Journal of Multivariate Analysis 102 (2011) 1454–1471

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

Tail order and intermediate tail dependence of multivariate copulas

Lei Hua *, Harry Joe

Department of Statistics, University of British Columbia, 333-6356 Agricultural Road, Vancouver, BC, Canada V6T1Z2

ARTICLE INFO

Article history:
Received 17 November 2009
Available online 27 May 2011

AMS subject classification:
62H99

Keywords:
Archimedean copula
Laplace transform
Max-infinitely divisible
Maximal moment
Reflection symmetry
Regular variation
Tail asymmetry

ABSTRACT

In order to study copula families that have tail patterns and tail asymmetry different from multivariate Gaussian and t copulas, we introduce the concepts of tail order and tail order functions. These provide an integrated way to study both tail dependence and intermediate tail dependence. Some fundamental properties of tail order and tail order functions are obtained. For the multivariate Archimedean copula, we relate the tail heaviness of a positive random variable to the tail behavior of the Archimedean copula constructed from the Laplace transform of the random variable, and extend the results of Charpentier and Segers [7] [A. Charpentier, J. Segers, Tails of multivariate Archimedean copulas, Journal of Multivariate Analysis 100 (7) (2009) 1521–1537] for upper tails of Archimedean copulas. In addition, a new one-parameter Archimedean copula family based on the Laplace transform of the inverse Gamma distribution is proposed; it possesses patterns of upper and lower tails not seen in commonly used copula families. Finally, tail orders are studied for copulas constructed from mixtures of max-infinitely divisible copulas.

1. Introduction

For statistical modeling with copulas, properties such as strengths of upper/lower tail dependence and reflection symmetry or direction of reflection asymmetry are important in deciding on appropriate copulas. For example, for the tail asymmetry phenomena of financial markets [33,32], copula families with a variety of tail behavior are useful for statistical modeling. Although, the multivariate Gaussian and t copula families have a wide range of dependence, they are not appropriate, when there is reflection or tail asymmetry. But copulas can be constructed from other methods to get different patterns of joint tail behavior. Then for the use of copulas, for the inference of joint tail probabilities, sensitivity analysis over different families can be performed.

The study of tail behavior of random vectors has received increasing attention, especially in the framework of quantitative risk management. Let $\mathbf{X} = (X_1, \ldots, X_d)^{\mathrm{T}}$ be a random vector with distribution function F and continuous univariate marginal distribution functions F_i, $i = 1, \ldots, d$. Due to Sklar's theorem [16,30],

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)), \tag{1}$$

in which the copula function $C : [0, 1]^d \to [0, 1]$ is uniquely determined by

$$C(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)), \tag{2}$$

where F_i^{-1} is the inverse function of F_i, $i = 1, \ldots, d$. The corresponding survivor function $\overline{C}$ is defined as $\overline{C}(u_1, \ldots, u_d) = 1 + \sum_{I \subset \{1,\ldots,d\}} (-1)^{|I|} C_I(u_i, i \in I)$, where C_I is the I-marginal of the copula C with $|I|$ the cardinality of the set I. In this paper, we will study tail behavior of the copula C.

* Corresponding author.
E-mail addresses: leihua@stat.ubc.ca (L. Hua), harry@stat.ubc.ca (H. Joe).

0047-259X/$ – see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmva.2011.05.011

There exist several related methodologies. The lower tail dependence parameter is defined as

$$\lambda_L = \lim_{u\to 0^+} u^{-1}C(u, \ldots, u),$$

and the upper tail dependence parameter is defined similarly with the survival function $\overline{C}$. As extensions, Juri and Wüthrich [20,21] studied tail dependence from a distributional point of view, Klüppelberg et al. [22] defined the so-called tail dependence function of $\mathbf{X}$ as

$$\lambda^{\mathbf{X}}(x_1, \ldots, x_d) = \lim_{t\to 0} t^{-1}\mathbb{P}[1 - F_1(X_1) \le tx_1, \ldots, 1 - F_d(X_d) \le tx_d],$$

and [31,18,25] further studied the properties of the tail dependence function and their applications for multivariate t copulas, vine copulas and heavy-tailed scale mixtures of multivariate distributions, respectively. We refer to the above papers for details and properties of tail dependence functions.

For the study of tail dependence behavior of random vectors, we not only have interest in the cases where the random vector is asymptotically dependent, but also where asymptotic independence exhibits. Ledford and Tawn [23] proposed the following model for a bivariate random vector $(X_1, X_2)^T$, where X_1 and X_2 are unit Fréchet distributed with cumulative distribution functions (cdf) $F_i(x) = e^{-1/x}, x \ge 0, i = 1, 2$, and are non-negatively associated,

$$\mathbb{P}[X_1 > r, X_2 > r] \sim \ell(r)r^{-1/\eta}, \quad r \to \infty, \tag{3}$$

where the notation "$g(x) \sim h(x), x \to x_0$" means that $\lim_{x\to x_0} g(x)/h(x) = 1$, and $\ell(r)$ is a slowly varying function and $1/2 \le \eta \le 1$. If we let $U_i = F_i(X_i), i = 1, 2$, where F_i is the cdf of the unit Fréchet and $r = (-\log(u))^{-1}$, then clearly

$$\lim_{u\to 1^-} \frac{\mathbb{P}[U_1 > u, U_2 > u]}{(\mathbb{P}[U_1 > u])^\kappa} = \lim_{r\to\infty} \frac{\mathbb{P}[X_1 > r, X_2 > r]}{(\mathbb{P}[X_1 > r])^\kappa} = \lim_{r\to\infty} \frac{\ell(r)r^{-1/\eta}}{[1 - \exp(-r^{-1})]^\kappa} = \lim_{r\to\infty} \frac{\ell(r)r^{-1/\eta}}{r^{-\kappa}}.$$

Thus the "tail order" κ that we will introduce in Definition 2 corresponds to $1/\eta$ of Ledford and Tawn's representation. If $\eta = 1$, *i.e.*, $\kappa = 1$ and $\ell(r) \nrightarrow 0$, X_1 and X_2 are upper tail dependent with upper tail dependence parameter $\lambda_U = \lim_{r\to\infty} \ell(r)$; if $1/2 < \eta < 1$, they are positively associated; if $\eta = 1/2$ and $\ell(r) \ge 1$, they are "near independence". A lot of research has been done following this direction. We refer to [23,24,10,15,34] for further development of this idea.

The relation (3) tells us that the power term $1/\eta$ dominates the speed of decay of the joint tail probability. We believe that the parameter $1/\eta$ plays an important role in the study of tail dependence behavior, and deserves a new name "tail order" that is explained in Section 2.1, based on copula functions. Moreover, analogously to the tail dependence function, we will propose the tail order function, which includes the information of the convergence along routes other than the diagonal.

In this paper, the emphasis is on the case where the tail order is between 1 and d for a d-dimensional random vector. We refer to this case as "intermediate tail dependence" under some positive dependence assumptions; this is explained before Example 1.

Our main contributions in this article span the following aspects: 1. We propose the concepts of tail order and tail order functions as an integrated way to study tail behavior of multivariate copulas. 2. We relate the tail heaviness of a positive random variable to the tail behavior of the Archimedean copula constructed by the Laplace transform of the random variable. In our opinion, it is an insightful way to better understand the tail behavior of Archimedean copulas. 3. Our theoretical study of tail behavior of Archimedean copulas leads to a new one-parameter Archimedean copula family, based on the Laplace transform of the inverse Gamma distribution, which shows patterns of upper and lower tails not seen in commonly used copula families.

The remainder of this paper is organized as follows. Section 2 introduces the concepts of tail order and tail order functions, and some properties of them. In particular, some results on relations of tail orders of marginal copulas are given. Sections 3 and 4 contain studies of intermediate tail dependence for Archimedean copulas and copulas constructed by mixture of max-id distributions, respectively. For multivariate Archimedean copulas, we have a more concrete result than [7] for the lower tail, and new results for the upper tail. Asymptotic behavior of Laplace transforms of positive random variables is studied in Section 3.1, and the new Archimedean copula family is presented in Section 3.4. Finally, Section 5 concludes with some topics of further research. The main proofs are put in the Appendix.

2. Tail orders: definitions and properties

In this section, we define the concepts of tail order and tail order functions, indicate their use for reflection asymmetry and derive some of their properties. The following notation is used throughout this paper: $\mathbf{1}_d = (1, \ldots, 1)$ with d elements, and then $u\mathbf{1}_d = (u, \ldots, u)$; $I_d = \{1, \ldots, d\}$; $\mathbb{N}_+ = \{1, 2, \ldots\}$; $[x] = \max\{t \text{ integer} : t \le x\}$; $\psi^{(i)}(s)$ is the ith order derivative of ψ evaluated at s; $A \subset B$ means that A is a subset (not necessary proper) of B.

2.1. Multivariate tail order and tail order functions

To avoid technicalities for tail orders, we assume conditions involving regular variation of tails of copula and other functions. Standard references on regular variation are [5,35].

Definition 1. A measurable function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is *regularly varying at* ∞ *with index* α (written $g \in \mathcal{R}_\alpha$) if for any $t > 0$,

$$\lim_{x\to\infty} \frac{g(xt)}{g(x)} = t^\alpha. \tag{4}$$

For the lower limit at 0^+, if for any $t > 0$, $\lim_{x\to 0^+} g(xt)/g(x) = t^\alpha$, then g is regularly varying at 0^+ and denoted by $g \in \mathcal{R}_\alpha(0^+)$. Note that $g(t) \in \mathcal{R}_\alpha \iff g(1/t) \in \mathcal{R}_{-\alpha}(0^+)$. If Eq. (4) holds with $\alpha = 0$ for any $t > 0$, then g is said to be slowly varying at ∞ and written as $g \in \mathcal{R}_0$. Similarly, $\mathcal{R}_0(0^+)$ is defined. We will usually use $\ell(x)$ to represent a slowly varying function, and a regularly varying function g can be written as $g(x) = x^\alpha \ell(x)$.

Definition 2. Suppose C is a d-dimensional copula. If there exists some $\kappa_L(C) > 0$ such that, with some $\ell(u) \in \mathcal{R}_0(0^+)$

$$C(u\mathbf{1}_d) \sim u^{\kappa_L(C)}\ell(u), \quad u \to 0^+,$$

then we refer to $\kappa_L(C)$ as the *lower tail order* of C and refer to $\lambda_L(C) = \lim_{u\to 0^+} \ell(u)$ as the *lower tail order parameter*, provided the limit exists. Similarly, the *upper tail order* is defined as $\kappa_U(C)$ such that

$$\overline{C}((1-u)\mathbf{1}_d) \sim u^{\kappa_U(C)}\ell(u), \quad u \to 0^+,$$

with the *upper tail order parameter* $\lambda_U(C) = \lim_{u\to 0^+} \ell(u)$, provided the limit exists.

When no confusion arises, we use the notation κ to represent lower or upper tail orders, and λ for tail order parameters. $\kappa_L(C) = 1$ [resp. $\kappa_U(C) = 1$] and $\ell(u) \nrightarrow 0$ corresponds to the usual definition of upper [resp. lower] tail dependence. We will assume that $\lim_{s\to 0^+} \ell(s) = h \in [0, \infty]$. But $h = 0$ or $h = \infty$ correspond to boundary cases, in which case more care is needed. In these boundary cases, the "speed" of decrease or increase of $\ell(u)$ affects the tail dependence behavior. For example, if $\ell(u) \to 0$, then a lower speed indicates a stronger tail dependence; if $\ell(u) \to +\infty$, then a higher speed indicates a stronger tail dependence. Note that with $\ell(u) \to h$, if $\kappa(C) = 1$ then $0 \le h \le 1$; if $\kappa(C) > 1$ then $0 \le h \le \infty$. Note also that $\kappa_L(C) = \kappa_U(C) = d$ for the d-dimensional independence copula. It is not possible for $\kappa < 1$ (refer to Proposition 2), but it is possible for $\kappa_L(C)$ and $\kappa_U(C)$ to be greater than d for copulas with negative dependence. For example, as a boundary case, for the bivariate counter-monotonic copula, $\kappa_L(C)$ and $\kappa_U(C)$ can be considered as $+\infty$ because $C(u, u)$ and $\overline{C}(1-u, 1-u)$ are zero for $0 < u < 1/2$.

The cases of $\kappa = 1$ or d have been well studied in the literature, while not much research exists for $1 < \kappa < d$. For the bivariate case, $1 < \kappa < 2$ represents some level of positive dependence in the tail, but not as strong as tail dependence. For multivariate cases, without any further conditions, the meaning of $1 < \kappa < d$ is complicated. We refer to the case $1 < \kappa < d$ as lower [resp. upper] intermediate tail dependence only when all marginal copulas (ultimately) possess positive lower [resp. upper] orthant dependence, of which a formal definition will be given in Definition 4. Unless otherwise specified, when a copula is said to possess intermediate tail dependence, the orthant dependence condition is assumed implicitly. The following is an example of intermediate tail dependence for Gaussian copulas.

