\big\|_{\gamma(H,Y)},$$

is bounded from $H^1(\mathbb{R}^n,Y)$ into $L^1(\mathbb{R}^n)$ and from $L^1(\mathbb{R}^n,Y)$ into $L^{1,\infty}(\mathbb{R}^n)$.

Proof. In order to show this property we use Banach valued Calderón–Zygmund theory [29].

As in (8) we can see that

$$P_t^{\mathcal{L}+\alpha}(x,y) \leqslant C\frac{t}{(t+|x-y|)^{n+1}}, \quad x,y\in\mathbb{R}^n \text{ and } t>0.$$

Hence, it follows that

$$\sup_{t>0}\big\|P_t^{\mathcal{L}+\alpha}(g)\big\|_Y \leqslant C\sup_{t>0}P_t\big(\|g\|_Y\big), \quad g\in L^p\big(\mathbb{R}^n,Y\big),\ 1\leqslant p<\infty,$$

and from well-known results we deduce that the maximal operator

$$P_*^{\mathcal{L}+\alpha}(g) = \sup_{t>0}\big\|P_t^{\mathcal{L}+\alpha}(g)\big\|_Y,$$

is bounded from $L^p(\mathbb{R}^n,Y)$ into $L^p(\mathbb{R}^n)$, for every $1<p<\infty$, and from $L^1(\mathbb{R}^n,Y)$ into $L^{1,\infty}(\mathbb{R}^n)$.

Moreover, according to [2, Theorem 1] the operator $\mathcal{G}_{\mathcal{L}+\alpha,Y}$ is bounded from

- $L^p(\mathbb{R}^n,Y)$ into $L^p(\mathbb{R}^n,\gamma(H,Y))$, $1<p<\infty$,
- $L^1(\mathbb{R}^n,Y)$ into $L^{1,\infty}(\mathbb{R}^n,\gamma(H,Y))$, and
- $H^1(\mathbb{R}^n,Y)$ into $L^1(\mathbb{R}^n,\gamma(H,Y))$.

Hence, if we define the operator $\mathbb{T}_\alpha^{\mathcal{L}}$ by

$$\mathbb{T}_\alpha^{\mathcal{L}}(f)(x,s,t) = P_s^{\mathcal{L}+\alpha}\mathcal{G}_{\mathcal{L}+\alpha,Y}(f)(x,t), \quad x\in\mathbb{R}^n,\ s,t>0,$$

it is bounded from $L^p(\mathbb{R}^n,Y)$ into $L^p(\mathbb{R}^n,L^\infty((0,\infty),\gamma(H,Y)))$, $1<p<\infty$, and from $H^1(\mathbb{R}^n,Y)$ into $L^{1,\infty}(\mathbb{R}^n,L^\infty((0,\infty),\gamma(H,Y)))$.

We are going to show that $\mathbb{T}_\alpha^{\mathcal{L}}$ is bounded from $L^1(\mathbb{R}^n,Y)$ into $L^{1,\infty}(\mathbb{R}^n,L^\infty((0,\infty),\gamma(H,Y)))$ and from $H^1(\mathbb{R}^n,Y)$ into $L^1(\mathbb{R}^n,L^\infty((0,\infty),\gamma(H,Y)))$.

We consider the function

$$\Omega_\alpha(x,y;s,t) = t\partial_t P_{t+s}^{\mathcal{L}+\alpha}(x,y), \quad x,y\in\mathbb{R}^n \text{ and } s,t>0. \tag{19}$$

It follows from (8) that

$$\big|\Omega_\alpha(x,y;s,t)\big| \leqslant C\frac{t}{(s+t+|x-y|)^{n+1}}, \quad x,y\in\mathbb{R}^n \text{ and } s,t>0. \tag{20}$$

Let $j=1,\dots,n$. By (11) we get

$$\begin{aligned}\big|\partial_{x_j}\Omega_\alpha(x,y;s,t)\big| &\leqslant Ct\int_0^\infty \frac{1}{u^{(n+4)/2}}e^{-c(|x-y|^2+(s+t)^2)/u}\,du \\ &\leqslant C\frac{t}{(s+t+|x-y|)^{n+2}}, \quad x,y\in\mathbb{R}^n \text{ and } s,t>0. \end{aligned}\tag{21}$$

By taking into account the symmetries we also have that

$$\left|\partial_{y_j}\Omega_\alpha(x,y;s,t)\right| \leqslant C\frac{t}{(s+t+|x-y|)^{n+2}}, \quad x,y\in\mathbb{R}^n \text{ and } s,t>0. \tag{22}$$

Let $N\in\mathbb{N}$ and $C([1/N,N],Y)$ be the space of continuous functions over the interval $[1/N,N]$ which take values in the Banach space Y. The function $\Omega_\alpha(x,y;s,t)$ satisfies the following Calderón–Zygmund type estimates

$$\left\|\Omega_\alpha(x,y;\cdot,\cdot)\right\|_{C([1/N,N],H)} \leqslant \left\|\Omega_\alpha(x,y;\cdot,\cdot)\right\|_{L^\infty((0,\infty),H)} \leqslant \frac{C}{|x-y|^n}, \quad x,y\in\mathbb{R}^n,\ x\neq y, \tag{23}$$

and

$$\begin{aligned}
&\left\|\nabla_x\Omega_\alpha(x,y;\cdot,\cdot)\right\|_{C([1/N,N],H)} + \left\|\nabla_y\Omega_\alpha(x,y;\cdot,\cdot)\right\|_{C([1/N,N],H)}\\
&\quad\leqslant \left\|\nabla_x\Omega_\alpha(x,y;\cdot,\cdot)\right\|_{L^\infty((0,\infty),H)} + \left\|\nabla_y\Omega_\alpha(x,y;\cdot,\cdot)\right\|_{L^\infty((0,\infty),H)}\\
&\quad\leqslant \frac{C}{|x-y|^{n+1}}, \quad x,y\in\mathbb{R}^n,\ x\neq y.
\end{aligned} \tag{24}$$

Note that the constant C does not depend on N. Indeed, by (20) we get

$$\begin{aligned}
\left\|\Omega_\alpha(x,y;\cdot,\cdot)\right\|_{L^\infty((0,\infty),H)} &\leqslant C\sup_{s>0}\left(\int_0^\infty \frac{t}{((s+t)^2+|x-y|^2)^{n+1}}\,dt\right)^{1/2}\\
&\leqslant C\left(\int_0^\infty \frac{dt}{(t+|x-y|)^{2n+1}}\right)^{1/2}\\
&\leqslant \frac{C}{|x-y|^n}, \quad x,y\in\mathbb{R}^n,\ x\neq y,
\end{aligned}$$

and (23) is established. In a similar way we can deduce (24) from (21) and (22).

Suppose now that $g\in L_c^\infty(\mathbb{R}^n)$. By (23) it is clear that

$$\int_{\mathbb{R}^n} \left\|\Omega_\alpha(x,y;\cdot,\cdot)\right\|_{C([1/N,N],H)}\left|g(y)\right|dy<\infty, \quad x\notin \operatorname{supp}(g).$$

We define

$$S_\alpha(g)(x) = \int_{\mathbb{R}^n} \Omega_\alpha(x,y;\cdot,\cdot)g(y)\,dy, \quad x\notin\operatorname{supp}(g),$$

where the integral is understood in the $C([1/N,N],H)$-Bochner sense. We have that

$$\big[S_\alpha(g)(x)\big](s,\cdot)=\int_{\mathbb{R}^n}\Omega_\alpha(x,y;s,\cdot)g(y)\,dy,\quad x\notin \operatorname{supp}(g)\text{ and } s\in[1/N,N].$$

Here the equality and the integral are understood in H and in the H-Bochner sense, respectively.

For every $h\in H$, we can write

$$\begin{aligned}&\Big\langle h,\int_{\mathbb{R}^n}\Omega_\alpha(x,y;s,\cdot)g(y)\,dy\Big\rangle_{H,H}\\&\quad=\int_{\mathbb{R}^n}\int_0^\infty\Omega_\alpha(x,y;s,t)h(t)\frac{dt}{t}g(y)\,dy\\&\quad=\int_0^\infty\int_{\mathbb{R}^n}\Omega_\alpha(x,y;s,t)g(y)\,dy\,h(t)\frac{dt}{t},\quad x\notin\operatorname{supp}(g)\text{ and } s\in[1/N,N].\end{aligned}$$

Hence, for every $x\notin\operatorname{supp}(g)$ and $s>0$,

$$\int_{\mathbb{R}^n}\Omega_\alpha(x,y;s,t)g(y)\,dy=\Big(\int_{\mathbb{R}^n}\Omega_\alpha(x,y;s,\cdot)g(y)\,dy\Big)(t),$$

as elements of H.

We have proved that

$$P_s^{\mathcal{L}+\alpha}\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(g)(x,\cdot)=\big[S_\alpha(g)(x)\big](s,\cdot),\quad x\notin\operatorname{supp}(g)\text{ and } s\in[1/N,N],$$

in the sense of equality in H.

Assume that $g=\sum_{j=1}^m b_jg_j$, where $b_j\in Y$ and $g_j\in L_c^\infty(\mathbb{R}^n)$, $j=1,\dots,m\in\mathbb{N}$. Then,

$$\begin{aligned}&P_s^{\mathcal{L}+\alpha}\mathcal{G}_{\mathcal{L}+\alpha,Y}(g)(x,\cdot)\\&\quad=\sum_{j=1}^m b_jP_s^{\mathcal{L}+\alpha}\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(g_j)(x,\cdot)=\sum_{j=1}^m b_j\big[S_\alpha(g_j)(x)\big](s,\cdot)\\&\quad=\Big(\int_{\mathbb{R}^n}\Omega_\alpha(x,y;\cdot,\cdot)g(y)\,dy\Big)(s,\cdot),\quad x\notin\operatorname{supp}(g)\text{ and } s\in[1/N,N],\end{aligned}$$

where the last integral is understood in the $C([1/N,N],\gamma(H,Y))$-Bochner sense.

According to Banach valued Calderón–Zygmund theory (see [29]) we deduce that the operator $\mathbb{T}_\alpha^{\mathcal{L}}$ can be extended from

- $L^2(\mathbb{R}^n, Y) \cap L^1(\mathbb{R}^n, Y)$ to $L^1(\mathbb{R}^n, Y)$ as a bounded operator from $L^1(\mathbb{R}^n, Y)$ into $L^{1,\infty}(\mathbb{R}^n, C([1/N, N], \gamma(H, Y)))$, and as
- a bounded operator from $H^1(\mathbb{R}^n, Y)$ into $L^1(\mathbb{R}^n, C([1/N, N], \gamma(H, Y)))$.

Moreover, if we denote by $\widetilde{\mathbb{T}}^{\mathcal{L}}_{\alpha,N}$ the extension of $\mathbb{T}^{\mathcal{L}}_{\alpha}$ to $L^1(\mathbb{R}^n, Y)$ there exists $C > 0$ independent of N such that

$$\big\|\widetilde{\mathbb{T}}^{\mathcal{L}}_{\alpha,N}\big\|_{L^1(\mathbb{R}^n,Y)\to L^{1,\infty}(\mathbb{R}^n,C([1/N,N],\gamma(H,Y)))} \leqslant C$$

and

$$\big\|\widetilde{\mathbb{T}}^{\mathcal{L}}_{\alpha,N}\big\|_{H^1(\mathbb{R}^n,Y)\to L^1(\mathbb{R}^n,C([1/N,N],\gamma(H,Y)))} \leqslant C.$$

Let $g \in L^1(\mathbb{R}^n, Y)$ and let $(g_k)_{k\in\mathbb{N}}$ be a sequence in $L^1(\mathbb{R}^n, Y) \cap L^2(\mathbb{R}^n, Y)$ such that

$$g_k \longrightarrow g, \quad \text{as } k \to \infty, \text{ in } L^1\big(\mathbb{R}^n, Y\big).$$

It is not difficult to see that

$$\mathbb{T}^{\mathcal{L}}_{\alpha}(g)(x, s, t) = \mathcal{G}_{\mathcal{L}+\alpha,Y}\big(P^{\mathcal{L}+\alpha}_s(g)\big)(x, t), \quad x \in \mathbb{R}^n \text{ and } s, t > 0,$$

and

$$\mathbb{T}^{\mathcal{L}}_{\alpha}(g_k)(x, s, t) = \mathcal{G}_{\mathcal{L}+\alpha,Y}\big(P^{\mathcal{L}+\alpha}_s(g_k)\big)(x, t), \quad x \in \mathbb{R}^n,\ s, t > 0 \text{ and } k \in \mathbb{N}.$$

