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The 3D Dimer and Ising problems revisited

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Abstract

We express the 3D Dimer partition function on a finite lattice as a linear combination of determinants of oriented adjacency matrices and the 3D Ising partition function as a linear combination of products over aperiodic closed walks. The methodology we use is embedding of cubic lattice on 2D surfaces of large genus.

1. Introduction

The motivation for this work is the renewed interest in the Ising and the Dimer models. Let us mention two such recent promising results. The first is a new relation (duality) between topological strings and the Dimer model, [1]. The second is the relation between the theory of discrete Riemann surfaces [5] and Theta functions and criticality of the Ising problem [2–4]. In this paper we present new formulas for the 3D Dimer and 3D Ising problems on a finite lattice. The formulas are obtained in a combinatorial way.

A foundation for the work described here has been given by the first author in [8,9]. The main contributions of this paper are:

1. Major simplification of the formulas of [8] for the 3D dimer problem, along with the proofs.
2. New formulas for the 3D Ising problem, based on Feynman's paths approach.
3. A unified compact treatment of both the 3D dimer and the 3D Ising problem.

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We consider the *Ising version of the Edwards–Anderson model*: A *coupling constant* J_{ij} is assigned to each bond $\{i, j\}$ of a lattice graph G; the coupling constant characterizes the interaction between the particles represented by sites i and j. A physical state of the system is an assignment of spin $\sigma_i \in \{+1, -1\}$ to each site i. The *Hamiltonian* (or energy function) is defined as $H(\sigma) = -\sum_{\{i,j\}\in E} J_{ij}\sigma_i\sigma_j$. The distribution of physical states over all possible energy levels is encapsulated in the *partition function* $Z(\beta) = \sum_\sigma \mathrm{e}^{-\beta H(\sigma)}$ from which all fundamental physical quantities may be derived.

We may reformulate the Ising problem in graph-theoretic terms as follows. A graph is a pair $G = (V, E)$ where V is a set of *vertices* and E is now the set of edges (not the energy). A graph with sufficient regularity properties may be called a *lattice graph*. We associate with each edge e of G a weight $w(e)$ and for a subset of edges $A \subset E$, $w(A)$ will denote the sum of the weights $w(e)$ associated with the edges in A.

An *even subgraph* of a graph $G = (V, E)$ is a set of edges $U \subset E$ such that each vertex of V is incident with an even number of edges from U. The *generating function of even subgraphs* $\mathcal{H}(G, x)$ equals the sum of $\alpha^{w(U)}$ over all even subgraphs U of G. A classic relation between the Ising partition function and the generating function of even subgraphs of the same graph states that

$$Z(\beta) = 2^n \left(\prod_{\{i,j\}\in E} \cosh(\beta J_{ij})\right) \mathcal{H}(G, \tanh(\beta J_{ij})).$$

A subset of edges $P \subset E$ is called a *perfect matching* or *dimer arrangement* if each vertex belongs to exactly one element of P. The *dimer partition function* on graph G may be viewed as a polynomial $\mathcal{P}(G, \alpha)$ which equals the sum of $\alpha^{w(P)}$ over all perfect matchings P of G. This polynomial is also called the *generating function of perfect matchings*.

The generating functions of even subgraphs and perfect matchings may be defined in a more general way as follows: associate a variable x_e with each edge e of graph G, let $x(A) = \prod_{e\in A} x_e$ and let for example the generating function of perfect matchings be the sum of $x(P)$ over all perfect matchings P of G. All results introduced in this paper also hold in this more general setting; the presentation using weights rather than variables is perhaps more natural.

Convention for the cubic lattice. This paper studies properties of finite cubic lattices. Let us now fix some notation for them.