Example 1 (*Gaussian Copula*)**.** Consider a multivariate Gaussian copula, constructed by

$$C_{\Phi_d}(u_1, \ldots, u_d) = \Phi_d(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d); \Sigma), \tag{5}$$

where $\Phi_d(\cdot; \Sigma)$ is the joint cdf of a standard d-variate Gaussian random vector with positive definite correlation matrix Σ. The multivariate Gaussian copula defined in (5) has intermediate tail dependence with the tail order $\kappa = \mathbf{1}_d\Sigma^{-1}\mathbf{1}_d^{\mathrm{T}}$, the sum of all elements of Σ^{-1}. It can be verified by noticing that $\lim_{u\to 0^+} C_{\Phi_d}(u\mathbf{1}_d)/u^\kappa = \lim_{t\to-\infty} \Phi_d(t\mathbf{1}_d)/[\Phi(t)]^\kappa$, and as $t \to -\infty$, $\Phi(t) \sim \phi(t)/|t|$ and $\Phi_d(t\mathbf{1}_d)$ is dominated by the exponent term $\exp(-t^2\mathbf{1}_d\Sigma^{-1}\mathbf{1}_d^{\mathrm{T}}/2)$ (Corollary 4.1 of [14]).

The bivariate Gaussian copula with $\rho > 0$ has intermediate tail dependence with the tail order $\kappa = 2/(1+\rho)$ and the slowly varying function at 0^+ being $\ell(u) = (-\log u)^{-\rho/(1+\rho)}$. A related result without using copula functions has been given in [23]. For dimension d with constant correlation ρ, the tail order is $\kappa = d/[1 + (d-1)\rho]$. For the trivariate case with $\rho_{12} = \rho_{23} = \rho$, we have $\kappa = [3 + \rho_{13} - 4\rho]/[1 + \rho_{13} - 2\rho^2]$.

Gaussian copulas are reflection symmetric and have intermediate tail dependence when correlations are positive. They are a subfamily of the elliptical copulas. Under some regularity conditions, tail orders of elliptical copulas will be determined by the tail behavior of corresponding radial random variable R. We refer the reader to a series of Hashorva's work for tail behavior of elliptical distributions, say [13].

Definition 3. Suppose C is a d-dimensional copula and $C(u\mathbf{1}_d) \sim u^\kappa \ell(u)$, $u \to 0^+$ for some $\ell(u) \in \mathcal{R}_0(0^+)$. The *lower tail order function* $b : \mathbb{R}_+^d \to \mathbb{R}_+$ is defined as

$$b(\mathbf{w}; C, \kappa) = \lim_{u\to 0^+} \frac{C(uw_j, 1 \le j \le d)}{u^\kappa \ell(u)},$$

provided the limit function exists. In parallel, if $\overline{C}((1-u)\mathbf{1}_d) \sim u^{\kappa}\ell(u)$, $u \to 0^+$ for some $\ell(u) \in \mathcal{R}_0(0^+)$, the *upper tail order function* $b^* : \mathbb{R}_+^d \to \mathbb{R}_+$ is defined as

$$b^*(\mathbf{w}; C, \kappa) = \lim_{u\to 0^+} \frac{\overline{C}(1-uw_j, 1 \le j \le d)}{u^{\kappa}\ell(u)},$$

provided the limit function exists. If $\ell(u) \to h \neq 0$, then $hb(\mathbf{w}; C, 1)$ and $hb^*(\mathbf{w}; C, 1)$ become the tail dependence functions in [18].

Note that the copula C that satisfies the conditions of the above definition is said to be multivariate regularly varying with limit function b or b^* [36]. Although the general theory and definitions must accommodate an arbitrary slowly varying function ℓ, in specific parametric families of copulas that have tractable forms, we find that either $\ell(u)$ is a constant or proportional to a power of $(-\log u)$.

Example 2 (*Extreme Value Copula*)**.** If a copula C satisfies

$$C(u_1^t, \ldots, u_d^t) = C^t(u_1, \ldots, u_d)$$

for any $(u_1, \ldots, u_d) \in [0,1]^d$ and $t > 0$, then we refer to C as an extreme value copula, denoted by C^{EV}. For any multivariate extreme value copula C^{EV}, there exists a function $A : [0,\infty)^d \to [0,\infty)$ such that $C^{EV}(u_1, \ldots, u_d) = \exp\{-A(-\log u_1, \ldots, -\log u_d)\}$, where A is convex, homogeneous of order 1 and satisfies $\max(x_1, \ldots, x_d) \le A(x_1, \ldots, x_d) \le x_1 + \cdots + x_d$. We refer to Chapter 6 of [16] for details of multivariate extreme value copulas. Thus,

$$C^{EV}(u\mathbf{1}_d) = \exp\{A(\mathbf{1}_d)\log u\} = u^{A(\mathbf{1}_d)}.$$

That is, for any extreme value copula C^{EV}, the lower tail order is $\kappa_L(C^{EV}) = A(\mathbf{1}_d)$ and there is intermediate lower tail dependence except for the boundary cases such as independence copula and comonotonicity copula, where $A(\mathbf{1}_d) = d$ and 1, respectively.

In order to get the lower tail order function of extreme value copulas, first consider the bivariate case, for which

$$\begin{aligned} C^{EV}(uw_1, uw_2) &= \exp\{-A(-\log uw_1, -\log uw_2)\} = \exp\left\{(\log u)A\left(1+\frac{\log w_1}{\log u}, 1+\frac{\log w_2}{\log u}\right)\right\} \\ &\sim \exp\left\{(\log u)\left[A(1,1) + A_1(1,1)\left(\frac{\log w_1}{\log u}\right) + A_2(1,1)\left(\frac{\log w_2}{\log u}\right)\right]\right\}, \quad u \to 0^+ \\ &= u^{A(1,1)} w_1^{A_1(1,1)} w_2^{A_2(1,1)}, \end{aligned}$$

where $A_i = \partial A/\partial x_i$, $i = 1, 2$. Therefore, the lower tail order function is $b(w_1, w_2) = w_1^{A_1(1,1)} w_2^{A_2(1,1)}$. Similarly, for a d-variate extreme value copula, $b(w_1, \ldots, w_d) = w_1^{A_1(\mathbf{1}_d)} \ldots w_d^{A_d(\mathbf{1}_d)}$. By Euler's formula for homogeneous functions, $A(\mathbf{1}_d) = \sum_{i=1}^d A_i(\mathbf{1}_d)$. Then it can be verified that b is homogeneous of order $A(\mathbf{1}_d)$.

In the bivariate case, $\kappa_U = 1$, $\lambda_U = 2 - A(1,1)$, and $\kappa_L = A(1,1)$. That is, a larger value of the upper tail dependence parameter implies stronger lower intermediate tail dependence.

We next mention how the upper and lower tail orders are useful to establish the direction of reflection asymmetry. Let C_R be the copula of $(1-U_1, \ldots, 1-U_d)$ when the copula of $(U_1, \ldots, U_d)$ is C, where U_i's are standard uniform variables. Reflection symmetry means that $C_R \equiv C$ and otherwise we say that there is reflection asymmetry. If $C(u\mathbf{1}_d) \ge C_R(u\mathbf{1}_d)$ for all $0 < u < u_0$, for some $0 < u_0 \le 1/2$, then the copula has more probability in the lower tail (reflection asymmetry with skewness to lower tail). If the inequality is reversed leading to $C(u\mathbf{1}_d) \le C_R(u\mathbf{1}_d)$, then the copula has more probability in the upper tail (reflection asymmetry with skewness to upper tail). For most existing parametric families of copulas, it is difficult to analytically compare $C(u\mathbf{1}_d)$ and $C_R(u\mathbf{1}_d)$, so the direction of reflection asymmetry is analytically easier via the upper and lower tail orders. For example, if $\kappa_L(C) > \kappa_U(C)$, then C has reflection asymmetry skewed to the upper tail (smaller κ means slower convergence to 0), and if $C(u\mathbf{1}_d) \sim \lambda_L u^{\kappa}$ and $C_R(u\mathbf{1}_d) \sim \lambda_U u^{\kappa}$ as $u \to 0^+$ with $\lambda_L > \lambda_U > 0$, then C has reflection asymmetry skewed to the lower tail. For many parametric copula families where we have done numerical computations, u_0 can be taken as 1/2.

The following are some elementary properties of the lower and upper tail order functions b and b^*. Obvious properties of tail order for C_R are the following: $\kappa_L(C) = \kappa_U(C_R)$, $\kappa_U(C) = \kappa_L(C_R)$, $b(\mathbf{w}; C, \kappa) = b^*(\mathbf{w}; C_R, \kappa)$ and $b^*(\mathbf{w}; C, \kappa) = b(\mathbf{w}; C_R, \kappa)$.

Proposition 1. *A lower tail order function $b(\mathbf{w}) = b(\mathbf{w}; C, \kappa)$ has following properties:*

1. *$b(\mathbf{1}) \equiv 1$, and $b(\mathbf{w}) = 0$ if there exists an $i \in I_d$ with $w_i = 0$;*
2. *$b(\mathbf{w})$ is increasing in w_i, $i \in I_d$;*

3. *for any fixed* $t > 0$,

$$b(t\mathbf{w}) = \lim_{u\to 0^+} \frac{C(tuw_j, 1 \le j \le d)}{u^\kappa \ell(u)} = t^\kappa \lim_{u\to 0^+} \frac{C(tuw_j, 1 \le j \le d)}{(tu)^\kappa \ell(tu)} = t^\kappa b(\mathbf{w}).$$

Thus, $b(\mathbf{w})$ *is homogeneous of order* κ.

If $b(\mathbf{w})$ is partially differentiable with respect to each w_i on $(0, +\infty)$, then by the Euler's formula on homogeneous functions, we can write

$$b(\mathbf{w}) = \frac{1}{\kappa} \sum_{j=1}^{d} \frac{\partial b}{\partial w_j} w_j, \quad \forall \mathbf{w} \in \mathbb{R}_+^d.$$

Remark 1. Since $C(u\mathbf{w}) \sim u^\kappa \ell(u) b(\mathbf{w}) = b(u\mathbf{w})\ell(u)$, $u \to 0^+$, the tail order function b captures the tail behavior of the copula C in different directions.

2.2. Further properties of tail orders

In this subsection, we obtain some general properties of tail orders of multivariate copulas, especially on inequalities on tail orders of marginal copulas. There is an "obvious" property in terms of concordance. For two multivariate cdfs F_1, F_2 with the same univariate marginals, we say that F_1 is less concordant than F_2, if $F_1(\mathbf{x}) \le F_2(\mathbf{x})$ and $\overline{F}_1(\mathbf{x}) \le \overline{F}_2(\mathbf{x})$ for any $\mathbf{x}$ in the support of F_1 and F_2. If C_1 is less concordant than C_2, then $\kappa_L(C_1) \ge \kappa_L(C_2)$ and $\kappa_U(C_1) \ge \kappa_U(C_2)$.

Next we introduce some concepts of positive dependence, under which multivariate copulas may have some particular properties on tail orders. We refer to [16,9] for details.

Definition 4. Suppose that $F(\mathbf{x})$ is the cdf of a d-variate random vector $\mathbf{X} = (X_1, \dots, X_d)^T$, then $\mathbf{X}$ or F is said to be

1. positive lower orthant dependent (PLOD) if $\mathbb{P}[X_i \le x_i, \forall i \in I_d] \ge \prod_{i=1}^d \mathbb{P}[X_i \le x_i]$ for any $\mathbf{x} \in \mathbb{R}^d$;
2. left tail decreasing in sequence (LTDS) if $\mathbb{P}[X_i \le x_i | X_1 \le x_1, \dots, X_{i-1} \le x_{i-1}]$ is decreasing in $x_1, \dots, x_{i-1}$ for all x_i, $i \in \{2, \dots, d\}$;
3. multivariate left tail decreasing (MLTD) if $(X_{i_1}, \dots, X_{i_d})$ is LTDS for all permutation $(i_1, \dots, i_d)$ of $(1, \dots, d)$.

Proposition 2. *Suppose a multivariate copula* $C(u_1, \dots, u_d)$ *has a lower tail order* $\kappa_L(C)$, *then* $\kappa_L(C) \ge 1$. *Moreover,*

1. *if C is (ultimately) positive lower orthant dependent (PLOD), then* $\kappa_L(C) \le d$;
2. *for any* $S_1 \subset S_2 \subseteq I_d$ *with* $|S_1| \ge 2$, $\kappa_L(C_{S_2}) - \kappa_L(C_{S_1}) \ge 0$. *In particular, if* $\kappa_L(C) = 1$, *then for any* $S \subset I_d$ *with* $|S| \ge 2$, $\kappa_L(C_S) = 1$; *if C is multivariate left tail decreasing (MLTD), then* $\kappa_L(C_{S_2}) - \kappa_L(C_{S_1}) \le |S_2| - |S_1|$.

Analogous results hold with κ_L *replaced by* κ_U, *and conditions of positive upper orthant dependence and multivariate right tail increasing.*

Remark 2. The above result says that when some regularity condition holds, marginality will keep the order of tail orders in the sense that marginals have smaller tail orders. However, marginality does not inherit the inequality between tail orders of lower and upper tails. For example, take the trivariate Archimedean copula with the ψ function in Example 3 in Section 3.1. Then $3^\alpha > 1 + \alpha$ for $0 < \alpha < 1$ so that $\kappa_L(C) > \kappa_U(C)$ (see Table 1). But $2^\alpha < 1 + \alpha$ for $0 < \alpha < 1$ so that for the bivariate marginals, $\kappa_L(C_I) < \kappa_U(C_I)$ with $|I| = 2$.