Hence, since $P^{\mathcal{L}+\alpha}_s$ is bounded from $L^1(\mathbb{R}^n, Y)$ into itself, for every $s > 0$, and $\mathcal{G}_{\mathcal{L}+\alpha,Y}$ is bounded from $L^1(\mathbb{R}^n, Y)$ into $L^{1,\infty}(\mathbb{R}^n, \gamma(H, Y))$ [2, Theorem 1],

$$\mathbb{T}^{\mathcal{L}}_{\alpha}(g_k)(\cdot, s, \cdot) \longrightarrow \mathbb{T}^{\mathcal{L}}_{\alpha}(g)(\cdot, s, \cdot), \quad \text{as } k \to \infty, \text{ in } L^{1,\infty}\big(\mathbb{R}^n, \gamma(H, Y)\big),$$

for every $s > 0$. Moreover, we can find a subsequence $(g_{k_\ell})_{\ell\in\mathbb{N}}$ of $(g_k)_{k\in\mathbb{N}}$ verifying that for every $s \in \mathbb{Q}$,

$$\mathbb{T}^{\mathcal{L}}_{\alpha}(g_{k_\ell})(x, s, \cdot) \longrightarrow \mathbb{T}^{\mathcal{L}}_{\alpha}(g)(x, s, \cdot), \quad \text{as } \ell \to \infty, \text{ in } \gamma(H, Y),$$

a.e. $x \in \mathbb{R}^n$. On the other hand,

$$\mathbb{T}^{\mathcal{L}}_{\alpha}(g_{k_\ell}) = \widetilde{\mathbb{T}}^{\mathcal{L}}_{\alpha,N}(g_{k_\ell}) \longrightarrow \widetilde{\mathbb{T}}^{\mathcal{L}}_{\alpha,N}(g), \quad \text{as } \ell \to \infty, \text{ in } L^{1,\infty}\big(\mathbb{R}^n, C\big([1/N, N], \gamma(H, Y)\big)\big),$$

and then, there exists a subsequence $(g_{k_{\ell_j}})_{j\in\mathbb{N}}$ of $(g_{k_\ell})_{\ell\in\mathbb{N}}$ such that, for every $s \in [1/N, N]$,

$$\mathbb{T}^{\mathcal{L}}_{\alpha}(g_{k_{\ell_j}})(x, s, \cdot) \longrightarrow \widetilde{\mathbb{T}}^{\mathcal{L}}_{\alpha,N}(g)(x, s, \cdot), \quad \text{as } j \to \infty, \text{ in } \gamma(H, Y),$$

a.e. $x \in \mathbb{R}^n$. Thus, for every $s \in [1/N, N] \cap \mathbb{Q}$,

$$\widetilde{\mathbb{T}}^{\mathcal{L}}_{\alpha,N}(g)(x,s,\cdot) = \mathbb{T}^{\mathcal{L}}_{\alpha}(g)(x,s,\cdot), \quad \text{a.e. } x \in \mathbb{R}^n, \text{ in } \gamma(H,Y).$$

Finally,

$$\begin{aligned}
&\left|\left\{x \in \mathbb{R}^n\colon \sup_{s>0}\left\|\mathbb{T}^{\mathcal{L}}_{\alpha}(g)(x,s,\cdot)\right\|_{\gamma(H,Y)} > \lambda\right\}\right| \\
&\quad\leqslant \left|\bigcup_{N\in\mathbb{N}}\left\{x \in \mathbb{R}^n\colon \sup_{s\in[1/N,N]}\left\|\mathbb{T}^{\mathcal{L}}_{\alpha}(g)(x,s,\cdot)\right\|_{\gamma(H,Y)} > \lambda\right\}\right| \\
&\quad= \lim_{N\to\infty}\left|\left\{x \in \mathbb{R}^n\colon \sup_{s\in[1/N,N]\cap\mathbb{Q}}\left\|\mathbb{T}^{\mathcal{L}}_{\alpha}(g)(x,s,\cdot)\right\|_{\gamma(H,Y)} > \lambda\right\}\right| \\
&\quad= \lim_{N\to\infty}\left|\left\{x \in \mathbb{R}^n\colon \sup_{s\in[1/N,N]\cap\mathbb{Q}}\left\|\widetilde{\mathbb{T}}^{\mathcal{L}}_{\alpha,N}(g)(x,s,\cdot)\right\|_{\gamma(H,Y)} > \lambda\right\}\right| \\
&\quad\leqslant \frac{C}{\lambda}\|g\|_{L^1(\mathbb{R}^n,Y)}, \quad \lambda > 0,
\end{aligned}$$

and we conclude that $\mathbb{T}^{\mathcal{L}}_{\alpha}$ is bounded from $L^1(\mathbb{R}^n, Y)$ into $L^{1,\infty}(\mathbb{R}^n, L^\infty((0,\infty), \gamma(H,Y)))$.

By proceeding in a similar way we can show that $\mathbb{T}^{\mathcal{L}}_{\alpha}$ is also bounded from $H^1(\mathbb{R}^n, Y)$ into $L^1(\mathbb{R}^n, L^\infty((0,\infty), \gamma(H,Y)))$. □

We now establish that $\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}$ is bounded from $H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$ into $H^1_{\mathcal{L}}(\mathbb{R}^n, \gamma(H,\mathbb{B}))$. According to Proposition 2.2 it is sufficient to show that $\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f) \in L^1(\mathbb{R}^n, \gamma(H,\mathbb{B}))$, for every $f \in H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$, and that the operator

$$T^{\mathcal{L}}_{\alpha}(f)(x) = \sup_{s>0}\left\|P^{\mathcal{L}+\alpha}_s \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f)(x,\cdot)\right\|_{\gamma(H,\mathbb{B})},$$

is bounded from $H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$ into $L^1(\mathbb{R}^n)$.

First of all, we are going to see that $\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}$ is a bounded operator from $H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$ into $L^1(\mathbb{R}^n, \gamma(H,\mathbb{B}))$. By [2, Theorem 1], $\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}$ is bounded from $H^1(\mathbb{R}^n, \mathbb{B})$ into $L^1(\mathbb{R}^n, \gamma(H,\mathbb{B}))$. Hence, if a is an atom for $H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$ such that $\int_{\mathbb{R}^n} a(x)\,dx = 0$, then

$$\left\|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(a)\right\|_{L^1(\mathbb{R}^n,\gamma(H,\mathbb{B}))} \leqslant C\|a\|_{L^1(\mathbb{R}^n,\mathbb{B})} \leqslant C,$$

where $C > 0$ does not depend on the atom a.

Suppose now that a is an atom for $H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$ such that $\operatorname{supp}(a) \subset B = B(x_0, r_0)$, where $x_0 \in \mathbb{R}^n$ and $\rho(x_0)/2 \leqslant r_0 \leqslant \rho(x_0)$, and that $\|a\|_{L^\infty(\mathbb{R}^n,\mathbb{B})} \leqslant |B|^{-1}$. Since $\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}$ is a bounded operator from $L^2(\mathbb{R}^n, \mathbb{B})$ into $L^2(\mathbb{R}^n, \gamma(H,\mathbb{B}))$ [2, Theorem 1], we have that

$$\int_{B^*} \big\|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(a)(x,\cdot)\big\|_{\gamma(H,\mathbb{B})}\,dx \leqslant \big|B^*\big|^{1/2}\Big(\int_{B^*} \big\|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(a)(x,\cdot)\big\|^2_{\gamma(H,\mathbb{B})}\,dx\Big)^{1/2}$$
$$\leqslant C|B|^{1/2}\Big(\int_B \big\|a(x)\big\|^2_{\mathbb{B}}\,dx\Big)^{1/2} \leqslant C, \tag{25}$$

being $B^* = B(x_0, 2r_0)$.

Moreover, if $y \in B$ and $x \notin B^*$, it follows that $|x-y| \geqslant r_0 \geqslant \rho(x_0)/2$ and $\rho(y) \sim \rho(x_0)$. Then, by taking into account (9), (10) and (12) we get

$$\int_{\mathbb{R}^n\setminus B^*} \big\|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(a)(x,\cdot)\big\|_{\gamma(H,\mathbb{B})}\,dx$$
$$\leqslant \int_{\mathbb{R}^n\setminus B^*}\int_B \big\|t\partial_t P_t^{\mathcal{L}+\alpha}(x,y)\big\|_H \big\|a(y)\big\|_{\mathbb{B}}\,dy\,dx$$
$$\leqslant C\int_{\mathbb{R}^n\setminus B^*}\int_B \frac{e^{-c(|x-y|^2+|y||x-y|)}}{|x-y|^n}\big\|a(y)\big\|_{\mathbb{B}}\,dy\,dx$$
$$\leqslant C\int_B \big\|a(y)\big\|_{\mathbb{B}}\sum_{j=0}^{\infty}\int_{2^{j-1}\rho(x_0)\leqslant|x-y|<2^j\rho(x_0)} \frac{dx\,dy}{|x-y|^{n+1/2}\rho(x_0)^{-1/2}}$$
$$\leqslant C\sum_{j=0}^{\infty}\frac{1}{(2^j\rho(x_0))^{1/2}\rho(x_0)^{-1/2}} \leqslant C\sum_{j=0}^{\infty}\frac{1}{2^{j/2}} \leqslant C. \tag{26}$$

From (25) and (26) we infer that

$$\big\|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(a)\big\|_{L^1(\mathbb{R}^n,\gamma(H,\mathbb{B}))} \leqslant C,$$

where $C > 0$ does not depend on a.

We consider $f = \sum_{j=1}^{\infty}\lambda_j a_j$, where a_j is an atom for $H^1_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})$ and $\lambda_j \in \mathbb{C}$, $j \in \mathbb{N}$, being $\sum_{j=1}^{\infty}|\lambda_j| < \infty$. The series converges in $L^1(\mathbb{R}^n,\mathbb{B})$. Hence, as a consequence of [2, Theorem 1], we have that

$$\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f) = \sum_{j=1}^{\infty}\lambda_j\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(a_j),$$

as elements of $L^{1,\infty}(\mathbb{R}^n,\gamma(H,\mathbb{B}))$. Also,

$$\big\|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f)\big\|_{L^1(\mathbb{R}^n,\gamma(H,\mathbb{B}))} \leqslant \sum_{j=1}^{\infty}|\lambda_j|\big\|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(a_j)\big\|_{L^1(\mathbb{R}^n,\gamma(H,\mathbb{B}))} \leqslant C\sum_{j=1}^{\infty}|\lambda_j|,$$

where $C > 0$ does not depend on f. Thus,

$$\left\|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f)\right\|_{L^1(\mathbb{R}^n,\gamma(H,\mathbb{B}))} \leqslant C\|f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})}.$$

Finally, to show that $T_\alpha^{\mathcal{L}}$ is bounded from $H^1_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})$ into $L^1(\mathbb{R}^n)$ we can proceed as above by considering the action of the operator on the two types of atoms of $H^1_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})$, and taking in mind the following facts, which can be deduced from the proof of Proposition 2.3:

- $T_\alpha^{\mathcal{L}}$ is bounded from $H^1(\mathbb{R}^n,\mathbb{B})$ into $L^1(\mathbb{R}^n)$,
- $T_\alpha^{\mathcal{L}}$ is bounded from $L^2(\mathbb{R}^n,\mathbb{B})$ into $L^2(\mathbb{R}^n)$,
- $P_s^{\mathcal{L}+\alpha}\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}$ can be associated to an integral operator with kernel Ω_α (see (19)) verifying that

$$\sup_{s>0}\left\|\Omega_\alpha(x,y,s,\cdot)\right\|_H \leqslant C\frac{e^{-c(|x-y|^2+|y||x-y|)}}{|x-y|^n}, \quad x,y\in\mathbb{R}^n,\ x\neq y,$$

- $T_\alpha^{\mathcal{L}}$ is bounded from $L^1(\mathbb{R}^n,\mathbb{B})$ into $L^{1,\infty}(\mathbb{R}^n)$.

2.3. Our next objective is to see that there exists $C > 0$ such that

$$\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})} \leqslant C\left\|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f)\right\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,\gamma(H,\mathbb{B}))}, \quad f\in BMO_{\mathcal{L}}\big(\mathbb{R}^n,\mathbb{B}\big). \tag{27}$$

In order to prove this we need to establish the following polarization equality.

Proposition 2.4. *Let $\mathbb{B}$ be a Banach space. If $a\in L_c^\infty(\mathbb{R}^n)\otimes\mathbb{B}^*$ and $f\in BMO_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})$, then*

$$\int_0^\infty\int_{\mathbb{R}^n}\big\langle\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}^*}(a)(x,t),\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f)(x,t)\big\rangle_{\mathbb{B}^*,\mathbb{B}}\frac{dx\,dt}{t} = \frac{1}{4}\int_{\mathbb{R}^n}\big\langle a(x),f(x)\big\rangle_{\mathbb{B}^*,\mathbb{B}}\,dx.$$

Proof. Firstly we consider $a\in L_c^\infty(\mathbb{R}^n)$ and $f\in BMO_{\mathcal{L}}(\mathbb{R}^n)$. In order to prove that

$$\int_0^\infty\int_{\mathbb{R}^n}\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a)(x,t)\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(f)(x,t)\frac{dx\,dt}{t} = \frac{1}{4}\int_{\mathbb{R}^n}a(x)f(x)\,dx, \tag{28}$$

we use the ideas developed in the proof of [13, Lemma 4].