Let m_1, m_2, k be positive integers, where k is even (we will use the bipartiteness of the lattice) and m_1 and m_2 odd. The cubic lattice $Q = Q(m_1 m_2 k)$ is the following graph:
The vertices are:

$$V_{xyz}, \quad x = 1, \ldots, m_1, y = 1, \ldots, m_2, z = 1, \ldots, k.$$

The edges are:

- the *vertical* edges $v_{xyz} = \{V_{xyz}, V_{xy(z+1)}\}$, $z = 1, \ldots, k-1$,
- the *width* edges $w_{xyz} = \{V_{xyz}, V_{x(y+1)z}\}$, $y = 1, \ldots, m_2 - 1$,
- the *horizontal* edges $h_{xyz} = \{V_{xyz}, V_{(x+1)yz}\}$, $x = 1, \ldots, m_1 - 1$.

Definition 1.1. We now fix as in Fig. 1 a *linear order* '$<$' on the set of the vertices of Q as

$$V_{11}, \bar{V}_{12}, \ldots, V_{1m_2}, \bar{V}_{2m_2}, V_{2(m_2-1)}, \ldots, V_{m_1 m_2}$$

where $V_{xy} = V_{xy1}, \ldots, V_{xyk}$ and $\bar{V}_{xy}$ denotes the reversal of V_{xy}.

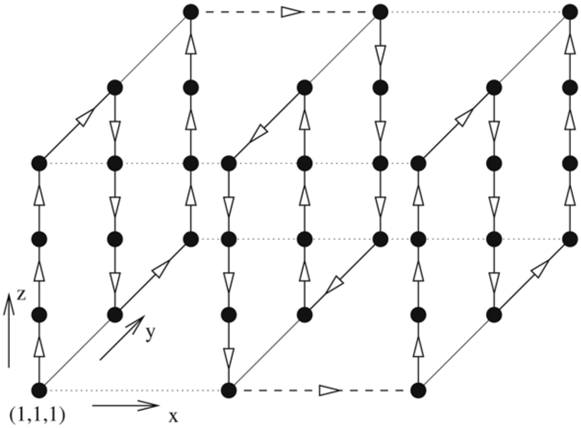

Fig. 1. Illustration of the linear order of vertices on a lattice $Q(3, 3, 4)$.

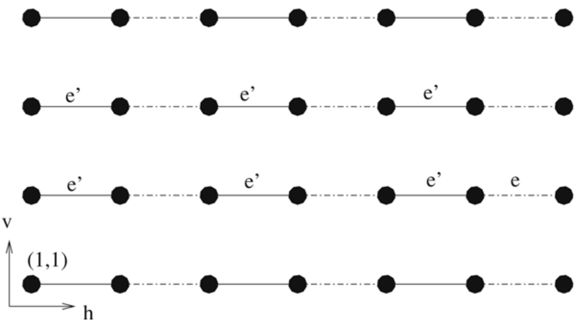

Fig. 2. For a fixed e ($h(L, e)$ is even) in a given layer L, all possible e' ($h(L, e')$ is odd) from the pairs $(e.e')$ are marked.

Definition 1.2. Let us now define a *set of pairs of edges* $R(Q)$. We let H be the set of all horizontal edges. For each $x \leq m_1$ we let W_x consist of all width edges with their x-coordinate equal to x. We call sets H and W_x *layers* of Q. It is convenient to depict the edges of a layer as the horizontal edges of a square grid, where the lower left corner is the smallest in the fixed ordering of vertices. The first column of the layer H is ordered according to the fixed ordering of vertices. The size of layer H is $m_1 \times km_2$, sizes of all W_x are $m_2 \times k$.

If e is an edge of such a layer L then let $h(L, e)$ ($v(L, e)$ respectively) denote the horizontal (vertical respectively) coordinate of the first vertex of e in L. For a layer L we let $R(L)$ be the set of all pairs (e, e') of edges of L such that $h(L, e') < h(L, e)$, $v(L, e) \leq v(L, e') \leq v(L, e) + 1$, and $h(L, e)$ is even while $h(L, e')$ is odd, example in Fig. 2. Finally we let $R(Q)$ be the union of $R(L)$ over all layers of Q.

2. Statement of the main results: The Dimer problem

Our result for the dimer problem is a simplification of the main result of [8]. The reader is strongly encouraged to check the original statement for a comparison. Our result goes far beyond Kasteleyn's conjecture [11] that the partition sum of dimer problem on the surface of genus g may be expressed as a linear combination of 4^g terms. We give compact formulas for this linear combination for dimer problem on a 3D lattice. The only work which goes in a similar direction is as far as we know [12]. Before we write the main theorems let us introduce some necessary notation.