Sometimes partial derivatives and the density have a simpler form than the copula cdf. We hope to know what tail properties will be inherited if we take partial derivatives of the copula. For example, for the lower tail, if

$$C(uw_1, \dots, uw_d) \sim u^\kappa \ell(u) b(w_1, \dots, w_d), \quad u \to 0^+,$$

then we want to differentiate both sides of the above with respect to the w_j's to get:

$$u \frac{\partial C(uw_1, \dots, uw_d)}{\partial w_j} \sim u^\kappa \ell(u) \frac{\partial b(w_1, \dots, w_d)}{\partial w_j}, \quad u \to 0^+,$$

and higher order derivatives up to:

$$u^d \frac{\partial^d C(uw_1, \dots, uw_d)}{\partial w_1 \cdots \partial w_d} \sim u^\kappa \ell(u) \frac{\partial^d b(w_1, \dots, w_d)}{\partial w_1 \cdots \partial w_d}, \quad u \to 0^+.$$

A sufficient condition is ultimate monotonicity of partial derivatives of the copula (eg: $\partial C/\partial u_j$ is ultimately monotone in u_j at 0^+, and similar conditions are sufficient for higher orders). A proof is similar to that in Theorem 1.7.2 (Monotone density theorem) in [5].

As an example of using the density to get the tail order, consider a multivariate Gaussian copula with positive definite correlation matrix Σ which satisfies $C_\Phi(u\mathbf{1}_d) \sim u^\kappa \ell(u) = u^\kappa(-\log u)^\zeta$, $u \to 0^+$. Then (as can be shown directly with the monotone density theorem), this would be equivalent to $c_\Phi(u\mathbf{1}_d) \sim hu^{\kappa-d}(-\log u)^\zeta$, $u \to 0^+$, where h is a constant. Thus, with ϕ_d for the multivariate Gaussian density,

$$1 = \lim_{u\to 0^+} \frac{c_\Phi(u\mathbf{1}_d)}{hu^{\kappa-d}(-\log u)^\zeta} = \lim_{u\to 0^+} \frac{\phi_d\left(\Phi^{-1}(u)\mathbf{1}_d;\, \Sigma\right)}{\phi^d(\Phi^{-1}(u))u^{\kappa-d}(-\log u)^\zeta h}$$
$$= \lim_{z\to -\infty} \frac{\phi_d\,(z\mathbf{1}_d;\, \Sigma)}{\phi^d(z)[\Phi(z)]^{\kappa-d}[-\log(\Phi(z))]^\zeta h} = \lim_{z\to -\infty} \frac{\phi_d\,(z\mathbf{1}_d;\, \Sigma)}{\phi^\kappa(z)|z|^{d-\kappa}[-\log(\phi(z)/|z|)]^\zeta h}. \quad (6)$$

Since the exponent terms dominate the numerator and denominator of (6), to cancel the exponent terms, a necessary condition is that $\kappa = \mathbf{1}_d\Sigma^{-1}\mathbf{1}_d^{\mathrm{T}}$, which turns out to be the tail order of the copula C_Φ. Also, to cancel the term of $|z|$ in (6), we need that $d - \kappa + 2\zeta = 0$, so $\zeta = (\kappa - d)/2$.

3. Intermediate tail dependence: Archimedean copulas

Archimedean copulas are reflection asymmetric except for the bivariate Frank copula, and have a variety of tail behavior. In this section, we will study the upper/lower tail orders and tail order functions for Archimedean copulas. A new family of one-parameter Archimedean copulas, that interpolates independence and comonotonicity, will be given that possesses intermediate upper and lower tail dependence, and has patterns of tail orders different from existing parametric families.

In the literature, a d-dimensional Archimedean copula C is often (e.g., [12,30]) defined as a copula of the form $C(u_1,\ldots,u_d) = \phi^{-1}(\phi(u_1)+\cdots+\phi(u_d))$, $(u_1,\ldots,u_d) \in [0,1]^d$. McNeil and Neslehova [29] showed that d-monotone is a sufficient and necessary condition on the Archimedean generator ϕ^{-1} so that the above form is a copula.

To get a better understanding of tail dependence (intermediate or very strong), we use the mixture of power or LT representation in [26,16]; for mixing distribution functions [resp. survival functions], the power is called *resilience* [resp. *frailty*] in [27]. Let

$$C_\psi(u_1,\ldots,u_d) = \psi(\psi^{-1}(u_1)+\cdots+\psi^{-1}(u_d)), \quad (u_1,\ldots,u_d) \in [0,1]^d, \quad (7)$$

where ψ is the LT of a positive random variable. Note that as $d \geq 3$ increases, Archimedean copulas extend less into the region of negative dependence (Sections 4.4 and 5.4 of [16]) and hence the restriction to LTs does not lose much generality. Since a LT is completely monotone, it can be used to construct copulas of any dimension.

Before getting to the main results, we provide some intuition on conditions on ψ for intermediate upper and lower tail dependence for C_ψ.

Let $G_1,\ldots,G_d$ be univariate cdfs. For $\eta > 0$, $G_1^\eta,\ldots,G_d^\eta$ are cdfs, and η is called a *resilience parameter*. As the parameter $\eta \to 0$, then random variables with distributions $G_1^\eta,\ldots,G_d^\eta$ tend toward the lower end-point of support of $G_1,\ldots,G_d$, and as $\eta \to \infty$, random variables with distributions $G_1^\eta,\ldots,G_d^\eta$ tend toward the upper end-point of support of $G_1,\ldots,G_d$. There is also a parallel for survival functions and frailty, where the conclusions are reversed when the frailty parameter goes to 0 or ∞.

In this way, an Archimedean copula C_ψ has a mixture representation with LT ψ. That is, $C_\psi(u_1,\ldots,u_d) = \int_0^\infty \prod_{j=1}^d G_j^\eta(u_j)\, dF_H(\eta)$, where F_H is the cdf of the resilience random variable H, $G_j(u) = \exp\{-\psi^{-1}(u)\}$ $(0 \leq u \leq 1)$ for all j, and $\psi(s) = \psi_H(s) = \int_0^\infty \mathrm{e}^{-s\eta} dF_H(\eta)$. The mixture means that: there are random variables $X_1,\ldots,X_d$ such that given $H = \eta$, they are conditionally independent with respective cdfs $G_1^\eta,\ldots,G_d^\eta$. If the random variable H has heavy tail at ∞, then there is a "chance" that $H = \eta$ is large and hence conditionally, $X_1,\ldots,X_d$ are all close to their upper endpoints of support (i.e., dependence in the upper tail). Hence conditions on the heaviness of the upper tail of the distribution of H lead to intermediate upper tail dependence. For the opposite tail, if the random variable H has concentration of density near 0, then there is a "chance" that $H = \eta$ is near zero and hence conditionally, $X_1,\ldots,X_d$ are all close to their lower endpoints of support (i.e., dependence in the lower tail). Hence conditions on the density of the lower tail of the distribution of H lead to intermediate lower tail dependence.

3.1. Laplace transform and univariate tail heaviness

In this subsection, we relate the asymptotic behavior of a LT to the maximal moment of the positive random variable with the given LT.

Definition 5. For a positive random variable Y with LT ψ, the *maximal non-negative moment* is

$$M_Y = M_\psi = \sup\{m \geq 0 : \mathbb{E}(Y^m) < \infty\}. \quad (8)$$

M_Y is 0 if no moments exist and M_Y is ∞ if all moments exist. A smaller value of M_Y means that Y has a heavier tail at ∞.

The next lemma shows that M_ψ is related to the behavior of ψ at 0 when $0 < M_\psi < 1$. The result for a general non-integer M_ψ such that $k < M_\psi < k+1$ will be derived subsequently.

Lemma 1. *Suppose $\psi(s)$ is the LT of a positive random variable Y, with $0 < M_Y < 1$. If $1-\psi(s)$ is regularly varying at 0^+, then $1-\psi(s) \in \mathcal{R}_{M_Y}(0^+)$.*

Remark 3. Even if $\mathbb{E}(Y) = \infty$, we may also have $M_Y = 1$. However, Lemma 1 does not hold in general for this case.

Remark 4. If we write $1-\psi(s) = s^M \ell(s)$ and $\ell(s) \to h_1$ with $0 < h_1 < \infty$ as $s \to 0^+$, then clearly $\psi(s) = 1 - h_1 s^M + o(s^M)$, $s \to 0^+$.

Proposition 3. *Suppose $\psi(s)$ is the LT of a positive random variable Y, with $k < M_Y < k+1$ where $k \in \{0\} \cup \mathbb{N}_+$. If $\left|\psi^{(k)}(0) - \psi^{(k)}(s)\right|$ is regularly varying at 0^+, then $\left|\psi^{(k)}(0) - \psi^{(k)}(s)\right| \in \mathcal{R}_{M_Y - k}(0^+)$. In particular, if the slowly varying component is $\ell(s)$ and $\lim_{s\to 0^+} \ell(s) = h'_{k+1}$ with $0 < h'_{k+1} < \infty$, then*

$$\psi(s) = 1 - h_1 s + h_2 s^2 - \cdots + (-1)^k h_k s^k + (-1)^{k+1} h_{k+1} s^{M_Y} + o(s^{M_Y}), \quad s \to 0^+, \tag{9}$$

where $0 < h_i < \infty$ for $i = 1, \ldots, k+1$.

The above results can be summarized as follows. If $M_\psi = \infty$, the LT $\psi(s)$ has an infinite Taylor expansion about $s = 0$. If M_ψ is finite and non-integer-valued, then with some regularity conditions, $\psi(s)$ has a Taylor expansion about $s = 0$ up to order $[M_\psi]$, and the next term after this has order M_ψ.

3.2. Upper tail

Based on the results in Section 3.1, we derive upper tail orders and corresponding tail order functions of multivariate Archimedean copulas; the results extend those of [7].

Proposition 4. *Let ψ be the LT of a positive random variable and assume that ψ satisfies the condition of Proposition 3. Assume that $k < M_\psi < k+1$ with some $k \in \{1, \ldots, d-1\}$, then the Archimedean copula C_ψ in (7) has upper intermediate tail dependence. The corresponding tail order is $\kappa_U = M_\psi$. If $\psi^{(i)}(0)$ is finite for all $i \in I_d$, then the upper tail order $\kappa_U = d$. If $\psi'(0)$ is infinite and $0 < M_\psi < 1$, then the upper tail order is $\kappa_U = 1$, and particularly for the bivariate case, $\lambda_U = 2 - 2^{M_\psi}$.*

Remark 5. If we know the value of a in (21) in the proof of Proposition 4, then the tail order parameter is

$$\lim_{u\to 1^-} \overline{C}_\psi(u,u)/(1-u)^{M_\psi} = 2a[-\psi'(0)]^{-1-M}(2^M - 1)/(1+M) = a[-\psi'(0)]^{-M_\psi}(2^{M_\psi} - 2)/M_\psi.$$

Example 3. Consider the LT of Example 4.2 in [19] with parameter $0 < \alpha < 1$ (see Joe–Ma in Table 1). We refer to this as the normalized integral of the positive stable LT. Note that $m = \psi'(0) = -1/\Gamma(1+\alpha^{-1})$ is finite, $\psi''(0) = \infty$ and $\psi(s) \sim 1 - s/\Gamma(1+\alpha^{-1})$ as $s \to 0^+$. We can write $\psi(s) = 1 + \psi'(0)s + o(s)$, $s \to 0^+$. Let $g(s) = \psi'(s) - \psi'(0) = (1 - \exp\{-s^\alpha\})/\Gamma(1+\alpha^{-1}) \sim s^\alpha/\Gamma(1+\alpha^{-1})$, $s \to 0^+$. Then clearly, $g(s) \in \mathcal{R}_\alpha(0^+)$ and can be written as $g(s) = s^\alpha \ell(s)$ with $\ell(s) \to 1/\Gamma(1+\alpha^{-1})$ as $s \to 0^+$. So $g(s) = as^\alpha + o(s^\alpha)$, $s \to 0^+$, with $a = 1/\Gamma(1+\alpha^{-1}) > 0$. By Proposition 4, the copula C_ψ has intermediate upper tail dependence when $0 < \alpha < 1$. Also, $\kappa_U = 1 + \alpha$ and

$$\lim_{u\to 0^+} \frac{\overline{C}_\psi(1-u, 1-u)}{u^{1+\alpha}} = \frac{2[\Gamma(1+\alpha^{-1})]^\alpha (2^\alpha - 1)}{1+\alpha}.$$

It can be shown numerically that the d-variate Archimedean copula with this one-parameter LT family is decreasing in concordance as α increases. As $\alpha \to 1^-$, numerically, the limit is close to the independence copula; as $\alpha \to 0^+$, the limit is close to the comonotonic copula.

In the next proposition, we state a result for the upper tail order function of Archimedean copulas.

Proposition 5. *Let C_ψ be a multivariate Archimedean copula with $\kappa_U = M_\psi$ being a non-integer in the interval $(1, d)$, and suppose ψ satisfies the condition of Proposition 3. With notation $M = M_\psi - [M_\psi]$ and $k = [M_\psi]$, the upper tail order parameter is*

$$\lambda_U(C_\psi) = \frac{Mh}{[-\psi'(0)]^{M_\psi} \prod_{j=0}^{k} (M_\psi - j)} \sum_{\emptyset \neq I \subset I_d} (-1)^{|I|+k+1} |I|^{M_\psi},$$

and the upper tail order function is

$$b^*(\mathbf{w}) = \frac{\sum_{\emptyset \neq I \subset I_d} (-1)^{|I|} \left(\sum_{i\in I} w_i\right)^{M_\psi}}{\sum_{\emptyset \neq I \subset I_d} (-1)^{|I|} |I|^{M_\psi}},$$

where $h = \lim_{s\to 0^+} \ell(s)$ with $|\psi^{(k)}(s) - \psi^{(k)}(0)| = s^M \ell(s)$ as $s \to 0^+$.