According to [13, Lemma 5] we can write

$$\int_0^\infty\int_{\mathbb{R}^n}\big|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a)(x,t)\big|\big|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(f)(x,t)\big|\frac{dx\,dt}{t} \leqslant C\left\|S_\alpha(a)\right\|_{L^1(\mathbb{R}^n)}\left\|I_\alpha(f)\right\|_{L^\infty(\mathbb{R}^n)}, \tag{29}$$

where

$$S_\alpha(a)(x) = \left(\int_0^\infty \int_{|x-y|<t} \left|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a)(y,t)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2}, \quad x \in \mathbb{R}^n,$$

and

$$I_\alpha(f)(x) = \sup_{B \ni x}\left(\frac{1}{|B|}\int_0^{r(B)} \int_B \left|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(f)(y,t)\right|^2 \frac{dy\,dt}{t}\right)^{1/2}, \quad x \in \mathbb{R}^n.$$

Here B represents a ball in $\mathbb{R}^n$ and $r(B)$ is its radius.

We are going to show that the area integral operator S_α is bounded from $H^1_{\mathcal{L}}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. According to [19, Theorem 8.2] S_0 is bounded from $H^1_{\mathcal{L}}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Then, it is sufficient to see that $S_\alpha - S_0$ is bounded from $L^1(\mathbb{R}^n)$ into itself.

By using Minkowski's inequality we obtain

$$\begin{aligned}
&\left(\int_0^\infty \int_{|x-y|<t} \left|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(g)(y,t) - \mathcal{G}_{\mathcal{L},\mathbb{C}}(g)(y,t)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2} \\
&\quad \leqslant \int_{\mathbb{R}^n} \left|g(z)\right| \left(\int_0^\infty \int_{|x-y|<t} \left|t\partial_t\left[P_t^{\mathcal{L}+\alpha}(y,z) - P_t^{\mathcal{L}}(y,z)\right]\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2} dz, \quad g \in L^1\left(\mathbb{R}^n\right).
\end{aligned}$$

Since,

$$\begin{aligned}
&t\partial_t\left[P_t^{\mathcal{L}+\alpha}(y,z) - P_t^{\mathcal{L}}(y,z)\right] \\
&\quad = \frac{t}{\sqrt{4\pi}} \int_0^\infty \frac{e^{-t^2/(4s)}}{s^{3/2}}\left(1 - \frac{t^2}{2s}\right)\left(e^{-\alpha s} - 1\right) W_s^{\mathcal{L}}(y,z)\,ds, \quad y,z \in \mathbb{R}^n,\ t > 0,
\end{aligned}$$

by employing Minkowski's inequality and (7) it follows that

$$\begin{aligned}
&\left(\int_0^\infty \int_{|x-y|<t} \left|t\partial_t\left[P_t^{\mathcal{L}+\alpha}(y,z) - P_t^{\mathcal{L}}(y,z)\right]\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2} \\
&\quad \leqslant C \int_0^\infty \frac{|e^{-\alpha s} - 1|}{s^{3/2}} \left(\int_0^\infty \int_{|x-y|<t} \left|te^{-t^2/(8s)} W_s^{\mathcal{L}}(y,z)\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2} ds \\
&\quad \leqslant C \int_0^\infty \frac{|e^{-(\alpha+n)s} - e^{-ns}|}{s^{3/2}(1 - e^{-4s})^{n/2}} \left(\int_0^\infty \int_{|x-y|<t} e^{-c(t^2+|y-z|^2)/s} \frac{dy\,dt}{t^{n-1}}\right)^{1/2} ds, \quad x,z \in \mathbb{R}^n.
\end{aligned}$$

By taking into account again that $|e^{-(\alpha+n)s} - e^{-ns}| \leqslant Cse^{-cs}$, $s \in (0,\infty)$, and that $t^2 + |z-y|^2 \geqslant (t^2 + |z-x|^2)/4$, when $|x-y| < t$, we can write

$$\left(\int_0^\infty \int_{|x-y|<t} \left|t\partial_t\left[P_t^{\mathcal{L}+\alpha}(y,z) - P_t^{\mathcal{L}}(y,z)\right]\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2}$$
$$\leqslant C\int_0^\infty \frac{e^{-cs}e^{-c|x-z|^2/s}}{s(1-e^{-4s})^{n/2}}\left(\int_0^\infty \int_{|x-y|<t} e^{-ct^2/s}\frac{dy\,dt}{t^{n-1}}\right)^{1/2} ds$$
$$\leqslant C\int_0^\infty \frac{e^{-cs}e^{-c|x-z|^2/s}}{s^{(n+1)/2}}\,ds, \quad x,z \in \mathbb{R}^n.$$

Then,

$$\int_{\mathbb{R}^n}\left(\int_0^\infty \int_{|x-y|<t} \left|t\partial_t\left[P_t^{\mathcal{L}+\alpha}(y,z) - P_t^{\mathcal{L}}(y,z)\right]\right|^2 \frac{dy\,dt}{t^{n+1}}\right)^{1/2} dx \leqslant C\int_0^\infty \frac{e^{-cs}}{s^{1/2}}\,ds \leqslant C,$$
$$z \in \mathbb{R}^n.$$

Hence, the operator $S_\alpha - S_0$ is bounded from $L^1(\mathbb{R}^n)$ into itself.

Our next objective is to see that $I_\alpha(f) \in L^\infty(\mathbb{R}^n)$. Let $x_0 \in \mathbb{R}^n$ and $r_0 > 0$. We denote by B the ball $B(x_0, r_0)$ and we decompose f as follows

$$f = (f - f_{B^*})\chi_{B^*} + (f - f_{B^*})\chi_{\mathbb{R}^n\setminus B^*} + f_{B^*} = f_1 + f_2 + f_3,$$

where $B^* = B(x_0, 2r_0)$.

According to [2, (4)], since $\gamma(H,\mathbb{C}) = H$, we can write

$$\frac{1}{|B|}\int_0^{r_0}\int_B \left|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(f_1)(y,t)\right|^2 \frac{dy\,dt}{t} \leqslant \frac{1}{|B|}\int_{\mathbb{R}^n}\int_0^\infty \left|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(f_1)(y,t)\right|^2 \frac{dt\,dy}{t}$$
$$\leqslant \frac{C}{|B|}\int_{B^*}\left|f(x) - f_{B^*}\right|^2 dx \leqslant C\|f\|^2_{BMO_{\mathcal{L}}(\mathbb{R}^n)}. \tag{30}$$

By using (8) we can proceed as in [13, p. 338] to obtain

$$\frac{1}{|B|}\int_0^{r_0}\int_B \left|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(f_2)(y,t)\right|^2 \frac{dy\,dt}{t} \leqslant C\|f\|^2_{BMO_{\mathcal{L}}(\mathbb{R}^n)}. \tag{31}$$

If $r_0 \geqslant \rho(x_0)$, since $\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(1) \in L^\infty(\mathbb{R}^n, H)$ (see Section 2.1), then

$$\frac{1}{|B|}\int_0^{r_0}\int_B \left|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(f_3)(y,t)\right|^2\frac{dy\,dt}{t} \leqslant \frac{|f_{B^*}|^2}{|B|}\int_B \left\|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(1)(y,\cdot)\right\|_H^2\,dy$$
$$\leqslant C|f_{B^*}|^2 \leqslant C\|f\|^2_{BMO_{\mathcal{L}}(\mathbb{R}^n)}. \tag{32}$$

Suppose now that $r_0 < \rho(x_0)$. According to (15), we have that

$$\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(1)(x,t) = \frac{t^2}{\sqrt{4\pi}}\int_0^\infty \frac{e^{-u}}{u^{3/2}}\partial_z W_z^{\mathcal{L}+\alpha}(1)(x)_{|z=t^2/(4u)}\,du, \quad x\in\mathbb{R}^n \text{ and } t>0.$$

By (14) it follows that, for every $x\in\mathbb{R}^n$ and $z>0$,

$$\left|\partial_z W_z^{\mathcal{L}+\alpha}(1)(x)\right| \leqslant C\frac{e^{-(\alpha+n)z}e^{-c(1-e^{-4z})|x|^2}}{(\rho(x))^2}$$
$$\leqslant C\frac{e^{-cz}\max\{e^{-cz/(\rho(x)^2)}, e^{-c/(\rho(x))^2}\}}{(\rho(x))^2} \leqslant C\frac{1}{(\rho(x))^{1/2}z^{3/4}}.$$

Then, we conclude that

$$\left|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(1)(x,t)\right| \leqslant C\left(\frac{t}{\rho(x)}\right)^{1/2}, \quad x\in\mathbb{R}^n \text{ and } t>0.$$

The arguments developed in [13, p. 339] allow us to obtain

$$\frac{1}{|B|}\int_0^{r_0}\int_B \left|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(f_3)(y,t)\right|^2\frac{dy\,dt}{t} \leqslant C\|f\|^2_{BMO_{\mathcal{L}}(\mathbb{R}^n)}. \tag{33}$$

Putting together (30), (31), (32) and (33) we get

$$\frac{1}{|B|}\int_0^{r_0}\int_B \left|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(f)(y,t)\right|^2\frac{dy\,dt}{t} \leqslant C\|f\|^2_{BMO_{\mathcal{L}}(\mathbb{R}^n)},$$

where C does not depend on B, and we prove that $I_\alpha(f)\in L^\infty(\mathbb{R}^n)$.

Since $a\in H^1_{\mathcal{L}}(\mathbb{R}^n)$, from (29) we deduce that

$$\int_0^\infty\int_{\mathbb{R}^n} \left|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a)(x,t)\right|\left|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(f)(x,t)\right|\frac{dx\,dt}{t} < \infty. \tag{34}$$

Then,

$$\int_0^\infty \int_{\mathbb{R}^n} \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a)(x,t)\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(f)(x,t)\frac{dx\,dt}{t}$$
$$= \lim_{N\to\infty} \int_{1/N}^N \int_{\mathbb{R}^n} \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a)(x,t)\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(f)(x,t)\frac{dx\,dt}{t}.$$

Let $N \in \mathbb{N}$. By interchanging the order of integration we obtain

$$\int_{1/N}^N \int_{\mathbb{R}^n} \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a)(x,t)\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(f)(x,t)\frac{dx\,dt}{t}$$
$$= \int_{\mathbb{R}^n} f(y) \int_{1/N}^N \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}\big(\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a)(\cdot,t_1)\big)(y,t)_{|t_1=t}\frac{dt\,dy}{t}.$$

We are going to justify this interchange in the order of integration. For that we will see that

$$\int_{1/N}^N \int_{\mathbb{R}^n} \big|f(y)\big| \int_{\mathbb{R}^n} \big|t\partial_t P_t^{\mathcal{L}+\alpha}(x,y)\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a)(x,t)\big|\frac{dx\,dy\,dt}{t} < \infty. \tag{35}$$

By using (8) it follows that

$$\int_{\mathbb{R}^n} \big|t\partial_t P_t^{\mathcal{L}+\alpha}(x,y)\big| \int_{\mathbb{R}^n} \big|t\partial_t P_t^{\mathcal{L}+\alpha}(x,z)\big|\big|a(z)\big|\,dz\,dx$$
$$\leqslant C\int_{\mathbb{R}^n} \big|a(z)\big| \int_{\mathbb{R}^n} \frac{t}{(|x-z|^2+t^2)^{(n+1)/2}}\frac{t}{(|x-y|^2+t^2)^{(n+1)/2}}\,dx\,dz$$
$$\leqslant C\int_{\mathbb{R}^n} \big|a(z)\big|\frac{t}{(t+|z-y|)^{n+1}}\,dz, \quad x, y \in \mathbb{R}^n \text{ and } t > 0.$$

Suppose that $\operatorname{supp}(a) \subset B = B(0,R)$. We have that

$$\int_{\mathbb{R}^n} \big|t\partial_t P_t^{\mathcal{L}+\alpha}(x,y)\big| \int_{\mathbb{R}^n} \big|t\partial_t P_t^{\mathcal{L}+\alpha}(x,z)\big|\big|a(z)\big|\,dz\,dx$$
$$\leqslant C\|a\|_{L^\infty(\mathbb{R}^n)} \leqslant \frac{C}{(1+|y|)^{n+1}}, \quad y \in B^* \text{ and } t > 0. \tag{36}$$

On the other hand, if $y \notin B^* = B(0, 2R)$ and $z \in B$, then $|z - y| \geqslant |y|/2$. Hence, we get

$$\int_{\mathbb{R}^n} \left|t\partial_t P_t^{\mathcal{L}+\alpha}(x, y)\right| \int_{\mathbb{R}^n} \left|t\partial_t P_t^{\mathcal{L}+\alpha}(x, z)\right|\left|a(z)\right| dz\, dx \leqslant CR^n \|a\|_{L^\infty(\mathbb{R}^n)} \frac{t}{(t + |y|)^{n+1}}, \quad y \notin B^* \text{ and } t > 0. \tag{37}$$

Since $f \in BMO_{\mathcal{L}}(\mathbb{R}^n)$, (36) and (37) imply (35).

By taking into account that $a \in L^2(\mathbb{R}^n)$ we can write, for every $x \in \mathbb{R}^n$ and $t > 0$,

$$\begin{aligned} &\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}\big(\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a)(\cdot, t_1)\big)(x, t)_{|t_1=t} \\ &\quad = -\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}\bigg(\sum_{k\in\mathbb{N}^n} t_1\sqrt{2|k| + n + \alpha}\, e^{-t_1\sqrt{2|k|+n+\alpha}} \langle a, h_k\rangle h_k\bigg)(x, t)_{|t_1=t} \\ &\quad = t^2 \sum_{k\in\mathbb{N}^n} \big(2|k| + n + \alpha\big) e^{-2t\sqrt{2|k|+n+\alpha}} \langle a, h_k\rangle h_k(x). \end{aligned}$$

Note that the last series converges uniformly in $(x, t) \in \mathbb{R}^n \times [a, b]$, for every $0 < a < b < \infty$. We have that

$$\begin{aligned} &\int_{1/N}^{N} \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}\big(\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a)(\cdot, t_1)\big)(y, t)_{|t_1=t} \frac{dt}{t} \\ &\quad = \sum_{k\in\mathbb{N}^n} \langle a, h_k\rangle h_k(y)\big(2|k| + n + \alpha\big) \int_{1/N}^{N} t e^{-2t\sqrt{2|k|+n+\alpha}}\, dt \\ &\quad = \sum_{k\in\mathbb{N}^n} \langle a, h_k\rangle h_k(y)\bigg[-\frac{1}{2}\sqrt{2|k| + n + \alpha}\bigg(N e^{-2N\sqrt{2|k|+n+\alpha}} - \frac{1}{N} e^{-\frac{2}{N}\sqrt{2|k|+n+\alpha}}\bigg) \\ &\qquad - \frac{1}{4}\big(e^{-2N\sqrt{2|k|+n+\alpha}} - e^{-\frac{2}{N}\sqrt{2|k|+n+\alpha}}\big)\bigg] \\ &\quad = -\frac{1}{4}\big[P_{2N}^{\mathcal{L}+\alpha}(a)(y) - P_{2/N}^{\mathcal{L}+\alpha}(a)(y)\big] - \frac{1}{4}\big[\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a)(y, 2N) - \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a)(y, 2/N)\big], \\ &\quad y \in \mathbb{R}^n. \end{aligned}$$

According to (8) it follows that

$$\begin{aligned} \sup_{t>0}\left|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a)(y, t)\right| &\leqslant C \sup_{t>0} \int_{\mathbb{R}^n} \frac{t|a(z)|}{(t + |z - y|)^{n+1}}\, dz \leqslant C\|a\|_{L^\infty(\mathbb{R}^n)} \\ &\leqslant \frac{C}{(1 + |y|)^{n+1}}, \quad y \in B^*, \end{aligned}$$

and by proceeding as in (8) and using (7), we get

$$\sup_{t>0}\left|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a)(y,t)\right| \leqslant C\sup_{t>0}\int_B \left|a(z)\right|\frac{te^{-c|y||z-y|}}{(t+|z-y|)^{n+1}}\,dz$$
$$\leqslant C\|a\|_{L^\infty(\mathbb{R}^n)}e^{-c|y|^2}\int_{\mathbb{R}^n}\frac{t}{(t+|z-y|)^{n+1}}\,dz$$
$$\leqslant \frac{C}{(1+|y|)^{n+1}}, \quad y\notin B^*.$$

In a similar way we can prove that

$$\sup_{t>0}\left|P_t^{\mathcal{L}+\alpha}(a)(y)\right| \leqslant \frac{C}{(1+|y|)^{n+1}}, \quad y\in\mathbb{R}^n.$$

We conclude that

$$\sup_{N\in\mathbb{N}}\left|\int_{1/N}^{N}\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}\big(\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a)(\cdot,t_1)\big)(y,t)_{|t_1=t}\frac{dt}{t}\right| \leqslant \frac{C}{(1+|y|)^{n+1}}, \quad y\in\mathbb{R}^n.$$

Hence, for every increasing sequence $(N_m)_{m\in\mathbb{N}}\subset\mathbb{N}$, we have that

$$\int_0^\infty\int_{\mathbb{R}^n}\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a)(x,t)\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(f)(x,t)\frac{dx\,dt}{t}$$
$$=\int_{\mathbb{R}^n}f(y)\lim_{m\to\infty}\int_{1/N_m}^{N_m}\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}\big(\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a)(\cdot,t_1)\big)(y,t)_{|t_1=t}\frac{dt\,dy}{t},$$

because $f\in BMO_{\mathcal{L}}(\mathbb{R}^n)$.

Then, (28) will be proved when we show that

$$\lim_{N\to\infty}\int_{1/N}^{N}\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}\big(\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a)(\cdot,t_1)\big)(y,t)_{|t_1=t}\frac{dt}{t}=\frac{a(y)}{4}, \quad \text{in } L^2\big(\mathbb{R}^n\big). \tag{38}$$

In order to see that (38) holds we use Plancherel equality to get

$$\left\|\int_{1/N}^{N}\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}\big(\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a)(\cdot,t_1)\big)(y,t)_{|t_1=t}\frac{dt}{t}-\frac{a(y)}{4}\right\|^2_{L^2(\mathbb{R}^n)}$$

$$= \sum_{k\in\mathbb{N}^n} \left|\langle a, h_k\rangle\right|^2 \left| \frac{\sqrt{2|k|+n+\alpha}}{2}\left(-Ne^{-2N\sqrt{2|k|+n+\alpha}} + \frac{1}{N}e^{-\frac{2}{N}\sqrt{2|k|+n+\alpha}}\right) - \frac{1}{4}\left(e^{-2N\sqrt{2|k|+n+\alpha}} - e^{-\frac{2}{N}\sqrt{2|k|+n+\alpha}}\right) - \frac{1}{4}\right|^2.$$

The dominated convergence theorem leads to

$$\lim_{N\to\infty} \left\| \int_{1/N}^{N} \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}\big(\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a)(\cdot,t_1)\big)(y,t)_{|t_1=t}\frac{dt}{t} - \frac{a(y)}{4}\right\|^2_{L^2(\mathbb{R}^n)} = 0.$$

Thus, the proof of (28) is finished.

Suppose now that $f \in BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$ and $a = \sum_{j=1}^m a_j b_j$, where $a_j \in L^\infty_c(\mathbb{R}^n)$ and $b_j \in \mathbb{B}^*$, $j = 1, \dots, m \in \mathbb{N}$. We have that

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} \big\langle \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}^*}(a)(x,t), \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f)(x,t)\big\rangle_{\mathbb{B}^*,\mathbb{B}} \frac{dx\,dt}{t} \\
&\quad = \sum_{j=1}^m \int_0^\infty \int_{\mathbb{R}^n} \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a_j)(x,t)\big\langle b_j, \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f)(x,t)\big\rangle_{\mathbb{B}^*,\mathbb{B}} \frac{dx\,dt}{t} \\
&\quad = \sum_{j=1}^m \int_0^\infty \int_{\mathbb{R}^n} \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a_j)(x,t)\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}\big(\langle b_j, f\rangle_{\mathbb{B}^*,\mathbb{B}}\big)(x,t) \frac{dx\,dt}{t}.
\end{aligned}$$

Since, $\langle b_j, f\rangle_{\mathbb{B}^*,\mathbb{B}} \in BMO_{\mathcal{L}}(\mathbb{R}^n)$, $j = 1, \dots, m$, the proof can be completed by using (28). □

We now prove (27). Let $f \in BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$. We denote by $\mathcal{A}$ the following linear space

$$\mathcal{A} = \operatorname{span}\big\{a\colon a \text{ is an atom in } H^1_{\mathcal{L}}\big(\mathbb{R}^n\big)\big\}.$$

We have that

$$\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})} = \sup_{\substack{a\in\mathcal{A}\otimes\mathbb{B}^* \\ \|a\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B}^*)}\leqslant 1}} \left| \int_{\mathbb{R}^n} \big\langle f(x), a(x)\big\rangle_{\mathbb{B},\mathbb{B}^*}\,dx\right|.$$

Note that, according to [21, Lemma 2.4] $\mathcal{A}\otimes\mathbb{B}^*$ is a dense subspace of $H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B}^*)$. Moreover, since $\mathbb{B}$ is UMD, $\mathbb{B}$ is reflexive and $\mathbb{B}^*$ is also a UMD space. Hence $(H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B}^*))^* = BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$.

By Proposition 2.4 we deduce that

$$\int_{\mathbb{R}^n} \langle a(x), f(x)\rangle_{\mathbb{B}^*,\mathbb{B}}\, dx = 4\int_0^\infty \int_{\mathbb{R}^n} \langle \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}^*}(a)(x,t), \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f)(x,t)\rangle_{\mathbb{B}^*,\mathbb{B}} \frac{dx\,dt}{t},$$

$a \in \mathcal{A} \otimes \mathbb{B}^*$.

Proposition 2.5. *Let Y be a Banach space. Suppose that $g \in BMO_{\mathcal{L}}(\mathbb{R}^n, Y)$ and $h \in H^1_{\mathcal{L}}(\mathbb{R}^n, Y^*)$ such that*

$$\int_{\mathbb{R}^n} \left|\langle h(x), g(x)\rangle_{Y^*,Y}\right| dx < \infty.$$

Then,

$$\left|\int_{\mathbb{R}^n} \langle h(x), g(x)\rangle_{Y^*,Y}\, dx\right| \leqslant C\|h\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,Y^*)} \|g\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,Y)}.$$

Proof. Note firstly that g defines an element T_g of $(H^1_{\mathcal{L}}(\mathbb{R}^n, Y^*))^*$ such that

$$T_g(a) = \int_{\mathbb{R}^n} \langle a(x), g(x)\rangle_{Y^*,Y}\, dx,$$

and

$$\left|\int_{\mathbb{R}^n} \langle a(x), g(x)\rangle_{Y^*,Y}\, dx\right| \leqslant C\|a\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,Y^*)} \|g\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,Y)},$$

provided that a is a linear combination of atoms in $H^1_{\mathcal{L}}(\mathbb{R}^n, Y^*)$. Moreover, it is well-known that the function $F(x) = \langle a(x), g(x)\rangle_{Y^*,Y}$, $x \in \mathbb{R}^n$, might not be integrable on $\mathbb{R}^n$ when $a \in H^1_{\mathcal{L}}(\mathbb{R}^n, Y^*)$. On the other hand, if $\tilde{g} \in L^\infty(\mathbb{R}^n, Y)$, then $\tilde{g} \in BMO_{\mathcal{L}}(\mathbb{R}^n, Y)$,

$$T_{\tilde{g}}(a) = \int_{\mathbb{R}^n} \langle a(x), \tilde{g}(x)\rangle_{Y^*,Y}\, dx, \quad a \in H^1_{\mathcal{L}}(\mathbb{R}^n, Y^*),$$

and

$$\left|\int_{\mathbb{R}^n} \langle a(x), \tilde{g}(x)\rangle_{Y^*,Y}\, dx\right| \leqslant C\|a\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,Y^*)} \|\tilde{g}\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,Y)}, \quad a \in H^1_{\mathcal{L}}(\mathbb{R}^n, Y^*).$$

Let $\ell \in \mathbb{N}$. We define the function $\Phi_\ell : Y \longrightarrow Y$ by

$$\Phi_\ell(b) = \begin{cases} \frac{\ell b}{\|b\|_Y}, & \|b\|_Y \geqslant \ell, \\ b, & \|b\|_Y < \ell. \end{cases}$$

Φ_ℓ is a Lipschitz function. Indeed, let $b_1, b_2 \in Y$. If $\|b_1\|_Y \geqslant \ell$ and $\|b_2\|_Y \geqslant \ell$, then

$$\begin{aligned} \big\|\Phi_\ell(b_1) - \Phi_\ell(b_2)\big\|_Y &= \left\| \frac{\ell b_1}{\|b_1\|_Y} - \frac{\ell b_2}{\|b_2\|_Y} \right\|_Y \leqslant \left\| b_1 - b_2 \frac{\|b_1\|_Y}{\|b_2\|_Y} \right\|_Y \\ &\leqslant \|b_1 - b_2\|_Y + \|b_2\|_Y \left| 1 - \frac{\|b_1\|_Y}{\|b_2\|_Y} \right| \leqslant 2\|b_1 - b_2\|_Y. \end{aligned}$$