Remark 6. For a d-variate Archimedean copula, the pattern of the upper tail order function also depends on the upper tail order κ. For example, in $d = 3$, the homogeneous function b^* is positively proportional to

$$-w_1^\kappa - w_2^\kappa - w_3^\kappa + (w_1 + w_2)^\kappa + (w_1 + w_3)^\kappa + (w_2 + w_3)^\kappa - (w_1 + w_2 + w_3)^\kappa, \quad 1 < \kappa < 2;$$
$$w_1^\kappa + w_2^\kappa + w_3^\kappa - (w_1 + w_2)^\kappa - (w_1 + w_3)^\kappa - (w_2 + w_3)^\kappa + (w_1 + w_2 + w_3)^\kappa, \quad 2 < \kappa < 3.$$

The signs of all terms depend on whether $1 < \kappa < 2$ or $2 < \kappa < 3$. The pattern of alternating signs extends to $d > 3$. This pattern, together with Lemma 2, also shows why we do not have a general form of the tail order function when M_ψ is a positive integer.

3.3. Lower tail

Next, we consider the lower tail. For intermediate lower tail dependence of Archimedean copulas, a general result has been obtained in Theorem 3.3 of [7]. We will derive a more concrete and usable result that involves the slowly varying function ℓ, and give an interpretation in terms of the (resilience) random variable H which has LT ψ.

The condition below on the LT $\psi(s)$ as $s \to \infty$ covers almost all of the LT families in the Appendix of [16], as well as other LT families that can be obtained by integration or differentiation. Suppose

$$\psi(s) \sim T(s) = a_1 s^q \exp\{-a_2 s^r\} \quad \text{and} \quad \psi'(s) \sim T'(s), \quad s \to \infty, \text{ with } a_1 > 0, a_2 \geq 0, \tag{10}$$

where $r = 0$ implies $a_2 = 0$ and $q < 0$, and $r > 0$ implies $r \leq 1$ and q can be 0, negative or positive. Note that $r > 1$ is not possible because of the complete monotonicity property of a LT.

The condition can be interpreted as follows. As $\psi(s)$ decreases to 0 more slowly as $s \to \infty$, then the random variable H with LT ψ has a heavier "tail" at 0. Let $z = \lim_{\eta \to 0} f_H(\eta) \in [0, \infty)$, where f_H is the density of H and is assumed well behaved near 0. As z increases, then the "tail" at 0 is heavier. If $z = 0$, then the tail is lighter as the rate of decrease to 0 is faster. If $z = \infty$, then the tail is heavier as the rate of increase to ∞ is faster. In terms of the LT and the condition in (10), as r increases (with fixed q), the tail of H at 0 gets lighter, and as q increases (with fixed r), the tail of H at 0 gets heavier.

The next proposition shows that lower tail dependence behavior is influenced by r.

Proposition 6. *Suppose a LT ψ satisfies the condition in (10) with $0 \leq r \leq 1$. If $r = 0$, then C_ψ has lower tail dependence or lower tail order is 1. If $r = 1$, then $\kappa_L(C_\psi) = d$. If $0 < r < 1$, then C_ψ has intermediate lower tail dependence with $1 < \kappa_L(C_\psi) = d^r < d$, $\ell(u) = d^q a_1^{1-\kappa} a_2^{-\zeta} (-\log u)^\zeta$ with $\zeta = (q/r)(1 - d^r)$, and the tail order function is $b(\mathbf{w}) = \prod_{i=1}^d w_i^{d^{r-1}}$.*

Remark 7. Condition (10) does not cover all possibilities. It is possible that as $s \to \infty$, $\psi(s)$ goes to 0 slower than anything of form (10). Examples are given by LT families LTF and LTG in [16], leading to Archimedean families such that $\lim_{u \to 0^+} C_\psi(u\mathbf{1}_d)/u = 1$ (for the bivariate case, see families BB2 and BB3 in [17,16]). Note that, for LTF, $\psi(s) = [1 + \delta^{-1} \log(1 + s)]^{-1/\theta}$ with $\delta > 0$ and $\theta > 0$ and as $s \to \infty$, $\psi(s) \sim \delta^{1/\theta} (\log s)^{-1/\theta}$; for LTG, $\psi(s) = \exp\{-[\delta^{-1} \log(1 + s)]^{1/\theta}\}$ with $\delta > 0$, $\theta > 1$ and as $s \to \infty$, $\psi(s) \sim \exp\{-\delta^{-1/\theta} (\log s)^{1/\theta}\}$.

Remark 8. Consider the pair (ϕ, ψ) of LTs where (a) $\phi'(0)$ is finite and $\psi(s) = \phi'(s)/\phi'(0)$ or (b) $\int_0^\infty \psi(v)dv$ is finite and $\phi(s) = \int_s^\infty \psi(v)dv / \int_0^\infty \psi(v)dv$. For the upper tail, we get $M_\psi = M_\phi - 1$ so that LT ψ has heavier tail and $\kappa_U(C_\psi)$ is smaller (stronger intermediate tail dependence) if $\kappa_U(C_\phi) < d$. Proposition 6 implies that $\kappa_L(C_\psi) = \kappa_L(C_\phi)$. But the second level of tail dependence strength comes from the slowly varying function $\ell(u) = d^q a_1^{1-\kappa} a_2^{-\zeta} (-\log u)^\zeta$. Since $C_\psi(u\mathbf{1}_d) \sim u^\kappa \ell(u)$, $u \to 0^+$, a smaller κ means stronger intermediate lower tail dependence at the first level, and a faster $\ell(u) \to +\infty$ or a slower $\ell(u) \to 0^+$ means stronger intermediate lower tail dependence at the second level. For the LT tail, $a_1 s^q \exp(-a_2 s^r)$, a smaller r means slower decrease to 0 as $s \to +\infty$ and the resilience random variable has more probability near 0 and C_ψ has more dependence in the lower tail. This can be shown by a smaller tail order d^r. A larger q means slower decrease to 0 as $s \to +\infty$, which also implies more lower tail dependence. This is seen from a faster increase of $\ell(u) \to +\infty$ as $u \to 0^+$ when $q < 0$ and increases, or a slower decrease of $\ell(u) \to 0^+$ as $u \to 0^+$ when $q > 0$ and increases. Note that when u is small enough, $(-\log u)^\zeta$ dominates $\ell(u)$.

3.4. A new parametric Archimedean copula

By applying the LT of the inverse Gamma distribution, we present a new one-parameter Archimedean copula that exhibits intermediate upper and lower tail dependence, and have essentially a full range of positive dependence from independence to comonotonicity.

Example 4 (*Archimedean Family Based on Inverse Gamma LT*). Let $Y = X^{-1}$ have the inverse Gamma ($I\Gamma$) distribution, where $X \sim \text{Gamma}(\alpha, 1)$ for $\alpha > 0$. Then it is straightforward to derive that $M_Y = \alpha$. The LT of the inverse Gamma distribution:

$$\psi(s; \alpha) = \frac{2}{\Gamma(\alpha)} s^{\alpha/2} K_\alpha(2\sqrt{s}), \quad s \geq 0, \ \alpha > 0, \tag{11}$$

Table 1
Tail order of some Archimedean copulas that interpolate independence and comonotonicity.

Copula/LT family	κ_L	κ_U
Frank; log-series LT $-\theta^{-1}\log[1-(1-e^{-\theta})e^{-s}]$ $(\theta>0)$	d	d
MTCJ[a]; gamma LT $(1+s)^{-1/\theta}$ $(\theta>0)$	1	d
Joe; Sibuya LT $1-(1-e^{-s})^{1/\theta}$ $(\theta>1)$	d	1
Gumbel; positive stable LT $\exp\{-s^{1/\theta}\}$ $(\theta>1)$	$d^{1/\theta}$	1
Joe–Hu; BB1 extension; $(1+s^{1/\delta})^{-1/\theta}$ $(\theta>0,\delta>1)$	1	1
Joe–Hu; BB7 extension; $1-[1-(1+s)^{-1/\delta}]^{1/\theta}$ $(\theta>1,\delta>0)$	1	1
Crowder; BB9 extension; $\exp\{-(\alpha^\theta+s)^{1/\theta}+\alpha\}$ $(\theta>1)$	$d^{1/\theta}$	d
Joe–Ma; $\int_s^\infty e^{-v^\alpha}dv/\Gamma(1+\alpha^{-1})$ $(0<\alpha<1)$	d^α	$1+\alpha$
new; LT of inverse gamma $2\Gamma^{-1}(\alpha)s^{\alpha/2}K_\alpha(2\sqrt{s})$ $(\alpha>0)$	$d^{1/2}$	$(d\wedge\alpha)\vee 1$

[a] Mardia–Takahasi–Cook–Johnson, see [11].

where K_α is the modified Bessel function of the second kind. (Please see the Appendix for the derivation of (11).) It can be shown numerically that the d-variate Archimedean copula with this one-parameter LT family is decreasing in concordance as α increases, with limits of the independence copula as $\alpha\to\infty$ and the comonotonic copula as $\alpha\to 0$.

Proposition 7. *Let C_ψ be an Archimedean copula constructed by (11). If $\alpha\in(0,+\infty)$ is not an integer, then the upper tail order is* $\max\{1,\min\{\alpha,d\}\}$. *The lower tail order is $\sqrt{d}$.*

Remark 9. For the bivariate case, $\kappa_U=\max\{1,\min\{\alpha,2\}\}$ and $\kappa_L=\sqrt{2}$. Hence there is reflection asymmetry with skewness to the upper tail for $0<\alpha<\sqrt{2}$ and skewness to the lower tail for $\alpha>\sqrt{2}$.

To conclude this subsection, we list in Table 1 the tail orders for some Archimedean copulas that interpolate independence and comonotonicity. A variety of tail behavior obtains from known parametric Archimedean families and the new Archimedean family. Note that the bivariate Frank copula is reflection symmetric. But for d-dimensional Frank copula with $d\geq 3$, it can be shown numerically that $\overline{C}_\psi(\frac{1}{2}\mathbf{1}_d)>C_\psi(\frac{1}{2}\mathbf{1}_d)$ for parameters $\theta>0$, although the lower tail order and upper tail order are the same. Some of the results in this table can be found in [7,15]. For all of the examples in Table 1, the upper and lower tail orders decrease or remain constant as the dependence parameter(s) leads to increased dependence/concordance.

4. Intermediate tail dependence: mixture of max-id copulas

As an extension of Archimedean copulas, we study in this section the tail orders for copulas that are constructed with mixtures of max-id copulas. Some results studied in [16] are extended to intermediate tail dependence. Let F be a d-variate cdf. If F^t is also a cdf function for all $t>0$, then F is max-id [17]. The class of copulas based on mixture of max-id distributions has led to interesting classes of bivariate two-parameter copula families with both upper and lower tail dependence (e.g., labeled as BB1, BB4, BB7 in [16]). As well, other forms of intermediate tail dependence behavior are possible. These types of copulas will give us more flexibility in choices of bivariate linking copulas in vines [1,18].

Here we generalize Theorems 4.13 and 4.16 in [16] to multivariate versions and intermediate tail dependence. In the earlier research on copulas, the analyses determined when tail dependence (tail order $\kappa=1$) can occur for different copula families; in that setting, the tail order occurred within the sufficient condition in Theorem 4.16 of [16]. Let K be a multivariate max-id copula and ψ be a LT of a positive random variable, and consider the copulas that are of the following form

$$C(u_1,\ldots,u_d)=\psi\left(-\log K\left(e^{-\psi^{-1}(u_1)},\ldots,e^{-\psi^{-1}(u_d)}\right)\right). \tag{12}$$

Proposition 8. *Suppose that a copula C be constructed by (12).*

1. *If ψ satisfies the condition of Proposition 3 with some $k\in\{1,\ldots,d-1\}$ and $\kappa_U(K_I)>1$ for any marginal copula K_I, then C has upper intermediate tail dependence and $\kappa_U(C)=\kappa_U(C_\psi)$.*
2. *If $1-\psi(s)\in\mathcal{R}_\beta(0^+)$, $\kappa_U(K)=1$ with marginal copula $K_I(u\mathbf{1}_{|I|})\sim u\ell_I(u)$, $u\to 0^+$ such that $\lim_{u\to 0^+}\ell_I(u)=h_I\in(0,1]$, $0<h_I^*=\sum_{\emptyset\neq J\subset I}(-1)^{|J|-1}h_J\leq 1$ and $0<\sum_{\emptyset\neq I\subset I_d}(-1)^{|I|-1}(h_I^*)^\beta\leq 1$, then $\kappa_U(C)=1$ with $\lambda_U(C)=\sum_{\emptyset\neq I\subset I_d}(-1)^{|I|-1}(h_I^*)^\beta$.*

Proposition 9. *Suppose that a copula C be constructed by (12) with $1\leq\alpha=\kappa_L(K)\leq d$. If $-\psi(s)/\psi'(s)\in\mathcal{R}_\beta$ with $0<\beta\leq 1$, and $1<\alpha^{1-\beta}<d$, then the copula C has lower intermediate tail dependence $\kappa_L(C)=\alpha^{1-\beta}$, and $\kappa_L(C)=\xi(\alpha,\beta)\cdot\kappa_L(C_\psi)$ with $\xi(\alpha,\beta)=(\alpha/d)^{1-\beta}\in(0,1]$. Also, $\kappa_L(K)=1$ implies that $\kappa_L(C)=1$.*

Remark 10. Note that $\kappa_L(C)$ is less than or equal to both $\kappa_L(K)$ and $\kappa_L(C_\psi)$. K can be the independence copula or have intermediate lower tail dependence. The lower tail order of the copula C is increasing in $\kappa_L(K)$. One consequence of Propositions 8 and 9 is that if $\kappa_U(K)=d$ and $\kappa_L(K)=d$ then $\kappa_U(C)=\kappa_U(C_\psi)$ and $\kappa_L(C)=\kappa_L(C_\psi)$. Hence if K is chosen as the parametric Frank copula family with parameter $\theta\geq 0$, then $C(u_1,\ldots,u_d;\theta)$ as given in (12) will be increasing in concordance as θ increases. The parameter θ affects dependence only, while the LT ψ controls the upper and lower tail orders.