Moreover, if $\|b_1\|_Y < \ell$ and $\|b_2\|_Y \geqslant \ell$, it follows that

$$\begin{aligned} \big\|\Phi_\ell(b_1) - \Phi_\ell(b_2)\big\|_Y &= \left\| b_1 - \frac{\ell b_2}{\|b_2\|_Y} \right\|_Y \leqslant \|b_1 - b_2\|_Y + \left\| b_2 - \frac{\ell b_2}{\|b_2\|_Y} \right\|_Y \\ &\leqslant \|b_1 - b_2\|_Y + \big| \|b_2\|_Y - \ell \big| \leqslant \|b_1 - b_2\|_Y + \|b_2\|_Y - \|b_1\|_Y \\ &\leqslant 2\|b_1 - b_2\|_Y. \end{aligned}$$

We define the function $g_\ell(x) = \Phi_\ell(g(x))$, $x \in \mathbb{R}^n$. We have that $g_\ell \in BMO_{\mathcal{L}}(\mathbb{R}^n, Y)$ and $\|g_\ell\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,Y)} \leqslant C\|g\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,Y)}$. Moreover,

$$\big|\langle h(x), g_\ell(x)\rangle_{Y^*,Y}\big| \leqslant \big|\langle h(x), g(x)\rangle_{Y^*,Y}\big|, \quad \text{a.e. } x \in \mathbb{R}^n.$$

By using convergence dominated theorem, since $\lim_{\ell\to\infty}\langle h(x), g_\ell(x)\rangle_{Y^*,Y} = \langle h(x), g(x)\rangle_{Y^*,Y}$ a.e. $x \in \mathbb{R}^n$, we deduce that

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \big\langle h(x), g(x)\big\rangle_{Y^*,Y}\, dx \right| &= \lim_{\ell\to\infty} \left| \int_{\mathbb{R}^n} \big\langle h(x), g_\ell(x)\big\rangle_{Y^*,Y}\, dx \right| \\ &\leqslant C \varlimsup_{\ell\to\infty} \|h\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,Y^*)} \|g_\ell\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,Y)} \\ &\leqslant C\|h\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,Y^*)} \|g\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,Y)}. \qquad \square \end{aligned}$$

Suppose that $a = \sum_{j=1}^m a_j b_j$, where a_j is an atom for $H^1_{\mathcal{L}}(\mathbb{R}^n)$ and $b_j \in \mathbb{B}^*$, $j = 1, \dots, m \in \mathbb{N}$. Then, according to Theorem 1.2 for $H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B}^*)$, we have that

$$\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}^*}(a) = \sum_{j=1}^m b_j \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a_j) \in H^1_{\mathcal{L}}\big(\mathbb{R}^n, \gamma\big(H, \mathbb{B}^*\big)\big).$$

If $(e_\ell)_{\ell=1}^\infty$ is an orthonormal basis in H by taking into account that $\gamma(H, \mathbb{B})^* = \gamma(H, \mathbb{B}^*)$ via trace duality we can write

$$
\begin{aligned}
&\big\langle \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}^*}(a)(x,\cdot),\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f)(x,\cdot)\big\rangle_{\gamma(H,\mathbb{B}^*),\gamma(H,\mathbb{B})}\\
&\quad=\sum_{j=1}^{m}\big\langle b_j\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a_j)(x,\cdot),\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f)(x,\cdot)\big\rangle_{\gamma(H,\mathbb{B}^*),\gamma(H,\mathbb{B})}\\
&\quad=\sum_{j=1}^{m}\sum_{\ell=1}^{\infty}\int_0^\infty e_\ell(t)\int_0^\infty \big\langle b_j\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a_j)(x,u),\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f)(x,t)\big\rangle_{\mathbb{B}^*,\mathbb{B}}e_\ell(u)\frac{du}{u}\frac{dt}{t}\\
&\quad=\sum_{j=1}^{m}\sum_{\ell=1}^{\infty}\int_0^\infty e_\ell(t)\int_0^\infty \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a_j)(x,u)\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}\big(\langle b_j,f\rangle_{\mathbb{B}^*,\mathbb{B}}\big)(x,t)e_\ell(u)\frac{du}{u}\frac{dt}{t}\\
&\quad=\sum_{j=1}^{m}\int_0^\infty \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a_j)(x,t)\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}\big(\langle b_j,f\rangle_{\mathbb{B}^*,\mathbb{B}}\big)(x,t)\frac{dt}{t}\\
&\quad=\sum_{j=1}^{m}\int_0^\infty \big\langle \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}^*}(a_jb_j)(x,t),\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f)(x,t)\big\rangle_{\mathbb{B}^*,\mathbb{B}}\frac{dt}{t}\\
&\quad=\int_0^\infty \big\langle \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}^*}(a)(x,t),\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f)(x,t)\big\rangle_{\mathbb{B}^*,\mathbb{B}}\frac{dt}{t},\quad \text{a.e. } x\in\mathbb{R}^n.
\end{aligned}
$$

Moreover, since $\langle b_j,f\rangle_{\mathbb{B}^*,\mathbb{B}}\in BMO_{\mathcal{L}}(\mathbb{R}^n)$, $j=1,\dots,m$, from (34) we deduce that

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\big|\big\langle \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}^*}(a)(x,\cdot),\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f)(x,\cdot)\big\rangle_{\gamma(H,\mathbb{B}^*),\gamma(H,\mathbb{B})}\big|\,dx\\
&\qquad\leqslant\sum_{j=1}^{m}\int_{\mathbb{R}^n}\int_0^\infty\big|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}(a_j)(x,t)\big|\big|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{C}}\big(\langle b_j,f\rangle_{\mathbb{B}^*,\mathbb{B}}\big)(x,t)\big|\frac{dt\,dx}{t}<\infty.
\end{aligned}
$$

Hence, according to Proposition 2.5 and the results proved in Section 2.2 we get

$$
\begin{aligned}
\left|\int_{\mathbb{R}^n}\big\langle a(x),f(x)\big\rangle_{\mathbb{B}^*,\mathbb{B}}\,dx\right|&=4\left|\int_{\mathbb{R}^n}\big\langle \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}^*}(a)(x,\cdot),\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f)(x,\cdot)\big\rangle_{\gamma(H,\mathbb{B}^*),\gamma(H,\mathbb{B})}\,dx\right|\\
&\leqslant C\big\|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}^*}(a)\big\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,\gamma(H,\mathbb{B}^*))}\big\|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f)\big\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,\gamma(H,\mathbb{B}))}\\
&\leqslant C\|a\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B}^*)}\big\|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f)\big\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,\gamma(H,\mathbb{B}))}.
\end{aligned}
$$

We conclude that

$$
\|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})}\leqslant C\big\|\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(f)\big\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,\gamma(H,\mathbb{B}))}.
$$

2.4. We are going to show that, for every $g \in H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$,

$$\|g\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})} \leqslant C \left\| \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(g) \right\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,\gamma(H,\mathbb{B}))}. \tag{39}$$

Suppose that $a \in \mathcal{A} \otimes \mathbb{B}$, where $\mathcal{A}$ is defined in Section 2.3. Since $\mathbb{B}$ is UMD, $(H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B}))^* = BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B}^*)$, and we have that

$$\|a\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})} = \sup_{\substack{f \in BMO_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B}^*) \\ \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B}^*)} \leqslant 1}} \left| \int_{\mathbb{R}^n} \langle f(x), a(x) \rangle_{\mathbb{B}^*,\mathbb{B}}\, dx \right|.$$

Moreover, for every $f \in BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B}^*)$, since $\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}^*}$ is bounded from $BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B}^*)$ into $BMO_{\mathcal{L}}(\mathbb{R}^n, \gamma(H, \mathbb{B}^*))$ (see Section 2.1), again by Proposition 2.5 it follows that

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \langle f(x), a(x) \rangle_{\mathbb{B}^*,\mathbb{B}}\, dx \right| &\leqslant C \left\| \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}^*}(f) \right\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,\gamma(H,\mathbb{B}^*))} \left\| \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(a) \right\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,\gamma(H,\mathbb{B}))} \\ &\leqslant C \|f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B}^*)} \left\| \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(a) \right\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,\gamma(H,\mathbb{B}))}. \end{aligned}$$

Hence,

$$\|a\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})} \leqslant C \left\| \mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}(a) \right\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,\gamma(H,\mathbb{B}))}.$$

Since $\mathcal{A} \otimes \mathbb{B}$ is a dense subspace in $H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$ and $\mathcal{G}_{\mathcal{L}+\alpha,\mathbb{B}}$ is bounded from $H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$ into $H^1_{\mathcal{L}}(\mathbb{R}^n, \gamma(H, \mathbb{B}))$ (see Section 2.2) we conclude that (39) holds for every $g \in H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$.

3. Proof of Theorem 1.3

3.1. We are going to prove that the operator $T^{\mathcal{L}}_{j,+}$ is bounded from $BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$ into $BMO_{\mathcal{L}}(\mathbb{R}^n, \gamma(H, \mathbb{B}))$. The corresponding property for $T^{\mathcal{L}}_{j,-}$ when $n \geqslant 3$ can be shown in a similar way.

We consider the function Ω defined by

$$\Omega(x, y, t) = \frac{t^2}{\sqrt{4\pi}} \int_0^\infty \frac{e^{-t^2/(4s)}}{s^{3/2}} (\partial_{x_j} + x_j) W^{\mathcal{L}}_s(x, y)\, ds, \quad x, y \in \mathbb{R}^n \text{ and } t > 0.$$

We have that

$$(\partial_{x_j} + x_j) W^{\mathcal{L}}_s(x, y) = \left(x_j - \frac{1}{2} \frac{1 + e^{-2s}}{1 - e^{-2s}} (x_j - y_j) - \frac{1}{2} \frac{1 - e^{-2s}}{1 + e^{-2s}} (x_j + y_j) \right) W^{\mathcal{L}}_s(x, y),$$
$$x, y \in \mathbb{R}^n \text{ and } s > 0.$$

Note that $|a| \leqslant |a+b| + |a-b|$, $a, b \in \mathbb{R}$. Then, it follows that, for every $x, y \in \mathbb{R}^n$ and $s > 0$,

$$\left|(\partial_{x_j} + x_j) W_s^{\mathcal{L}}(x, y)\right| \leqslant C \frac{1}{\sqrt{1-e^{-2s}}} \left(\frac{e^{-2s}}{1-e^{-4s}}\right)^{n/2} \times \exp\left(-\frac{1}{8}\left(\frac{1+e^{-2s}}{1-e^{-2s}}|x-y|^2 + \frac{1-e^{-2s}}{1+e^{-2s}}|x+y|^2\right)\right).$$

As in (7) we obtain, for every $x, y \in \mathbb{R}^n$ and $t > 0$,

$$\begin{aligned}\left|\Omega(x, y, t)\right| &\leqslant C t^2 e^{-c(|x-y|^2 + |y||x-y|)} \int_0^\infty \frac{e^{-c(t^2+|x-y|^2)/s} e^{-ns}}{s^{3/2}(1-e^{-4s})^{(n+1)/2}}\, ds \\ &\leqslant C t^2 e^{-c(|x-y|^2 + |y||x-y|)} \int_0^\infty \frac{e^{-c(t^2+|x-y|^2)/s}}{s^{(n+4)/2}}\, ds \\ &\leqslant C \frac{t^2 e^{-c(|x-y|^2 + |y||x-y|)}}{(t+|x-y|)^{n+2}}. \qquad (40)\end{aligned}$$

Hence, it follows that

$$\begin{aligned}\left\|\Omega(x, y, \cdot)\right\|_H &\leqslant C e^{-c(|x-y|^2 + |y||x-y|)} \left(\int_0^\infty \frac{t^3}{(t+|x-y|)^{2n+4}}\, dt\right)^{1/2} \\ &\leqslant C \frac{e^{-c(|x-y|^2 + |y||x-y|)}}{|x-y|^n}, \quad x, y \in \mathbb{R}^n,\ x \neq y. \qquad (41)\end{aligned}$$

Let $i = 1, \ldots, n$. We can write, if $i \neq j$,

$$\partial_{x_i}(\partial_{x_j} + x_j) W_s^{\mathcal{L}}(x, y) = -\left(x_j - \frac{1}{2}\frac{1+e^{-2s}}{1-e^{-2s}}(x_j - y_j) - \frac{1}{2}\frac{1-e^{-2s}}{1+e^{-2s}}(x_j + y_j)\right) \times \left(\frac{1}{2}\frac{1+e^{-2s}}{1-e^{-2s}}(x_i - y_i) + \frac{1}{2}\frac{1-e^{-2s}}{1+e^{-2s}}(x_i + y_i)\right) W_s^{\mathcal{L}}(x, y),$$

$x, y \in \mathbb{R}^n$ and $s > 0$,

and

$$\partial_{x_j}(\partial_{x_j} + x_j) W_s^{\mathcal{L}}(x, y) = -\left\{\frac{2e^{-4s}}{1-e^{-4s}} + \left(x_j - \frac{1}{2}\frac{1+e^{-2s}}{1-e^{-2s}}(x_j - y_j) - \frac{1}{2}\frac{1-e^{-2s}}{1+e^{-2s}}(x_j + y_j)\right)\right.$$

$$\times\left(\frac{1}{2}\frac{1+e^{-2s}}{1-e^{-2s}}(x_j-y_j)+\frac{1}{2}\frac{1-e^{-2s}}{1+e^{-2s}}(x_j+y_j)\right)\Bigg\}W_s^{\mathcal{L}}(x,y),$$

$x, y\in\mathbb{R}^n$ and $s>0$.

Then, we get, for each $x, y\in\mathbb{R}^n$ and $s>0$,

$$\left|\partial_{x_i}(\partial_{x_j}+x_j)W_s^{\mathcal{L}}(x,y)\right|\leqslant C\frac{1}{1-e^{-2s}}\left(\frac{e^{-2s}}{1-e^{-4s}}\right)^{n/2}\times\exp\left(-\frac{1}{8}\left(\frac{1+e^{-2s}}{1-e^{-2s}}|x-y|^2+\frac{1-e^{-2s}}{1+e^{-2s}}|x+y|^2\right)\right).$$

By proceeding as above we obtain

$$\left\|\partial_{x_i}\Omega(x,y,\cdot)\right\|_H\leqslant\frac{C}{|x-y|^{n+1}},\quad x,y\in\mathbb{R}^n,\ x\neq y. \tag{42}$$

In a similar way we can see that

$$\left\|\partial_{y_i}\Omega(x,y,\cdot)\right\|_H\leqslant\frac{C}{|x-y|^{n+1}},\quad x,y\in\mathbb{R}^n,\ x\neq y. \tag{43}$$

Putting together (42) and (43) we conclude that

$$\left\|\nabla_x\Omega(x,y,\cdot)\right\|_H+\left\|\nabla_y\Omega(x,y,\cdot)\right\|_H\leqslant\frac{C}{|x-y|^{n+1}},\quad x,y\in\mathbb{R}^n,\ x\neq y.$$

According to [2, Theorem 2] the operator $T_{j,+}^{\mathcal{L}}$ is bounded from $L^2(\mathbb{R}^n,\mathbb{B})$ into $L^2(\mathbb{R}^n,\gamma(H,\mathbb{B}))$. Moreover, the same argument we have used in Section 2.1 allows us to show that, for every $f\in L_c^\infty(\mathbb{R}^n,\mathbb{B})$,

$$T_{j,+}^{\mathcal{L}}(f)(x,t)=\left(\int_{\mathbb{R}^n}\Omega(x,y,\cdot)f(y)\,dy\right)(t),\quad \text{a.e. } x\notin\operatorname{supp}(f).$$

By taking into account (13), for each $x\in\mathbb{R}^n$ and $s>0$, we obtain that

$$(\partial_{x_j}+x_j)W_s^{\mathcal{L}}(1)(x)=\frac{1}{\pi^{n/2}}\left(\frac{e^{-2s}}{1+e^{-4s}}\right)^{n/2}\left(1-\frac{1-e^{-4s}}{1+e^{-4s}}\right)x_j\exp\left(-\frac{1-e^{-4s}}{2(1+e^{-4s})}|x|^2\right).$$

Hence, Minkowski's inequality leads to

$$\left\|T_{j,+}^{\mathcal{L}}(1)(x,\cdot)\right\|_H\leqslant C\int_0^\infty\frac{e^{-s}}{\sqrt{s}}\left\|t(\partial_{x_j}+x_j)W_{t^2/4s}^{\mathcal{L}}(1)(x)\right\|_H\,ds$$

$$\leqslant C \int_0^\infty e^{-s} \left\| \sqrt{u}(\partial_{x_j} + x_j) W_u^{\mathcal{L}}(1)(x) \right\|_H ds \leqslant C, \quad x \in \mathbb{R}^n.$$

In a similar way we can see that $\nabla_x T_{j,+}^{\mathcal{L}}(1) \in L^\infty(\mathbb{R}^n, H)$.

By using Theorem 1.1 we conclude that $T_{j,+}^{\mathcal{L}}$ is bounded from $BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$ into $BMO_{\mathcal{L}}(\mathbb{R}^n, \gamma(H, \mathbb{B}))$.

3.2. We are going to see that $T_{j,+}^{\mathcal{L}}$ is a bounded operator from $H_{\mathcal{L}}^1(\mathbb{R}^n, \mathbb{B})$ into $H_{\mathcal{L}}^1(\mathbb{R}^n, \gamma(H, \mathbb{B}))$. The boundedness property of $T_{j,-}^{\mathcal{L}}$ can be proved in a similar way, for $n \geqslant 3$.

In Section 3.1 we saw that $T_{j,+}^{\mathcal{L}}$ is a Calderón–Zygmund operator. Hence, it follows that $T_{j,+}^{\mathcal{L}}$ can be extended from $L^2(\mathbb{R}^n, \mathbb{B}) \cap L^1(\mathbb{R}^n, \mathbb{B})$ to $L^1(\mathbb{R}^n, \mathbb{B})$ as a bounded operator from $H^1(\mathbb{R}^n, \mathbb{B})$ into $L^1(\mathbb{R}^n, \gamma(H, \mathbb{B}))$ and from $L^1(\mathbb{R}^n, \mathbb{B})$ into $L^{1,\infty}(\mathbb{R}^n, \gamma(H, \mathbb{B}))$. Moreover, according to [2, Theorem 2], $T_{j,+}^{\mathcal{L}}$ is a bounded operator from $H^1(\mathbb{R}^n, \mathbb{B})$ into $L^1(\mathbb{R}^n, \gamma(H, \mathbb{B}))$ and from $L^1(\mathbb{R}^n, \mathbb{B})$ into $L^{1,\infty}(\mathbb{R}^n, \gamma(H, \mathbb{B}))$. By using (41), the procedure developed in Section 2.2 allows us to see that the operator $T_{j,+}^{\mathcal{L}}$ is bounded from $H_{\mathcal{L}}^1(\mathbb{R}^n, \mathbb{B})$ into $L^1(\mathbb{R}^n, \gamma(H, \mathbb{B}))$.

We consider the maximal operator S defined by

$$S(f)(x) = \sup_{s>0} \left\| P_s^{\mathcal{L}+2}\big(T_{j,+}^{\mathcal{L}}(f)\big)(x, \cdot) \right\|_{\gamma(H,\mathbb{B})}.$$

According to Proposition 2.2 the proof of our objective will be finished when we establish that the operator S is bounded from $H_{\mathcal{L}}^1(\mathbb{R}^n, \mathbb{B})$ into $L^1(\mathbb{R}^n)$.

The maximal operator $\mathcal{M}_*$ given by

$$\mathcal{M}_*(g) = \sup_{s>0} \left\| P_s^{\mathcal{L}+2}(g) \right\|_{\gamma(H,\mathbb{B})}$$

is known to be bounded from $L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))$ into $L^p(\mathbb{R}^n)$, for every $1 < p < \infty$, and from $L^1(\mathbb{R}^n, \gamma(H, \mathbb{B}))$ into $L^{1,\infty}(\mathbb{R}^n)$. Since $T_{j,+}^{\mathcal{L}}$ is bounded from $L^p(\mathbb{R}^n, \mathbb{B})$ into $L^p(\mathbb{R}^n, \gamma(H, \mathbb{B}))$, $1 < p < \infty$, from $L^1(\mathbb{R}^n, \mathbb{B})$ into $L^{1,\infty}(\mathbb{R}^n, \gamma(H, \mathbb{B}))$, and from $H^1(\mathbb{R}^n, \mathbb{B})$ into $L^1(\mathbb{R}^n, \gamma(H, \mathbb{B}))$, the operator $\mathbb{S}$ defined by

$$\mathbb{S}(f)(x, s, t) = P_s^{\mathcal{L}+2}\big(T_{j,+}^{\mathcal{L}}(f)(\cdot, t)\big)(x)$$

is bounded from $L^p(\mathbb{R}^n, \mathbb{B})$ into $L^p(\mathbb{R}^n, L^\infty((0, \infty), \gamma(H, \mathbb{B})))$, $1 < p < \infty$, and from $H^1(\mathbb{R}^n, \mathbb{B})$ into $L^{1,\infty}(\mathbb{R}^n, L^\infty((0, \infty), \gamma(H, \mathbb{B})))$.

According to [33, Lemmas 4.1 and 4.2] we have that, for every $f \in L_c^\infty(\mathbb{R}^n) \otimes \mathbb{B}$,

$$\mathbb{S}(f)(x, s, t) = t(\partial_{x_j} + x_j) P_{s+t}^{\mathcal{L}}(f)(x), \quad x \in \mathbb{R}^n \text{ and } s, t > 0.$$

We consider the function

$$\mathcal{Y}(x,y,s,t)=t\frac{s+t}{\sqrt{4\pi}}\int_0^\infty \frac{e^{-(s+t)^2/(4u)}}{u^{3/2}}(\partial_{x_j}+x_j)W_u^{\mathcal{L}}(x,y)\,du,$$
$$x,y\in\mathbb{R}^n,\ x\neq y \text{ and } s,t>0.$$

By proceeding as in (41) we can see that

$$\left\|\mathcal{Y}(x,y,\cdot,\cdot)\right\|_{L^\infty((0,\infty),H)}\leqslant C\frac{e^{-c(|x-y|^2+|y||x-y|)}}{|x-y|^n},\quad x,y\in\mathbb{R}^n,\ x\neq y, \tag{44}$$

and

$$\left\|\nabla_x\mathcal{Y}(x,y,\cdot,\cdot)\right\|_{L^\infty((0,\infty),H)}+\left\|\nabla_y\mathcal{Y}(x,y,\cdot,\cdot)\right\|_{L^\infty((0,\infty),H)}\leqslant\frac{C}{|x-y|^{n+1}},$$
$$x,y\in\mathbb{R}^n,\ x\neq y.$$

Moreover, as in Section 2.2 we can see that, for every $g\in L_c^\infty(\mathbb{R}^n)\otimes\mathbb{B}$,

$$\mathbb{S}(g)(x,s,t)=\left(\int_{\mathbb{R}^n}\mathcal{Y}(x,y,\cdot,\cdot)g(y)\,dy\right)(s,t),\quad x\notin \operatorname{supp}(g),$$

being the integral understood in the $L^\infty((0,\infty),\gamma(H,\mathbb{B}))$-Bochner sense.

Vector valued Calderón–Zygmund theory implies that the operator $\mathbb{S}$ can be extended from $L^2(\mathbb{R}^n,\mathbb{B})\cap L^1(\mathbb{R}^n,\mathbb{B})$ to $L^1(\mathbb{R}^n,\mathbb{B})$ as a bounded operator from $L^1(\mathbb{R}^n,\mathbb{B})$ to $L^{1,\infty}(\mathbb{R}^n,L^\infty((0,\infty),\ \gamma(H,\mathbb{B})))$ and from $H^1(\mathbb{R}^n,\mathbb{B})$ into $L^1(\mathbb{R}^n,L^\infty((0,\infty),\gamma(H,\mathbb{B})))$. In order to see that $\mathbb{S}$ is in fact bounded from $L^1(\mathbb{R}^n,\mathbb{B})$ into $L^{1,\infty}(\mathbb{R}^n,L^\infty((0,\infty),\gamma(H,\mathbb{B})))$ and from $H^1(\mathbb{R}^n,\mathbb{B})$ into $L^1(\mathbb{R}^n,L^\infty((0,\infty),\gamma(H,\mathbb{B})))$, we can proceed as at the end of the proof of Proposition 2.3.

By taking into account that

- (44) holds,
- $\mathbb{S}$ is bounded from $L^1(\mathbb{R}^n,\mathbb{B})$ into $L^{1,\infty}(\mathbb{R}^n,L^\infty((0,\infty),\gamma(H,\mathbb{B})))$,
- $\mathbb{S}$ is bounded from $H^1(\mathbb{R}^n,\mathbb{B})$ into $L^1(\mathbb{R}^n,L^\infty((0,\infty),\gamma(H,\mathbb{B})))$,

we can prove, by using the procedure employed in the final part of Section 2.2, that $\mathbb{S}$ is bounded from $H^1_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})$ into $L^1(\mathbb{R}^n,L^\infty((0,\infty),\gamma(H,\mathbb{B})))$.

Thus the proof of Theorem 1.3 for $T^{\mathcal{L}}_{j,+}$ is finished.