When we take K as the independence copula or the Frank copula with positive dependence, and the LT has tail of the form $\psi(s) \sim a_1 s^q \exp\{-a_2 s^r\}$, $s \to \infty$, $0 \le r < 1$, where a_1, a_2 are some positive constants, then we can construct a new family of Archimedean copulas that satisfies the condition of Proposition 9.

In dimensions $d \ge 3$, Archimedean and mixture of max-id copula families cannot achieve the range of dependence available from vine copulas [3,1,18]. But for $d = 2$, the mixture of max-id approach can lead to more candidates, with a variety of upper and lower tail behavior, to be used as bivariate linking copulas in vines. For instance, from Table 1, the preceding subsections and propositions, the Joe–Ma ψ function, which is the normalized integral of the positive stable LT, combined with the bivariate Gaussian copula with $\rho \ge 0$ can lead to a two-parameter family with more flexible upper and lower tail orders. Note that, from Theorem 2.6 of [16], the bivariate Gaussian density is TP_2 if $\rho \ge 0$, and hence max-id.

5. Discussion

We have shown how the concept of tail order is useful to quantify the strength of upper and lower tail dependence, as well as the direction of reflection asymmetry. One- and two-parameter families that are Archimedean copulas and mixture of max-id copulas together can cover a wide range of tail orders. The interpretation through the latent resilience variable shows why Archimedean copulas can obtain a full range of tail orders by varying the density of the resilience at 0 and ∞. In order to get our results for Archimedean copulas, we needed Proposition 3 which, on its own, contributes knowledge about LTs.

Archimedean copulas only have exchangeable dependence but their bivariate versions can be used within vines. Vine copulas [3,1,18] in dimension d, which include multivariate Gaussian and t copulas as special cases, are built from $d(d-1)/2$ bivariate linking copulas, of which $d-1$ are bivariate marginal copulas and the remainder are conditional bivariate copulas with the number of conditioning variables between 1 to $d-2$. By choosing bivariate linking copulas with flexible tail orders and reflecting symmetry/asymmetry, we can get vine copulas to cover a wide range of tail behavior, as well as dependence structures. Tail orders of vine copulas in terms of the tail orders of the bivariate linking copulas will be studied in future research. For vine copulas, we are also interested in conditions that retain consistent relation of upper and lower tail orders for all margins.

Acknowledgments

The first author was supported by the Alexander Graham Bell Canada Graduate Scholarship. The second author was supported by an NSERC Discovery Grant. The authors appreciate the comments from three referees and an associate editor. Their suggestions greatly improved the presentation of this paper.

Appendix. Proofs

Derivation of LT of the inverse Gamma distribution: With $Y = X^{-1}$ and $X \sim \text{Gamma}(\alpha, 1)$, the LT is derived as

$$\psi(s) = \psi(s;\alpha) = \mathbb{E}(e^{-sY}) = \mathbb{E}(e^{-s/X}) = [\Gamma(\alpha)]^{-1}\int_0^\infty e^{-s/x}x^{\alpha-1}e^{-x}dx.$$

From the GIG(ν, χ, φ) density [28],

$$\int_0^\infty w^{\nu-1}\exp\left\{-\frac{1}{2}(\chi w^{-1} + \varphi w)\right\} dw = 2(\chi/\varphi)^{\nu/2} K_\nu(\sqrt{\chi\varphi}).$$

Note that $K_\nu = K_{-\nu}$. Hence with $\chi = 2s, \varphi = 2, \nu = \alpha$

$$\psi(s;\alpha) = 2\Gamma^{-1}(\alpha)(2s/2)^{\alpha/2}K_\alpha(\sqrt{2s\cdot 2}) = 2\Gamma^{-1}(\alpha)s^{\alpha/2}K_\alpha(2\sqrt{s}). \quad \Box$$

Proof of Proposition 2. Assuming $C(u\mathbf{1}_d) \sim u^{\kappa_L(C)}\ell(u)$, $u \to 0^+$, with $\ell(u) \in \mathcal{R}_0(0^+)$, for any copula C and $0 \le u \le 1$, $C(u\mathbf{1}_d) \le u$. Therefore, $\kappa_L(C) \ge 1$. To prove the first statement, by the condition of PLOD, we have $C(u\mathbf{1}_d) \ge u^d$ for any $0 \le u \le 1$ and thus, $\kappa_L(C) \le d$.

To prove the second statement, choosing $S_1 \subset S_2$ with $|S_2| - |S_1| = j \in \mathbb{N}_+$. Let us consider the case where $j = 1$ first, for some $l \in \{1, \dots, |S_2|\}$ and any $0 \le u \le 1$,

$$\begin{aligned} C_{S_2}(u\mathbf{1}_{|S_2|}) &= \mathbb{P}[U_l \le u | U_1 \le u, \dots, U_{l-1} \le u, U_{l+1} \le u, \dots, U_{|S_2|} \le u] \\ &\quad \times \mathbb{P}[U_1 \le u, \dots, U_{l-1} \le u, U_{l+1} \le u, \dots, U_{|S_2|} \le u] \\ &\ge \mathbb{P}[U_l \le u] \times \mathbb{P}[U_1 \le u, \dots, U_{l-1} \le u, U_{l+1} \le u, \dots, U_{|S_2|} \le u] \\ &= uC_{S_1}(u\mathbf{1}_{|S_1|}). \end{aligned}$$

The inequality is due to the MLTD of C. Clearly, $\kappa_L(C_{S_2}) - \kappa_L(C_{S_1}) \le 1$. Since $\mathbb{P}[U_l \le u | U_1 \le u, \ldots, U_{l-1} \le u, U_{l+1} \le u, \ldots, U_{|S_2|} \le u] \le 1$, $C_{S_2}(u\mathbf{1}_{|S_2|}) \le C_{S_1}(u\mathbf{1}_{|S_1|})$ and thus, $\kappa_L(C_{S_2}) - \kappa_L(C_{S_1}) \ge 0$. An iterated argument will prove the case for a general j: $0 \le \kappa_L(C_{S_2}) - \kappa_L(C_{S_1}) \le |S_2| - |S_1|$. If $\kappa_L(C) = 1$, then for any $S \subset I_d$ with $|S| \ge 2$, we have $1 \le \kappa_L(C_S) \le \kappa_L(C) = 1$, which completes the proof. For $\kappa_L(C) = 1$, note that the MLTD condition is not needed. □

Proof of Lemma 1. Let Z be an exponential random variable, independent of Y, with mean 1. Choose any fixed m with $0 < m < 1$. Then $\mathbb{E}(Z^{-m}) = \Gamma(1-m)$, and if we define $W_m = (Y/Z)^m$, then for any $w > 0$,

$$\mathbb{P}[W_m \ge w] = \mathbb{P}[Z \le Y w^{-1/m}] = \int_0^\infty \left(1 - \exp\{-y w^{-1/m}\}\right) F_Y(dy) = 1 - \psi(w^{-1/m}),$$

where F_Y is the cdf of Y. Therefore, $\mathbb{E}(Y^m) < \infty$ implies $\mathbb{E}(W_m) < \infty$ and $\lim_{w\to\infty} w[1 - \psi(w^{-1/m})] = 0$, i.e.,

$$\lim_{s\to 0^+} [1 - \psi(s)]/s^m = 0. \tag{13}$$

If $1 - \psi(s)$ is regularly varying at 0^+, then we can write $1 - \psi(s) = s^\alpha \ell(s)$ with $\alpha \ne 0$, where $\ell(s) \in \mathcal{R}_0(0^+)$. Then, (13) implies that $\lim_{s\to 0^+} s^{\alpha - m}\ell(s) = 0$. Let $\epsilon > 0$ be arbitrarily small. If $m = M_Y - \epsilon$, then we have $\mathbb{E}(Y^{M_Y - \epsilon}) < \infty$ and thus $\lim_{s\to 0^+} s^{\alpha - M_Y + \epsilon}\ell(s) = 0$. Therefore, $\alpha \ge M_Y - \epsilon$.

Also by a result on page 49 of [8], (13) implies that for any $0 < \delta < 1$, $\mathbb{E}\left(Y^{m(1-\delta)}\right) < \infty$. If we assume that there exists an $\epsilon > 0$ with $m = M_Y + \epsilon$ such that, $\lim_{s\to 0^+} s^{\alpha - M_Y - \epsilon}\ell(s) = 0$, then for any small $\delta > 0$, $\mathbb{E}\left(Y^{(M_Y+\epsilon)(1-\delta)}\right) < \infty$. Then we may choose some $\delta_\epsilon < \epsilon/(\epsilon + M_Y)$, and get $\mathbb{E}\left(Y^{(M_Y+\epsilon)(1-\delta_\epsilon)}\right) < \infty$ with $(M_Y + \epsilon)(1 - \delta_\epsilon) > M_Y$, which gives rise to a contradiction. Thus, for any $\epsilon > 0$, we must have $\lim_{s\to 0^+} s^{\alpha - M_Y - \epsilon}\ell(s) \ne 0$, and hence, $\alpha - M_Y - \epsilon \le 0$. So,

$$M_Y - \epsilon \le \alpha \le M_Y + \epsilon,$$

which completes the proof. □

Proof of Proposition 3. This proof extends that in Lemma 1, which corresponds to the case where $k = 0$. For a positive integer j, let $Z_j \sim \text{Gamma}(j+1, 1)$ so that $\mathbb{E}(Z_j^{-m}) = \Gamma(j+1-m)/\Gamma(j+1)$ if $0 < m < j+1$. Let $W_{m,j} = (Y/Z_j)^m$ where Y is independent of Z_j. Then for $0 < m < j+1$, $\mathbb{E}(W_{m,j}) < \infty$ if and only if $\mathbb{E}(Y^m) < \infty$. Next, similar to the proof of Lemma 1, if Y has LT ψ and moments up to order k, for $j \in \{0, 1, \ldots, k\}$ and $0 < m < j+1$,

$$\begin{aligned}
\Pr\left[W_{m,j} = (Y/Z_j)^m \ge w\right] &= \Pr(Z_j \le Y w^{-1/m}) = \int_0^\infty F_{Z_j}(y w^{-1/m})\, dF_Y(y) \\
&= \int_0^\infty \left[1 - \sum_{i=0}^{j} \frac{y^i w^{-i/m}}{i!} \exp\{-y w^{-1/m}\}\right] dF_Y(y) = 1 - \sum_{i=0}^{j} \frac{w^{-i/m}}{i!} (-1)^i \psi^{(i)}(w^{-1/m}).
\end{aligned}$$

Suppose $0 < m < \min\{j+1, M_Y\}$. Then $\mathbb{E}(Y^m) < \infty$ implies that

$$w\left[1 - \sum_{i=0}^{j} \frac{w^{-i/m}}{i!} (-1)^i \psi^{(i)}(w^{-1/m})\right] \to 0, \quad w \to \infty,$$

i.e.,

$$s^{-m}\left[1 - \sum_{i=0}^{j} \frac{s^i}{i!} (-1)^i \psi^{(i)}(s)\right] \to 0, \quad s \to 0^+. \tag{14}$$

Assuming ψ has derivatives at zero up to kth order, then for positive integer $j \le k$, the main term in (14) is

$$\begin{aligned}
1 - \sum_{i=0}^{j} \frac{s^i}{i!} (-1)^i \psi^{(i)}(s) &= 1 - \sum_{i=0}^{j-1} \frac{s^i}{i!} (-1)^i \left[\sum_{l=0}^{j-i} s^l \psi^{(i+l)}(0)/l! + o(s^{j-i})\right] - \frac{s^j}{j!} (-1)^j \psi^{(j)}(s) \\
&= 1 - \sum_{i=0}^{j-1} \frac{s^i}{i!} (-1)^i \sum_{l=i}^{j} s^{l-i} \psi^{(l)}(0)/(l-i)! - \frac{s^j}{j!} (-1)^j \psi^{(j)}(s) + o(s^j) \\
&= 1 - \sum_{l=0}^{j} \psi^{(l)}(0) \frac{s^l}{l!} \left\{\sum_{i=0}^{l\wedge(j-1)} (-1)^i \frac{l!}{i!(l-i)!}\right\} - \frac{s^j}{j!} (-1)^j \psi^{(j)}(s) + o(s^j) \\
&= 1 - \psi(0) - \psi^{(j)}(0) \frac{s^j}{j!} \left[-(-1)^j\right] - \frac{s^j}{j!} (-1)^j \psi^{(j)}(s) + o(s^j) \\
&= (-1)^{j-1} \frac{s^j}{j!} \left[\psi^{(j)}(s) - \psi^{(j)}(0)\right] + o(s^j).
\end{aligned} \tag{15}$$