4. Proof of Theorem 1.4

Theorems 1.2 and 1.3 show that (i) implies (ii) and (i) implies (iii).

Suppose that (ii) is true for some $j = 1, \dots, n$. Let $f = \sum_{i=1}^m f_i b_i$, where $f_i \in H^1_{\mathcal{L}}(\mathbb{R}^n)$ and $b_i \in \mathbb{B}$, $i = 1, \dots, m \in \mathbb{N}$. We denote by $R^{\mathcal{L}}_{j,+}$ the j-th Riesz transform in the Hermite setting (see Appendix A for definitions). According to Proposition A.2,

$$R^{\mathcal{L}}_{j,+}(f) = \sum_{i=1}^m b_i R^{\mathcal{L}}_{j,+}(f_i) \in H^1_{\mathcal{L}}\big(\mathbb{R}^n\big) \otimes \mathbb{B}.$$

By applying [33, Lemmas 4.1 and 4.2] we get, for every atom a for $H^1_{\mathcal{L}}(\mathbb{R}^n)$,

$$T^{\mathcal{L}}_{j,+}(a) = -\mathcal{G}_{\mathcal{L}+2,\mathbb{C}} R^{\mathcal{L}}_{j,+}(a).$$

Moreover, $T^{\mathcal{L}}_{j,+}$ and $\mathcal{G}_{\mathcal{L}+2,\mathbb{C}} \circ R^{\mathcal{L}}_{j,+}$ are bounded operators from $H^1_{\mathcal{L}}(\mathbb{R}^n)$ into $H^1_{\mathcal{L}}(\mathbb{R}^n, H)$ (see Theorem 1.3, Proposition A.2 and Theorem 1.2). Then, we have that

$$T^{\mathcal{L}}_{j,+}(g) = -\mathcal{G}_{\mathcal{L}+2,\mathbb{C}} R^{\mathcal{L}}_{j,+}(g), \quad g \in H^1_{\mathcal{L}}\big(\mathbb{R}^n\big),$$

and this implies

$$T^{\mathcal{L}}_{j,+}(f) = -\mathcal{G}_{\mathcal{L}+2,\mathbb{B}} R^{\mathcal{L}}_{j,+}(f).$$

We can write

$$\begin{aligned}\big\| R^{\mathcal{L}}_{j,+}(f)\big\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})} &\leqslant C \big\| \mathcal{G}_{\mathcal{L}+2,\mathbb{B}} R^{\mathcal{L}}_{j,+}(f)\big\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,\gamma(H,B))} = C \big\| T^{\mathcal{L}}_{j,+} f\big\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,\gamma(H,\mathbb{B}))} \\ &\leqslant C \| f\|_{H^1_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})}.\end{aligned}$$

Since $H^1_{\mathcal{L}}(\mathbb{R}^n) \otimes \mathbb{B}$ is a dense subspace of $H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$ [21, Lemma 2.4], $H^1(\mathbb{R}^n, \mathbb{B}) \subset H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B})$ and $H^1_{\mathcal{L}}(\mathbb{R}^n, \mathbb{B}) \subset L^1(\mathbb{R}^n, \mathbb{B})$ [27, Theorem 4.1] implies that $R^{\mathcal{L}}_{j,+}$ can be extended to $L^2(\mathbb{R}^n, \mathbb{B})$ as a bounded operator from $L^2(\mathbb{R}^n, \mathbb{B})$ into itself. Then, from [1, Theorem 2.3] we deduce that $\mathbb{B}$ is UMD.

Assume now (iii) holds for some $j = 1, \dots, n$. By proceeding as above, this time applying Proposition A.1, we can see that, for every $f \in L^\infty_c(\mathbb{R}^n) \otimes \mathbb{B}$,

$$\big\| R^{\mathcal{L}}_{j,+}(f)\big\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})} \leqslant C \| f\|_{BMO_{\mathcal{L}}(\mathbb{R}^n,\mathbb{B})}. \tag{45}$$

Let $\mathbb{E}$ be a finite dimensional subspace of $\mathbb{B}$. By taking into account that $L^\infty_c(\mathbb{R}^n) \otimes \mathbb{E} = L^\infty_c(\mathbb{R}^n, \mathbb{E}) \subset BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{E})$ and $BMO_{\mathcal{L}}(\mathbb{R}^n, \mathbb{E}) \subset BMO(\mathbb{R}^n, \mathbb{E})$, from (45) and [27, Theorem 4.1] we deduce that $R^{\mathcal{L}}_{j,+}$ can be extended to $L^2(\mathbb{R}^n, \mathbb{E})$ as a bounded operator from $L^2(\mathbb{R}^n, \mathbb{E})$ into itself and

$$\big\| R^{\mathcal{L}}_{j,+}(f)\big\|_{L^2(\mathbb{R}^n,\mathbb{E})} \leqslant C \| f\|_{L^2(\mathbb{R}^n,\mathbb{E})}, \quad f \in L^2\big(\mathbb{R}^n, \mathbb{E}\big),$$

where $C > 0$ does not depend on $\mathbb{E}$. Hence,

$$\left\| R^{\mathcal{L}}_{j,+}(f) \right\|_{L^2(\mathbb{R}^n,\mathbb{B})} \leqslant C \| f \|_{L^2(\mathbb{R}^n,\mathbb{B})}, \quad f \in L^2(\mathbb{R}^n) \otimes \mathbb{B}.$$

From [1, Theorem 2.3] it follows that $\mathbb{B}$ is UMD.

The proof of the result when $T^{\mathcal{L}}_{j,+}$ and $\mathcal{G}_{\mathcal{L}+2,\mathbb{B}}$ are replaced by $T^{\mathcal{L}}_{j,-}$ and $\mathcal{G}_{\mathcal{L}-2,\mathbb{B}}$, respectively, can be made similarly, for every $n \geqslant 3$.

Appendix A

The Hermite operator L can be written as follows

$$L = -\frac{1}{2}\left[(\nabla + x)(\nabla - x) + (\nabla - x)(\nabla + x)\right].$$

This decomposition suggests to call Riesz transforms in the Hermite setting to the operators formally defined by

$$R^{\mathcal{L}}_{j,\pm} = (\partial_{x_j} \pm x_j)\mathcal{L}^{-1/2}, \quad j = 1, \ldots, n \tag{46}$$

(see [33] and [37]).

Let $j = 1, \ldots, n$. We denote by e_j the j-th coordinate vector in $\mathbb{R}^n$. It is well known that

$$(\partial_{x_j} + x_j)h_k = (2k_j)^{1/2} h_{k-e_j}, \qquad (\partial_{x_j} - x_j)h_k = -(2k_j + 2)^{1/2} h_{k+e_j}, \tag{47}$$

for every $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$.

The negative square root $\mathcal{L}^{-1/2}$ of $\mathcal{L}$ is defined by

$$\mathcal{L}^{-1/2}(f)(x) = \int_0^\infty P^{\mathcal{L}}_t(f)(x)\,dt, \quad f \in L^2(\mathbb{R}^n).$$

We have that

$$\mathcal{L}^{-1/2}(f) = \sum_{k \in \mathbb{N}^n} \frac{1}{\sqrt{2|k| + n}} \langle f, h_k \rangle h_k, \quad f \in L^2(\mathbb{R}^n). \tag{48}$$

Equalities (46), (47) and (48) lead to define the Riesz transforms $R^{\mathcal{L}}_{j,\pm}$ by

$$R^{\mathcal{L}}_{j,+}(f) = \sum_{k \in \mathbb{N}^n} \sqrt{\frac{2k_j}{2|k| + n}} \langle f, h_k \rangle h_{k-e_j}, \quad f \in L^2(\mathbb{R}^n),$$

and

$$R_{j,-}^{\mathcal{L}}(f) = -\sum_{k\in\mathbb{N}^n}\sqrt{\frac{2k_j+2}{2|k|+n}}\langle f, h_k\rangle h_{k+e_j}, \quad f\in L^2(\mathbb{R}^n).$$

Plancherel equality imply that $R_{j,\pm}^{\mathcal{L}}$ is bounded from $L^2(\mathbb{R}^n)$ into itself. L^p-boundedness properties of $R_{j,\pm}^{\mathcal{L}}$ were established by Stempak and Torrea in [33] (see also [39]). They use Calderón–Zygmund theory and show that $R_{j,\pm}^{\mathcal{L}}$ are singular integrals associated to the Calderón–Zygmund kernels

$$R_{j,\pm}^{\mathcal{L}}(x,y) = \int_0^\infty (\partial_{x_j} \pm x_j) P_t^{\mathcal{L}}(x,y)\,dt, \quad x,y\in\mathbb{R}^n,\ x\neq y. \tag{49}$$

$R_{j,\pm}^{\mathcal{L}}$ can be extended from $L^2(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ as a bounded operator from $L^p(\mathbb{R}^n)$ into itself, $1<p<\infty$, and from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$ [33, Corollary 3.4]. We continue denoting by $R_{j,\pm}^{\mathcal{L}}$ the extended operators.

In the following propositions we analyze the behavior of $R_{j,\pm}^{\mathcal{L}}$, $j=1,\dots,n$, in the spaces $BMO_{\mathcal{L}}(\mathbb{R}^n)$ and $H_{\mathcal{L}}^1(\mathbb{R}^n)$.

Proposition A.1. *Let $j=1,\dots,n$. Then, the Riesz transforms $R_{j,\pm}^{\mathcal{L}}$ are bounded from $BMO_{\mathcal{L}}(\mathbb{R}^n)$ into itself.*

Proof. We only analyze $R_{j,+}^{\mathcal{L}}$. The operator $R_{j,-}^{\mathcal{L}}$ can be studied similarly. In [3, Section 4.3] it was shown that the operator $R_{j,+}^{\mathcal{L}} - x_j\mathcal{L}^{-1/2}$ is bounded from $BMO_{\mathcal{L}}(\mathbb{R}^n)$ into itself.

We consider now the operator $T_j = x_j\mathcal{L}^{-1/2}$. By (4) we can write

$$T_j(f)(x) = \frac{x_j}{\sqrt{\pi}}\int_0^\infty W_t^{\mathcal{L}}(f)(x)\frac{dt}{\sqrt{t}} = \int_0^\infty M_j(x,y)f(y)\,dy, \quad f\in L^2(\mathbb{R}^n),$$

where

$$M_j(x,y) = \frac{x_j}{\sqrt{\pi}}\int_0^\infty W_t^{\mathcal{L}}(x,y)\frac{dt}{\sqrt{t}}, \quad x,y\in\mathbb{R}^n,\ x\neq y.$$

According to [8, Lemma 3] the operator T_j is bounded from $L^2(\mathbb{R}^n)$ into itself.

We are going to show that

$$\left|M_j(x,y)\right| \leqslant C\frac{e^{-c(|x-y|^2+|x||x-y|)}}{|x-y|^n}, \quad x,y\in\mathbb{R}^n \text{ and } x\neq y,$$

and

$$\left|\nabla_x M_j(x,y)\right| + \left|\nabla_y M_j(x,y)\right| \leqslant \frac{C}{|x-y|^{n+1}}, \quad x, y \in \mathbb{R}^n \text{ and } x \neq y.$$

By using (7) we deduce

$$\begin{aligned}
\left|M_j(x,y)\right| &\leqslant C|x|e^{-c(|x-y|^2+|x||x-y|)} \int_0^\infty \frac{e^{-c(|x-y|^2/t+(1-e^{2t})|x+y|^2)}}{t^{(n+1)/2}}\,dt \\
&\leqslant C\big(|x+y|+|x-y|\big)e^{-c(|x-y|^2+|x||x-y|)} \\
&\quad \times \left(\int_0^1 \frac{e^{-c(|x-y|^2/t+t|x+y|^2)}}{t^{(n+1)/2}}\,dt + e^{-c|x+y|^2}\right) \\
&\leqslant Ce^{-c(|x-y|^2+|x||x-y|)}\left(\int_0^1 \frac{e^{-c|x-y|^2/t}}{t^{(n+2)/2}}\,dt + 1\right) \\
&\leqslant C\frac{e^{-c(|x-y|^2+|x||x-y|)}}{|x-y|^n}, \quad x, y \in \mathbb{R}^n, x \neq y.
\end{aligned}$$

Let $i = 1, \ldots, n$. For every $x, y \in \mathbb{R}^n$, $x \neq y$, and $i \neq j$, we have that

$$\partial_{x_i} M_j(x,y) = -\frac{1}{2\sqrt{\pi}} x_j \int_0^\infty \left(\frac{1+e^{-2t}}{1-e^{-2t}}(x_i-y_i) + \frac{1-e^{-2t}}{1+e^{-2t}}(x_i+y_i)\right) W_t^{\mathcal{L}}(x,y)\frac{dt}{\sqrt{t}},$$

and

$$\begin{aligned}
\partial_{x_j} M_j(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \bigg(1 - x_j\frac{1+e^{-2t}}{2(1-e^{-2t})}(x_j-y_j) \\
- x_j\frac{1-e^{-2t}}{2(1+e^{-2t})}(x_j+y_j)\bigg) W_t^{\mathcal{L}}(x,y)\frac{dt}{\sqrt{t}}.
\end{aligned}$$

Hence, by (7) we get

$$\begin{aligned}
&\left|\partial_{x_i} M_j(x,y)\right| \\
&\quad \leqslant C\int_0^\infty \left(1 + |x|\frac{|x-y|}{1-e^{-2t}} + \big(|x+y|+|x-y|\big)\big(1-e^{-2t}\big)|x+y|\right) W_t^{\mathcal{L}}(x,y)\frac{dt}{\sqrt{t}}
\end{aligned}$$

$$\leqslant C\int_0^\infty \frac{e^{-c|x-y|^2/t}e^{-ct}}{(1-e^{-4t})^{(n+2)/2}}\frac{dt}{\sqrt{t}} \leqslant C\int_0^\infty \frac{e^{-c|x-y|^2/t}}{t^{(n+3)/2}}\,dt$$
$$\leqslant \frac{C}{|x-y|^{n+1}}, \quad x,y\in\mathbb{R}^n,\ x\neq y.$$

In a similar way we can see that, for every $i=1,\dots,n$,

$$\left|\partial_{y_i}M_j(x,y)\right| \leqslant \frac{C}{|x-y|^{n+1}}, \quad x,y\in\mathbb{R}^n,\ x\neq y.$$

According to (13) we can write

$$T_j(1)(x) = \frac{x_j}{\pi^{(n+1)/2}}\int_0^\infty \left(\frac{e^{-2t}}{1+e^{-4t}}\right)^{n/2}\exp\left(-\frac{1-e^{-4t}}{2(1+e^{-4t})}|x|^2\right)\frac{dt}{\sqrt{t}}, \quad x\in\mathbb{R}^n.$$

It follows that

$$\left|T_j(1)(x)\right| \leqslant C|x|\left(\int_0^1 \frac{e^{-ct|x|^2}}{\sqrt{t}}\,dt + e^{-c|x|^2}\int_1^\infty e^{-nt}\,dt\right) \leqslant C, \quad x\in\mathbb{R}^n.$$

Moreover, for every $i=1,\dots,n$, $i\neq j$, we have that

$$\partial_{x_i}T_j(1)(x) = -\frac{x_jx_i}{\sqrt{\pi}}\int_0^\infty \frac{1-e^{-4t}}{1+e^{-4t}}W_t^{\mathcal{L}}(1)(x)\frac{dt}{\sqrt{t}}, \quad x\in\mathbb{R}^n,$$

and

$$\partial_{x_j}T_j(1)(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty \left(1-\frac{1-e^{-4t}}{1+e^{4t}}x_j^2\right)W_t^{\mathcal{L}}(1)(x)\frac{dt}{\sqrt{t}}, \quad x\in\mathbb{R}^n.$$

Then, we can deduce that $\nabla T_j(1)\in L^\infty(\mathbb{R}^n)$.

By [3, Theorem 1.1] we conclude that T_j can be extended to $BMO_{\mathcal{L}}(\mathbb{R}^n)$ as a bounded operator from $BMO_{\mathcal{L}}(\mathbb{R}^n)$ into itself. □

Proposition A.2. *Let $j=1,\dots,n$. Then, the Riesz transforms $R_{j,\pm}^{\mathcal{L}}$ can be extended from $L^2(\mathbb{R}^n)\cap H_{\mathcal{L}}^1(\mathbb{R}^n)$ to $H_{\mathcal{L}}^1(\mathbb{R}^n)$ as bounded operators from $H_{\mathcal{L}}^1(\mathbb{R}^n)$ into itself.*

Proof. We study the operator $R_{j,+}^{\mathcal{L}}$. $R_{j,-}^{\mathcal{L}}$ can be analyzed in a similar way.

By taking in mind Proposition 2.2 it is enough to see that $R^{\mathcal{L}}_{j,+}$ can be extended as a bounded operator from $H^1_{\mathcal{L}}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ and that the operator G defined by

$$G(f)(x,t) = P_t^{\mathcal{L}+2}\big(R^{\mathcal{L}}_{j,+}(f)\big)(x),$$

is bounded from $H^1_{\mathcal{L}}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n, L^\infty(0,\infty))$.

In [33, Theorem 3.3] it was proved that the Riesz transform $R^{\mathcal{L}}_{j,+}$ is a Calderón–Zygmund operator associated to the kernel given in (49). Thus, in order to see that $R^{\mathcal{L}}_{j,+}$ can be extended as a bounded operator from $H^1_{\mathcal{L}}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, we only need to show that

$$\big|R^{\mathcal{L}}_{j,+}(x,y)\big| \leqslant C\frac{e^{-c(|x-y|^2+|y||x-y|)}}{|x-y|^n}, \quad x,y\in\mathbb{R}^n,\ x\neq y \tag{50}$$

and then reasoning as at the end of Section 2.2. Estimation (50) follows from (40).

Now we establish that G can be extended as a bounded operator from $H^1_{\mathcal{L}}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n, L^\infty(0,\infty))$. We observe that (see [33, (4.1) and (4.3)])

$$\begin{aligned} G(f)(x,t) &= \sum_{k\in\mathbb{N}^n}\sqrt{\frac{2k_j}{2|k|+n}}e^{-t\sqrt{2|k|+n}}\langle f,h_k\rangle h_{k-e_j}(x) \\ &= R^{\mathcal{L}}_{j,+}\big(P_t^{\mathcal{L}}(f)\big)(x), \quad f\in L^2\big(\mathbb{R}^n\big). \end{aligned}$$

We consider the function

$$\mathcal{C}(x,y,t) = \int_0^\infty (\partial_{x_j}+x_j)P^{\mathcal{L}}_{t+s}(x,y)\,ds, \quad x,y\in\mathbb{R}^n \text{ and } t\in(0,\infty).$$

This function $\mathcal{C}$ satisfies the following $L^\infty(0,\infty)$-Hermite–Calderón–Zygmund conditions:

$$\big\|\mathcal{C}(x,y,\cdot)\big\|_{L^\infty(0,\infty)} \leqslant C\frac{e^{-c(|x-y|^2+|y||x-y|)}}{|x-y|^n}, \quad x,y\in\mathbb{R}^n,\ x\neq y \tag{51}$$

and

$$\begin{aligned} &\big\|\nabla_x\mathcal{C}(x,y,\cdot)\big\|_{L^\infty(0,\infty)} + \big\|\nabla_y\mathcal{C}(x,y,\cdot)\big\|_{L^\infty(0,\infty)} \leqslant \frac{C}{|x-y|^{n+1}}, \\ &x,y\in\mathbb{R}^n,\ x\neq y. \end{aligned} \tag{52}$$

Indeed, by (40) it follows that

$$\left\|\mathcal{C}(x,y,\cdot)\right\|_{L^\infty(0,\infty)} \leqslant e^{-c(|x-y|^2+|y||x-y|)} \sup_{t>0} \int_0^\infty \frac{t+s}{(t+s+|x-y|)^{n+2}}\,ds$$

$$\leqslant C \sup_{t>0} \frac{e^{-c(|x-y|^2+|y||x-y|)}}{(t+|x-y|)^n}$$

$$\leqslant C \frac{e^{-c(|x-y|^2+|y||x-y|)}}{|x-y|^n}, \quad x,y \in \mathbb{R}^n,\ x \neq y.$$

In order to show (52) we can proceed in a similar way.

Suppose now that $f \in C_c^\infty(\mathbb{R}^n)$. We can write

$$G(f)(x,t) = \int_{\mathbb{R}^n} \mathcal{C}(x,y,t) f(y)\,dy, \quad x \notin \operatorname{supp}(f) \text{ and } t > 0.$$

Let $x \notin \operatorname{supp}(f)$. Note that, for every $y \in \mathbb{R}^n$, the function $g_{x,y}(t) = \mathcal{C}(x,y,t) f(y)$, $t \in (0,\infty)$ is continuous, $\lim_{t\to\infty} g_{x,y}(t) = 0$, and there exists the limit $\lim_{t\to 0^+} g_{x,y}(t)$.

We denote by $C_0([0,\infty))$ the space of continuous functions on $[0,\infty)$ that converge to zero in infinity. $C_0([0,\infty))$ is endowed with the supremum norm. The dual space of $C_0([0,\infty))$ is the space of complex measures $\mathcal{M}([0,\infty))$ on $[0,\infty)$.

By (51) we have that $\int_{\mathbb{R}^n} \|\mathcal{C}(x,y,\cdot)\|_{L^\infty(0,\infty)} |f(y)|\,dy < \infty$. We define

$$L_x(f) = \int_{\mathbb{R}^n} \mathcal{C}(x,y,\cdot) f(y)\,dy,$$

where the last integral is understood in the $C_0([0,\infty))$-Bochner sense. Let $\mu \in \mathcal{M}([0,\infty))$. We can write

$$\left\langle \mu, L_x(f)\right\rangle_{\mathcal{M}([0,\infty)),C_0([0,\infty))} = \int_{[0,\infty)} L_x(f)(s)\,d\mu(s) = \int_{\mathbb{R}^n} \int_{[0,\infty)} \mathcal{C}(x,y,s)\,d\mu(s) f(y)\,dy$$

$$= \int_{[0,\infty)} \int_{\mathbb{R}^n} \mathcal{C}(x,y,s) f(y)\,dy\,d\mu(s).$$

Then,

$$L_x(f)(t) = \int_{\mathbb{R}^n} \mathcal{C}(x,y,t) f(y)\,dy, \quad t \in [0,\infty),$$

and we conclude that

$$G(f)(x,t)=\biggl(\int_{\mathbb{R}^n}\mathcal{C}(x,y,\cdot)f(y)\,dy\biggr)(t),\quad t\in(0,\infty),$$

where the integral is understood in the $C_0([0,\infty))$-Bochner sense.

$R^{\mathcal{L}}_{j,+}$ is bounded from $L^2(\mathbb{R}^n)$ into itself. Moreover, the maximal operator

$$P^{\mathcal{L}+2}_*(g)=\sup_{t>0}\bigl|P^{\mathcal{L}+2}_t(g)\bigr|$$

is also bounded from $L^2(\mathbb{R}^n)$ into itself. Hence, G is bounded operator from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n, L^\infty(0,\infty))$.

According to Banach valued Calderón–Zygmund theory we deduce that G can be extended to $L^1(\mathbb{R}^n)$ as a bounded operator from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n, L^\infty(0,\infty))$ and from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n, L^\infty(0,\infty))$.

By proceeding now as in the final part of Section 2.2, (51) allows us to conclude that G can be extended to $H^1_{\mathcal{L}}(\mathbb{R}^n)$ as a bounded operator from $H^1_{\mathcal{L}}(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n, L^\infty(0,\infty)))$. We denote by $\widetilde{G}$ this extension.

Suppose that $f\in H^1_{\mathcal{L}}(\mathbb{R}^n)$. There exist a sequence $(a_i)_{i\in\mathbb{N}}$ of atoms for $H^1_{\mathcal{L}}(\mathbb{R}^n)$ and a sequence $(\lambda_i)_{i\in\mathbb{N}}$ of complex numbers such that $\sum_{i=1}^\infty|\lambda_i|<\infty$ and $f=\sum_{i=1}^\infty\lambda_i a_i$. Since this series converge in $H^1_{\mathcal{L}}(\mathbb{R}^n)$, we have that

$$\widetilde{G}(f)=\sum_{i=1}^\infty\lambda_i G(a_i),\quad \text{in } L^1\bigl(\mathbb{R}^n, L^\infty(0,\infty)\bigr).$$

Then, for every $t>0$,

$$\widetilde{G}(f)(\cdot,t)=\sum_{i=1}^\infty\lambda_i G(a_i)(\cdot,t)=\sum_{i=1}^\infty\lambda_i P^{\mathcal{L}+2}_t\bigl(R^{\mathcal{L}}_{j,+}(a_i)\bigr),\quad \text{in } L^1\bigl(\mathbb{R}^n\bigr).$$

Moreover, for every $t>0$,

$$P^{\mathcal{L}+2}_t R^{\mathcal{L}}_{j,+}(f)=\sum_{i=1}^\infty\lambda_i P^{\mathcal{L}+2}_t\bigl(R^{\mathcal{L}}_{j,+}(a_i)\bigr),\quad \text{in } L^1\bigl(\mathbb{R}^n\bigr).$$

We conclude that

$$\widetilde{G}(f)(\cdot,t)=P^{\mathcal{L}+2}_t\bigl(R^{\mathcal{L}}_{j,+}(f)\bigr),\quad t>0,$$

and the proof of this property is finished. □

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