Hence (14) implies that

$$s^{j-m}\left[\psi^{(j)}(s)-\psi^{(j)}(0)\right]\to 0,\quad s\to 0^+,$$

if j is a non-negative integer less than M_Y and $m<M_Y$. In particular, if k is a non-negative integer such that $k<m<M_Y<k+1$, then

$$s^{k-m}\left[\psi^{(k)}(s)-\psi^{(k)}(0)\right]\to 0,\quad s\to 0^+.$$

If $\left|\psi^{(k)}(0)-\psi^{(k)}(s)\right|$ is regularly varying at 0^+, we write $\left|\psi^{(k)}(0)-\psi^{(k)}(s)\right|=s^{\alpha}\ell(s)$. For any $\epsilon>0$, a similar argument in the proof of Lemma 1 will prove that $\alpha\geq M_Y-k-\epsilon$. Now we prove the other direction. We assume that there exists an $\epsilon>0$ with $m=M_Y+\epsilon$ such that,

$$\lim_{s\to 0^+} s^{\alpha+k-M_Y-\epsilon}\ell(s)=0, \tag{16}$$

that is, $s^{k-M_Y-\epsilon}\left[\psi^{(k)}(s)-\psi^{(k)}(0)\right]\to 0$ as $s\to 0$. Since ψ is completely monotonic, $\psi^{(k)}(0)-\psi^{(k)}(s)$ is either negative or positive as $s\to 0^+$. That is, $(-1)^k[\psi^{(k)}(0)-\psi^{(k)}(s)]>0$. The following argument is for an even k, and similar when k is odd. Then by the Karamata's theorem (refer to [36]), regular variation of $\left|\psi^{(k)}(0)-\psi^{(k)}(s)\right|$ implies that

$$-\psi^{(k-1)}(x)+\psi^{(k-1)}(0)+x\psi^{(k)}(0)=\int_0^x\left[\psi^{(k)}(0)-\psi^{(k)}(s)\right]ds\sim(\alpha+1)^{-1}x^{\alpha+1}\ell(x),\quad x\to 0^+. \tag{17}$$

Since $\int_0^x[\psi^{(k)}(0)-\psi^{(k)}(s)]ds$ is again regularly varying, we can take the integration on both sides repeatedly and obtain for $j=0,1,\ldots,k$,

$$-\psi^{(k-j)}(x)+\sum_{i=0}^{j}\frac{x^{j-i}}{(j-i)!}\psi^{(k-i)}(0)\sim\left(\alpha\prod_{i=0}^{j}\frac{1}{\alpha+i}\right)x^{\alpha+j}\ell(x),\quad x\to 0^+. \tag{18}$$

Multiplying both sides of (18) by $\frac{x^{k-j}}{(k-j)!}(-1)^{k-j}$ leads to

$$\begin{aligned}LHS_j &=: -\frac{x^{k-j}}{(k-j)!}(-1)^{k-j}\psi^{(k-j)}(x)+\sum_{i=0}^{j}\frac{(-1)^{k-j}}{(k-j)!(j-i)!}x^{k-i}\psi^{(k-i)}(0)\\ &\sim \frac{(-1)^{k-j}}{(k-j)!}\left(\alpha\prod_{i=0}^{j}\frac{1}{\alpha+i}\right)x^{k+\alpha}\ell(x),\quad x\to 0^+.\end{aligned} \tag{19}$$

Then we add the left-hand side of (19) for $j=0,\ldots,k$, and after rearranging the summand, we have

$$\sum_{j=0}^{k}LHS_j=-\sum_{i=0}^{k}\frac{x^i}{i!}(-1)^i\psi^{(i)}(x)+1+\sum_{i=0}^{k-1}\sum_{j=i}^{k}\frac{(-1)^{k-j}}{(k-j)!(j-i)!}x^{k-i}\psi^{(k-i)}(0). \tag{20}$$

By the binomial theorem, for each given $i\in(0,\ldots,k-1)$, $\sum_{j=i}^{k}\frac{(-1)^{k-j}}{(k-j)!(j-i)!}\equiv 0$. Then, from (19) and (20) we can conclude that

$$\sum_{j=0}^{k}LHS_j=1-\sum_{i=0}^{k}\frac{x^i}{i!}(-1)^i\psi^{(i)}(x)=O\left(x^{k+\alpha}\ell(x)\right).$$

Therefore, multiplying both sides of the above by $s^{-M_Y-\epsilon}$ and using (16),

$$s^{-M_Y-\epsilon}\left[1-\sum_{i=0}^{k}\frac{s^i}{i!}(-1)^i\psi^{(i)}(s)\right]\to 0,\quad s\to 0^+.$$

Then for any small $\delta>0$, $\mathbb{E}\left(Y^{(M_Y+\epsilon)(1-\delta)}\right)<\infty$. Then we may choose some $\delta_\epsilon<\epsilon/(\epsilon+M_Y)$, and get $\mathbb{E}\left(Y^{(M_Y+\epsilon)(1-\delta_\epsilon)}\right)<\infty$ with $(M_Y+\epsilon)(1-\delta_\epsilon)>M_Y$, which gives rise to a contradiction to the fact that M_Y is the maximal moment. Thus, for any $\epsilon>0$, we must have $\lim_{s\to 0^+}s^{\alpha+k-M_Y-\epsilon}\ell(s)\neq 0$, and hence, $\alpha\leq M_Y-k+\epsilon$. Thus, $\alpha=M_Y-k$.

To prove the last statement of the proposition, since $1-\phi(s)=1-\psi^{(k)}(s)/\psi^{(k)}(0)=\left[\psi^{(k)}(0)-\psi^{(k)}(s)\right]/\psi^{(k)}(0)\in\mathscr{R}_{M_Y-k}(0^+)$, by Remark 4, we have

$$\phi(s)=\psi^{(k)}(s)/\psi^{(k)}(0)=1-h'_{k+1}s^{M_Y-k}+o(s^{M_Y-k}).$$

Then, by integration, we will have

$$\begin{aligned}\psi(s) &= 1+\psi^{(1)}(0)s+\frac{1}{2}\psi^{(2)}(0)s^2+\cdots+(-1)^{k+1}h_{k+1}s^{k+M_Y-k}+o(s^{k+M_Y-k})\\ &= 1-h_1s+h_2s^2-\cdots+(-1)^{k+1}h_{k+1}s^{M_Y}+o(s^{M_Y})\quad s\to 0^+,\end{aligned}$$

where $0<h_i<\infty$. The integration is due to Lemma 31 of [6]. □

Proof of Proposition 4. We provide the proof only for the bivariate case. For $d\geq 3$, the intermediate upper tail dependence can be studied analogously, and the (omitted) proof is similar but with more complicated notation.

Let $\psi'(0)=m$ with $-\infty<m<0$, then by Proposition 3, as $s\to 0^+$, letting $M=M_\psi-1$,

$$g(s)=\psi'(s)-m=as^M+o(s^M),\quad (0<a<\infty;\ 0<M<1).$$

Since $g'(s)=\psi''(s)$ is increasing as $s\to 0^+$, if we write $g(s)\sim as^M\ell(s)$, $s\to 0^+$, where $\ell(s)\in\mathcal{R}_0(0^+)$ and $\ell(s)\to 1$ as $s\to 0^+$. Note that

$$g(s)=\psi'(s)-m=\int_0^s\psi''(x)dx=\int_0^s g'(x)dx.$$

By the Monotone Density Theorem (Theorem 1.7.2 of [5]),

$$\psi''(s)=g'(s)\sim aMs^{M-1}\ell(s),\quad s\to 0^+. \tag{21}$$

Thus, $\lim_{s\to 0^+}\psi''(2s)/\psi''(s)=2^{M-1}$. Observe that for $0<\zeta<1$,

$$\begin{aligned}\lim_{u\to 1^-}\frac{\overline{C}_\psi(u,u)}{(1-u)^{1+\zeta}} &= \lim_{u\to 1^-}\frac{1-2u+\psi(2\psi^{-1}(u))}{(1-u)^{1+\zeta}} = \lim_{u\to 1^-}\frac{-2+2\psi'(2\psi^{-1}(u))/\psi'(\psi^{-1}(u))}{-(1+\zeta)(1-u)^{\zeta}}\\ &= \lim_{u\to 1^-}\frac{4\psi''(2\psi^{-1}(u))/[\psi'(\psi^{-1}(u))]^2-2\psi''(\psi^{-1}(u))\psi'(2\psi^{-1}(u))/[\psi'(\psi^{-1}(u))]^3}{\zeta(1+\zeta)(1-u)^{\zeta-1}}\\ &= \lim_{s\to 0^+}\frac{4\psi''(2s)/[\psi'(s)]^2-2\psi''(s)\psi'(2s)/[\psi'(s)]^3}{\zeta(1+\zeta)(1-\psi(s))^{\zeta-1}}\quad (\text{letting } s=\psi^{-1}(u))\\ &= \lim_{s\to 0^+}\frac{4m^{-2}\psi''(2s)-2m^{-2}\psi''(s)}{\zeta(1+\zeta)(1-\psi(s))^{\zeta-1}} = \lim_{s\to 0^+}\frac{4m^{-2}\frac{\psi''(2s)}{\psi''(s)}-2m^{-2}}{\zeta(1+\zeta)\frac{(1-\psi(s))^{\zeta-1}}{\psi''(s)}}\\ &= \lim_{s\to 0^+}\frac{2m^{-2}(2^M-1)}{\zeta(1+\zeta)\frac{(1-\psi(s))^{\zeta-1}}{\psi''(s)}}.\end{aligned}$$

By Proposition 3, there is a constant $h>0$ such that

$$1-\psi(s)=-ms-hs^{M+1}+o(s^{M+1}),\quad s\to 0^+.$$

Then, as $s\to 0^+$,

$$[1-\psi(s)]^{\zeta-1}\sim(-m)^{\zeta-1}s^{\zeta-1}.$$

In addition, it has been shown that $\psi''(s)\sim aMs^{M-1}\ell(s)$ as $s\to 0^+$ and $2m^{-2}(2^M-1)$ is finite. Hence, the intermediate tail dependence exists if and only if $\zeta=M$ and $\kappa_U=1+M=M_\psi$.

The proof for the case of $k=0$ is similar, by applying Proposition 3. □

Lemma 2. *Let $d\geq 2$ be a positive integer and let j be a positive integer that is less than d. Let*

$$S_{dj}(w_1,\ldots,w_d)=\sum_{\emptyset\neq I\subset I_d}(-1)^{|I|-1}\left(\sum_{i\in I}w_i\right)^j. \tag{22}$$

Then $S_{dj}\equiv 0$.

Proof of Lemma 2. When $j=1$, by the binomial theorem, for any $n\in\mathbb{N}_+$, $\sum_{i=0}^n(-1)^i\binom{n}{i}=0$, so that $S_{d1}\equiv 0$ for $d\geq 2$.

For $1<j<d$, S_{dj} is a symmetric homogeneous function of order j, and its first order partial derivatives are homogeneous of order $j-1$. By recursion with Euler's formula for homogeneous functions to the jth order partial derivatives

$$S_{dj}(\mathbf{w}) = \frac{1}{j}\sum_{i=1}^d\frac{\partial S_{dj}(\mathbf{w})}{\partial w_i}w_i=\frac{1}{j(j-1)}\sum_{i_1=1}^d\sum_{i_2=1}^d\frac{\partial^2 S_{dj}(\mathbf{w})}{\partial w_{i_1}\partial w_{i_2}}w_{i_1}w_{i_2}$$

$$= \frac{1}{j!}\sum_{i_1=1}^{d}\cdots\sum_{i_j=1}^{d}\frac{\partial^j S_{dj}(\mathbf{w})}{\partial w_{i_1}\cdots\partial w_{i_j}} w_{i_1}\cdots w_{i_j}. \tag{23}$$

We will show that all the jth order partial derivatives are 0. Because of symmetry, we consider only terms for which $w_{i_1}\ldots w_{i_j} = w_1^{n_1}\cdots w_p^{n_p}$ where $1 \le p \le j < d$, $n_1 > 0, \ldots, n_p > 0$ and $n_1 + \cdots + n_p = j$. Then

$$\frac{\partial^j S_{dj}(\mathbf{w})}{\partial^{n_1} w_1\cdots\partial^{n_p} w_p} = j!(-1)^{p-1} + \sum_{\emptyset\ne J\subset\{p+1,\ldots,d\}}(-1)^{p+|J|-1} j! = j!(-1)^{p-1}\left(1+\sum_{\emptyset\ne J\subset\{p+1,\ldots,d\}}(-1)^{|J|}\right)$$

$$= j!(-1)^{p-1}\sum_{i=0}^{d-p}(-1)^i\binom{d-p}{i} = 0,$$

which completes the proof. □

Note that (22) is not zero for $j = d$ because (23) would include a non-zero term such as $\partial^d S_{dj}/\partial w_1\cdots\partial w_d = (-1)^{d-1}d!$. In fact, there are $d!$ non-zero terms in (23) when $(i_1,\ldots,i_d)$ is a permutation of $(1,\ldots,d)$, and $S_{dd}(w_1,\ldots,w_d) = (-1)^{d-1}d!\prod_{i=1}^d w_i$.

Proof of Proposition 5. Consider

$$\lim_{u\to0^+}\frac{\mathbb{P}\left[\bigcap_{i\in I_d}\{U_i \ge 1-uw_i\}\right]}{u^{k+M}} = \sum_{\emptyset\ne I\subset I_d}(-1)^{|I|-1}\lim_{u\to0^+}\frac{\mathbb{P}\left[\bigcup_{i\in I}\{U_i\ge 1-uw_i\}\right]}{u^{k+M}}.$$

By Proposition 3, since the function $w\mapsto 1-\psi(w)\in\mathcal{R}_1(0^+)$, then we have $w\mapsto\psi^{-1}(1-w)\in\mathcal{R}_1(0^+)$. Thus,

$$\begin{aligned}\lim_{u\to0^+}\frac{\mathbb{P}\left[\bigcup_{i\in I}\{U_i\ge 1-uw_i\}\right]}{u^{k+M}} &= \lim_{u\to0^+}\frac{1-\psi[\psi^{-1}(1-uw_1)+\cdots+\psi^{-1}(1-uw_d)]}{u^{k+M}}\\ &= \lim_{u\to0^+}\frac{1-\psi[\psi^{-1}(1-uw_1)+\cdots+\psi^{-1}(1-uw_d)]}{\{1-\psi[\psi^{-1}(1-u)]\}^{k+M}}\\ &= \lim_{u\to0^+}\frac{1-\psi\left[\psi^{-1}(1-u)\left(\frac{\psi^{-1}(1-uw_1)}{\psi^{-1}(1-u)}+\cdots+\frac{\psi^{-1}(1-uw_d)}{\psi^{-1}(1-u)}\right)\right]}{\{1-\psi[\psi^{-1}(1-u)]\}^{k+M}}\\ &= \lim_{u\to0^+}\frac{1-\psi\left[\psi^{-1}(1-u)\left(\sum_{i\in I}w_i\right)\right]}{\{1-\psi[\psi^{-1}(1-u)]\}^{k+M}}.\end{aligned}$$

Let $s = \psi^{-1}(1-u)$ and

$$Q(\mathbf{w}) = \sum_{\emptyset\ne I\subset I_d}(-1)^{|I|-1}\lim_{s\to0^+}\frac{1-\psi\left[s\left(\sum_{i\in I}w_i\right)\right]}{\{1-\psi(s)\}^{k+M}}. \tag{24}$$

To obtain the limit in (24), we may use the l'Hopital's rule. For the first k derivatives of the numerator, for fixed $\mathbf{w}$ and $\psi^{(j)}(0)$ finite for $j = 1,\ldots,k$,

$$\lim_{s\to0}\sum_{\emptyset\ne I\subset I_d}(-1)^{|I|-1}\psi^{(j)}\left[s\left(\sum_{i\in I}w_i\right)\right]\left(\sum_{i\in I}w_i\right)^j = 0,\quad j\in\{1,\ldots,k\},$$

because by Lemma 2,

$$\sum_{\emptyset\ne I\subset I_d}(-1)^{|I|-1}\left(\sum_{i\in I}w_i\right)^j = 0,\quad j\in\{1,\ldots,k\},\ 1\le k<d.$$

Then by the l'Hopital's rule ($k+1$ applications)

$$Q(\mathbf{w}) = \sum_{\emptyset\neq I\subset I_d}(-1)^{|I|-1}\lim_{s\to 0^+}\frac{-\psi^{(k+1)}\left[s\left(\sum_{i\in I}w_i\right)\right]\left(\sum_{i\in I}w_i\right)^{k+1}\Big/\psi^{(k+1)}(s)}{[-\psi^{(1)}(s)]^{k+1}\left\{\prod_{j=0}^{k}(k+M-j)\right\}(1-\psi(s))^{M-1}\Big/\psi^{(k+1)}(s)}.$$

Since $g(s) = |\psi^{(k)}(s)-\psi^{(k)}(0)| = hs^M + o(s^M)$ with $h>0$, we can write $g(s)\sim s^M\ell(s)$, $s\to 0^+$ with a slowly varying function $\ell(s)\to h$ as $s\to 0^+$. Note that, $g(s)=\int_0^s|\psi^{(k+1)}(x)|dx = \int_0^s g'(x)dx$, and $g'(s)$ is monotonic as $s\to 0^+$, then by the Monotone density theorem, $|\psi^{(k+1)}(s)| = g'(s)\sim Ms^{M-1}\ell(s)$ and $\psi^{(k+1)}(s)\sim(-1)^{k+1}Ms^{M-1}\ell(s)$. Therefore,

$$Q(\mathbf{w}) = \frac{Mh}{[-\psi^{(1)}(0)]^{k+M}\prod_{j=0}^{k}(k+M-j)}\sum_{\emptyset\neq I\subset I_d}(-1)^{|I|+k+1}\left(\sum_{i\in I}w_i\right)^{k+M}.$$

Then, the upper tail order function is

$$b^*(\mathbf{w}) = \frac{Q(\mathbf{w})}{Q(\mathbf{1}_d)} = \frac{\sum_{\emptyset\neq I\subset I_d}(-1)^{|I|}\left(\sum_{i\in I}w_i\right)^{k+M}}{\sum_{\emptyset\neq I\subset I_d}(-1)^{|I|}\,|I|^{k+M}}.$$

Note that this is a homogeneous function in $\mathbf{w}$ of order $\kappa_U = M_\psi = k+M$. This completes the proof. □

Proof of Proposition 6. If $r=0$, then $\psi^{-1}(t)\sim(t/a_1)^{1/q}$ as $t\to 0^+$ (where $q<0$). If $r>0$, then for large s and small t, in

$$\log\psi(s) = \log t\sim\log a_1 + q\log s - a_2 s^r,\quad s\to\infty,$$

the third term dominates, so that

$$\psi^{-1}(t)\sim[(-\log t)/a_2]^{1/r},\quad t\to 0^+.$$

Next, consider $C_\psi(u\mathbf{1}_d) = \psi(d\psi^{-1}(u))$.

For $r=0$, one gets $\psi(d\psi^{-1}(u))\sim\psi(da_1^{-1/q}u^{1/q})\sim d^q u$, as $u\to 0^+$, with $d^q\in(0,1)$, so that $\kappa_L = 1$.

For $0<r<1$, suppose

$$\Delta_{L,\kappa} = \lim_{u\to 0^+}\frac{\psi\left(d\psi^{-1}(u)\right)}{u^\kappa(-\log u)^\zeta} > 0.$$

Then by l'Hopital's rule,

$$\Delta_{L,\kappa} = \lim_{u\to 0^+}\frac{d\psi'\left(d\psi^{-1}(u)\right)/\psi'\left(\psi^{-1}(u)\right)}{\kappa u^{\kappa-1}(-\log u)^\zeta} = \lim_{s\to\infty}\frac{d\psi'(ds)/\psi'(s)}{\kappa[\psi(s)]^{\kappa-1}[-\log\psi(s)]^\zeta}. \tag{25}$$

By condition (10), the dominating term of $\psi'(s)$ or $T'(s)$ is

$$\psi'(s)\sim -a_1a_2rs^{q+r-1}\exp\{-a_2s^r\},\quad s\to\infty.$$

Consider the limit of the right-hand side of (25) without the factor d/κ:

$$\psi'(ds)/\psi'(s)\sim d^{q+r-1}\exp\{-a_2(d^r-1)s^r\},\quad s\to\infty;$$

$$\begin{aligned}[\psi(s)]^{\kappa-1}[-\log\psi(s)]^\zeta &\sim [a_1s^q]^{\kappa-1}\exp\{-a_2(\kappa-1)s^r\}\cdot[a_2s^r - q\log s - \log a_1]^\zeta\\ &\sim a_1^{\kappa-1}a_2^\zeta s^{q(\kappa-1)+r\zeta}\exp\{-a_2(\kappa-1)s^r\},\quad s\to\infty.\end{aligned}$$

Hence $\kappa = d^r$, $q(\kappa-1)+r\zeta = 0$ or $\zeta = (q/r)(1-\kappa) = (q/r)(1-d^r)$ and

$$\Delta_{L,\kappa} = \frac{dd^{q+r-1}}{\kappa a_1^{\kappa-1}a_2^\zeta} = \frac{d^q}{a_1^{\kappa-1}a_2^\zeta}.$$

So $\ell(u) = \Delta_{L,\kappa}\cdot(-\log u)^\zeta = d^qa_1^{1-\kappa}a_2^{-\zeta}(-\log u)^\zeta$.

Under the condition in (10), it can be verified that $-\psi(s)/\psi'(s)\in\mathcal{R}_{1-r}$, which satisfies the condition in Theorem 3.3 of [7]. So the tail order function is obtained. □

Proof of Proposition 7. When $0 < \nu < 1$,

$$K_\nu(s) \sim \frac{1}{2}\left(\Gamma(\nu)\,(s/2)^{-\nu} + \Gamma(-\nu)\,(s/2)^{\nu}\right). \tag{26}$$

We refer to the website of Wolfram Research [4] for asymptotic behavior of modified Bessel function of the second kind. For $0 < \alpha < 1$,

$$\psi(s;\alpha) \sim 1 + \frac{\Gamma(-\alpha)}{\Gamma(\alpha)} s^\alpha, \quad s \to 0^+,$$

and $1 - \psi(s;\alpha) \sim -s^\alpha \Gamma(-\alpha)/\Gamma(\alpha) \in \mathcal{R}_\alpha(0^+)$. This is consistent with Lemma 1.

Now, let us consider the case where α is non-integer with $\alpha > 1$. For an integer j with $0 < j < \alpha$,

$$\begin{aligned}\psi^{(j)}(s;\alpha) &= [\Gamma(\alpha)]^{-1}(-1)^j \int_0^\infty e^{-s/x} x^{\alpha-j-1} e^{-x}\,dx = (-1)^j \frac{\Gamma(\alpha-j)}{\Gamma(\alpha)} \mathbb{E}(e^{-s/X'}), \quad X' \sim \mathrm{Gamma}(\alpha-j, 1)\\ &= (-1)^j 2\Gamma^{-1}(\alpha) s^{(\alpha-j)/2} K_{\alpha-j}(2\sqrt{s}).\end{aligned}$$

So, $\phi(s;\alpha,j) = \psi^{(j)}(s;\alpha)/\psi^{(j)}(0;\alpha)$ is the LT of $Z = 1/X' \sim \mathrm{I}\Gamma(\alpha-j,1)$ with $M_Z = \alpha - j$. When ν is non-integer with $|\nu| > 1$, the behavior near 0 of K_ν is

$$K_\nu(x) \sim x^{-|\nu|} 2^{|\nu|-1} \Gamma(|\nu|) \left[1 + \frac{x^2}{4(1-|\nu|)}\right].$$

So, for integer $j < \alpha - 1$

$$(-1)^j \psi^{(j)}(s;\alpha) = 2\Gamma^{-1}(\alpha) s^{(\alpha-j)/2} K_{\alpha-j}(2\sqrt{s}) \sim \frac{\Gamma(\alpha-j)}{\Gamma(\alpha)}\left(1 + \frac{s}{1-\alpha+j}\right), \quad s \to 0;$$

for $k < \alpha < k+1$, where $k \in \mathbb{N}_+$, then by (26)

$$(-1)^k \psi^{(k)}(s;\alpha) = 2\Gamma^{-1}(\alpha) s^{(\alpha-k)/2} K_{\alpha-k}(2\sqrt{s}) \sim \frac{\Gamma(\alpha-k)}{\Gamma(\alpha)} + \frac{\Gamma(-\alpha+k)}{\Gamma(\alpha)} s^{\alpha-k}, \quad s \to 0.$$

Therefore, $|\psi^{(k)}(s) - \psi^{(k)}(0)| \in \mathcal{R}_{\alpha-k}(0^+)$, which is consistent with Proposition 3. Then, by Proposition 3, there is a positive constant h_{k+1} such that

$$\psi(s) = 1 + \psi^{(1)}(0)s + \frac{1}{2}\psi^{(2)}(0)s^2 + \cdots + (-1)^k \psi^{(k)}(0)s^k/k! + (-1)^{k+1} h_{k+1} s^\alpha + o(s^\alpha).$$

The upper tail order of the d-variate Archimedean copula C_ψ follows from Propositions 4 and 5. Therefore, if $\alpha \in (0, +\infty)$ is not an integer, the upper tail order is $\max\{1, \min\{\alpha, d\}\}$.

Next we investigate the lower tail. From [2], p. 378: for large z,

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}\, e^{-z}\left\{1 + \frac{4\nu^2 - 1}{8z} + O(z^{-2})\right\}.$$

Hence,

$$\psi(s;\alpha) = 2\Gamma^{-1}(\alpha) s^{\alpha/2} K_\alpha(2\sqrt{s}) \sim 2\Gamma^{-1}(\alpha) s^{\alpha/2} \sqrt{\frac{\pi}{4s^{1/2}}}\, e^{-2s^{1/2}} = \pi^{1/2}\Gamma^{-1}(\alpha) s^{\alpha/2-1/4} e^{-2s^{1/2}}, \quad s \to \infty.$$

Also,

$$\psi^{(1)}(s;\alpha) = -2\Gamma^{-1}(\alpha) s^{(\alpha-1)/2} K_{\alpha-1}(2\sqrt{s}) \sim -\pi^{1/2}\Gamma^{-1}(\alpha) s^{\alpha/2-3/4} e^{-2s^{1/2}}, \quad s \to \infty.$$

For the d-dimensional Archimedean copula, then by Proposition 6 with $a_1 = \pi^{1/2}\Gamma^{-1}(\alpha)$, $q = \alpha/2 - 1/4$, $a_2 = 2$ and $r = 1/2$ in (10), as $u \to 0$,

$$\psi\left(d\psi^{-1}(u)\right) \sim \frac{d^{\alpha/2-1/4}}{a_1^{\kappa-1} 2^\zeta} (-\log u)^\zeta u^{\sqrt{d}}, \quad a_1 = \pi^{1/2}\Gamma^{-1}(\alpha),\ \zeta = (\alpha - 1/2)(1 - \sqrt{d}).$$

Thus $\kappa_L(C_\psi) = \sqrt{d}$. □

Proof of Proposition 8. Suppose that K is a multivariate max-id copula such that, for any index set $\emptyset \neq I \subset I_d$, $\overline{K}_I((1-s)\mathbf{1}_{|I|}) \sim s^{a_I} \ell_I(s)$, $s \to 0^+$, with $1 < a_I$ and $\ell_I(s) \in \mathcal{R}_0(0^+)$. Note that

$$K((1-s)\mathbf{1}_d) = 1 + \sum_{\emptyset \neq I \subset I_d} (-1)^{|I|} \overline{K}_I\left((1-s)\mathbf{1}_{|I|}\right), \tag{27}$$

where $\overline{K}_I$ is the survivor function of the I-marginal copula K_I and let $\overline{K}_{\{i\}}(1-s) = s$ for any $i \in I_d$. Letting $s = 1-\exp\{-\psi^{-1}(u)\}$, as $u \to 1^-$, i.e., $s \to 0^+$, since $a_I > 1$ for any $\emptyset \neq I \subset I_d$, $1 - ds$ dominates the right-hand side of (27), and thus,

$$-\log K\left(e^{-\psi^{-1}(u)}\mathbf{1}_d\right) = -\log K((1-s)\mathbf{1}_d) \sim -\log(1-ds) \sim ds = d(1-\exp\{-\psi^{-1}(u)\}) \sim d\psi^{-1}(u).$$

Therefore,

$$C(u\mathbf{1}_d) \sim \psi\left(d\psi^{-1}(u)\right) = C_\psi(u\mathbf{1}_d), \quad u \to 1^-.$$

By Proposition 4, we know that C has intermediate upper tail dependence, and moreover, $\kappa_U(C) = \kappa_U(C_\psi)$. This proves (a).

To prove (b), note from Proposition 2 that $\kappa_U(K_I) = 1$ for any marginal copula K_I. Assuming $\ell_I(s) \to h_I \in (0, 1]$ as $s \to 0^+$ and $h_{\{i\}} = 1$ for all i,

$$-\log K_I\left(e^{-\psi^{-1}(u)}\mathbf{1}_{|I|}\right) = -\log K_I((1-s)\mathbf{1}_{|I|}) \sim -\log\left(1-h_I^* s\right) \sim h_I^*\psi^{-1}(u),$$

where $h_I^* = \sum_{\emptyset\neq J\subset I}(-1)^{|J|-1}h_J$. By the construction of (12), for any I-marginal copula of C,

$$C_I(u\mathbf{1}_{|I|}) = \psi\left(-\log K_I(e^{-\psi^{-1}(u)}\mathbf{1}_{|I|})\right) \sim \psi\left(h_I^*\psi^{-1}(u)\right).$$

Thus, as $s \to 0^+$, i.e., $u \to 1^-$,

$$\overline{C}(u\mathbf{1}_d) = 1 + \sum_{\emptyset\neq I\subset I_d}(-1)^{|I|}C_I(u\mathbf{1}_{|I|}) \sim 1 + \sum_{\emptyset\neq I\subset I_d}(-1)^{|I|}\psi\left(h_I^*\psi^{-1}(u)\right) = \sum_{\emptyset\neq I\subset I_d}(-1)^{|I|-1}\left(1-\psi\left[h_I^*\psi^{-1}(u)\right]\right).$$

If $1-\psi(x) \in \mathcal{R}_\beta(0^+)$, then clearly,

$$\overline{C}((1-u)\mathbf{1}_d) \sim u\sum_{\emptyset\neq I\subset I_d}(-1)^{|I|-1}(h_I^*)^\beta, \quad u \to 0^+.$$

So, $\kappa_U(C) = 1$ and $\lambda_U(C) = \sum_{\emptyset\neq I\subset I_d}(-1)^{|I|-1}(h_I^*)^\beta$. □

Proof of Proposition 9. Suppose that a d-variate max-id copula $K(s\mathbf{1}_d) \sim s^\alpha\ell(s)$ as $s \to 0^+$ and let $s = \exp\{-\psi^{-1}(u)\}$. As $u \to 0^+$, thus $s \to 0^+$,

$$-\log K\left(e^{-\psi^{-1}(u)}\mathbf{1}_d\right) = -\log K(s\mathbf{1}_d) \sim -\log(s^\alpha\ell(s)) \sim -\alpha\log s = \alpha\psi^{-1}(u).$$

Therefore,

$$C(u\mathbf{1}_d) = \psi\left(-\log K\left(e^{-\psi^{-1}(u)}\mathbf{1}_d\right)\right) \sim \psi\left(\alpha\psi^{-1}(u)\right), \quad u \to 0^+.$$

With some modification of the proof of Theorem 3.3 of [7], we can prove the rest. For purpose of notational convenience, we include the modification in the following. Letting $\Lambda = (\alpha + d\nu)/(1+\nu)$ and $\omega(s) = -\psi(s)/\psi'(s)$, then we know that

$$\lim_{s\to 0^+}\frac{\psi^{-1}(sx)-\psi^{-1}(s)}{\omega(\psi^{-1}(s))} = -\log(x),$$

and if $y(t) \to y \in \mathbb{R}$ as $t \to \infty$, then

$$\lim_{t\to\infty}\frac{\psi(t+y(t)\omega(t))}{\psi(t)} = \exp(-y).$$

For any $t > 0$, write $\psi(\alpha\psi^{-1}(ut)) = \psi\left[\alpha\psi^{-1}(u) + y(u,t)\omega(\alpha\psi^{-1}(u))\right]$, where

$$y(u,t) = \left(\frac{\alpha[\psi^{-1}(ut)-\psi^{-1}(u)]}{\omega(\psi^{-1}(u))}\right)\frac{\omega(\psi^{-1}(u))}{\omega(\alpha\psi^{-1}(u))}.$$

As $u \to 0^+$, $y(u,t) \to -\alpha\log(t)\alpha^{-\beta} = -\alpha^{1-\beta}\log(t)$. Therefore,

$$\lim_{u\to 0^+}\frac{\psi^{-1}(\alpha\psi^{-1}(ut))}{\psi^{-1}(\alpha\psi^{-1}(u))} = \exp(\alpha^{1-\beta}\log(t)) = t^{\alpha^{1-\beta}},$$

and thus $C(u\mathbf{1}_d) \in \mathcal{R}_{\alpha^{1-\beta}}(0^+)$. We have also known from Theorem 3.3 of [7] that $\kappa_L(C_\psi) = d^{1-\beta}$. This completes the proof. □

References

[1] K. Aas, C. Czado, A. Frigessi, H. Bakken, Pair-copula constructions of multiple dependence, Insurance: Mathematics & Economics 44 (2) (2009) 182–198.
[2] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, in: National Bureau of Standards Applied Mathematics Series, For sale by the Superintendent of Documents, vol. 55, US Government Printing Office, Washington, DC, 1964.
[3] T. Bedford, R.M. Cooke, Probability density decomposition for conditionally dependent random variables modeled by vines, Annals of Mathematics and Artificial Intelligence 32 (1–4) (2001) 245–268.
[4] http://functions.wolfram.com/Bessel-TypeFunctions/BesselK/introductions/Bessels/ShowAll.html (accessed on 11.03.11).
[5] N.H. Bingham, C.M. Goldie, J.L. Teugels, Regular Variation, in: Encyclopedia of Mathematics and its Applications, vol. 27, Cambridge University Press, Cambridge, 1987.
[6] K.W. Breitung, Asymptotic Approximations for Probability Integrals, in: Lecture Notes in Mathematics, vol. 1592, Springer-Verlag, Berlin, 1994.
[7] A. Charpentier, J. Segers, Tails of multivariate Archimedean copulas, Journal of Multivariate Analysis 100 (7) (2009) 1521–1537.
[8] K.L. Chung, A Course in Probability Theory, second ed., Academic, New York, 1974.
[9] A. Colangelo, M. Scarsini, M. Shaked, Some notions of multivariate positive dependence, Insurance: Mathematics & Economics 37 (1) (2005) 13–26.
[10] S. Coles, J. Heffernan, J. Tawn, Dependence measures for extreme value analyses, Extremes 2 (1999) 339–365.
[11] R.D. Cook, M.E. Johnson, A family of distributions for modelling non-elliptically symmetric multivariate data, Journal of the Royal Statistical Society B 43 (1981) 210–218.
[12] C. Genest, R.J. MacKay, Copules archimédiennes et familles de lois bidimensionnelles dont les marges sont données, The Canadian Journal of Statistics 14 (2) (1986) 145–159.
[13] E. Hashorva, Asymptotic properties of type I elliptical random vectors, Extremes 10 (4) (2007) 175–206.
[14] E. Hashorva, J. Hüsler, On multivariate Gaussian tails, Annals of the Institute of Statistical Mathematics 55 (3) (2003) 507–522.
[15] J.E. Heffernan, A directory of coefficients of tail dependence, Extremes 3 (3) (2000) 279–290. 2001.
[16] H. Joe, Multivariate Models and Dependence Concepts, in: Monographs on Statistics and Applied Probability, vol. 73, Chapman & Hall, London, 1997.
[17] H. Joe, T. Hu, Multivariate distributions from mixtures of max-infinitely divisible distributions, Journal of Multivariate Analysis 57 (2) (1996) 240–265.
[18] H. Joe, H. Li, A.K. Nikoloulopoulos, Tail dependence functions and vine copulas, Journal of Multivariate Analysis 101 (2010) 252–270.
[19] H. Joe, C. Ma, Multivariate survival functions with a min-stable property, Journal of Multivariate Analysis 75 (1) (2000) 13–35.
[20] A. Juri, M.V. Wüthrich, Copula convergence theorems for tail events, Insurance: Mathematics & Economics 30 (3) (2002) 405–420.
[21] A. Juri, M.V. Wüthrich, Tail dependence from a distributional point of view, Extremes 6 (3) (2003) 213–246. 2004.
[22] C. Klüppelberg, G. Kuhn, L. Peng, Semi-parametric models for the multivariate tail dependence function — the asymptotically dependent case, Scandinavian Journal of Statistics 35 (4) (2008) 701–718.
[23] A.W. Ledford, J.A. Tawn, Statistics for near independence in multivariate extreme values, Biometrika 83 (1) (1996) 169–187.
[24] A.W. Ledford, J.A. Tawn, Modelling dependence within joint tail regions, Journal of the Royal Statistical Society. Series B 59 (2) (1997) 475–499.
[25] H. Li, Y. Sun, Tail dependence for heavy-tailed scale mixtures of multivariate distributions, Journal of Applied Probability 46 (4) (2009) 925–937.
[26] A.W. Marshall, I. Olkin, Families of multivariate distributions, Journal of the American Statistical Association 83 (403) (1988) 834–841.
[27] A.W. Marshall, I. Olkin, Life Distributions: Structure of Nonparametric, Semiparametric, and Parametric Families, in: Springer Series in Statistics, Springer, New York, 2007.
[28] A.J. McNeil, R. Frey, P. Embrechts, Quantitative Risk Management, in: Princeton Series in Finance, Princeton University Press, Princeton, NJ, 2005.
[29] A.J. McNeil, J. Nešlehová, Multivariate Archimedean copulas, d-monotone functions and l_1-norm symmetric distributions, The Annals of Statistics 37 (5B) (2009) 3059–3097.
[30] R.B. Nelsen, An Introduction to Copulas, second ed., in: Springer Series in Statistics, Springer, New York, 2006.
[31] A.K. Nikoloulopoulos, H. Joe, H. Li, Extreme value properties of multivariate t copulas, Extremes 12 (2009) 129–148.
[32] T. Okimoto, New evidence of asymmetric dependence structures in international equity markets, Journal of Financial and Quantitative Analysis 43 (03) (2008) 787–815.
[33] A.J. Patton, Modelling asymmetric exchange rate dependence, International Economic Review 47 (2) (2006) 527–556.
[34] A. Ramos, A. Ledford, A new class of models for bivariate joint tails, Journal of the Royal Statistical Society. Series B (Statistical Methodology) 71 (1) (2009) 219–241.
[35] S.I. Resnick, Extreme Values, Regular Variation, and Point Processes, in: Applied Probability. A Series of the Applied Probability Trust, vol. 4, Springer-Verlag, New York, 1987.
[36] S.I. Resnick, Heavy-tail Phenomena, in: Springer Series in Operations Research and Financial Engineering, Springer, New York, 2007.