Determinants. Q is a bipartite graph, which means that its vertices may be partitioned into two sets V_1, V_2 such that if e is an edge of Q then $|e \cap V_1| = |e \cap V_2| = 1$. We have $|V_1| = |V_2| = m_1 m_2 k/2$. Let $\mathcal{Z}$ be the square $(|V_1| \times |V_2|)$ matrix defined by $\mathcal{Z}_{ij} = 0$ if $\{i, j\}$ is not an edge of Q. If $e = \{ij\}$ is an edge of Q ($i \in V_1$, $j \in V_2$) then $\mathcal{Z}_{ij} = \alpha^{w(ij)}$ if $i < j$ (in fixed linear order) and $\mathcal{Z}_{ij} = -\alpha^{w(ij)}$ if $i > j$.

We will consider the matrix $\mathcal{Z}$ with its rows and columns ordered in agreement with the fixed ordering and we will assume that $V_{111} \in V_1$. A *signing* of a matrix is obtained by multiplying some of the entries of the matrix by -1.

An orientation of a graph $G = (V, E)$ is a *digraph* $D = (V, A)$ obtained from G by assigning an orientation to each edge of G, i.e. by ordering the elements of each edge of G. The elements of A are called *arcs*. We say that signing Z of $\mathcal{Z}$ corresponds to orientation D of Q if $Z_{ij} = -\mathcal{Z}_{ij}$ for $(ij) \in D$ and $i > j$. For arc (ij) of orientation D of Q we let $\mathrm{sign}(D, (ij)) = 1$ if $i < j$, and $\mathrm{sign}(D, (ij)) = -1$ otherwise.

Theorem 1. *The dimer partition function* $\mathcal{P}(Q, \alpha)$ *is*

$$\mathcal{P}(Q, \alpha) = 2^{-C} \sum_{D} (-1)^{|\{(e,e') \in R(Q); \mathrm{sign}(D,e) = \mathrm{sign}(D,e') = -1\}|} \det(Z(D)),$$

where the sum is over all orientations of Q with all vertical edges positive, and $C = km_1(m_2 - 1)/2 + km_2(m_1 - 1)/2$.

Knowing the proof Theorem 1 can be easily rewritten as

Theorem 2.

$$\mathcal{P}(Q, \alpha) = 2^{C} \alpha^{w(M)} - (2^{C} + 1)\mathcal{E},$$

where M is the unique perfect matching of Q consisting of vertical edges only and $\mathcal{E}$ equals the average of $\det(Z(D))$ *over all orientations D of Q satisfying:* $|\{(e, e') \in R(Q); \mathrm{sign}(D, e) = \mathrm{sign}(D, e') = -1\}|$ *is even (and again all vertical edges are positive).*

There are 2^{2C} orientations to be sum over in Theorem 1 and $2^{C-1}(2^C + 1)$ terms in Theorem 2 to be averaged over. Therefore exact numerical analysis is probably not possible. We tested Theorem 1 for small planar lattices. However it could be interesting to study statistical properties of determinants $\det(Z(D))$. Let us remark here also that Theorem 1 for a planar graph (2D Ising model, $m_1 = 1$) is still nontrivial, a question remains how to reduce it to Kasteleyn's single determinant.

3. Statement of the main results: The Ising problem

The theory described in Section 4 combined with the results of [6,9] yields an expression of the 3D Ising partition function as a linear combination of products over aperiodic closed walks on Q. We get a remarkably simple formula, analogous to the statement for the dimer partition function.

Products over aperiodic closed walks. Let $G = (V, E)$ be a planar graph embedded in the plane and for each edge e let x_e be an associate variable. Let $A = (V, A(G))$ be an arbitrary orientation of G. If $e \in E$ then a_e will denote the orientation of e in $A(G)$ and a_e^{-1} will be the reversed directed edge to a_e. We let $x_{a_e} = x_{a_e^{-1}} = x_e$. A circular sequence $p =$

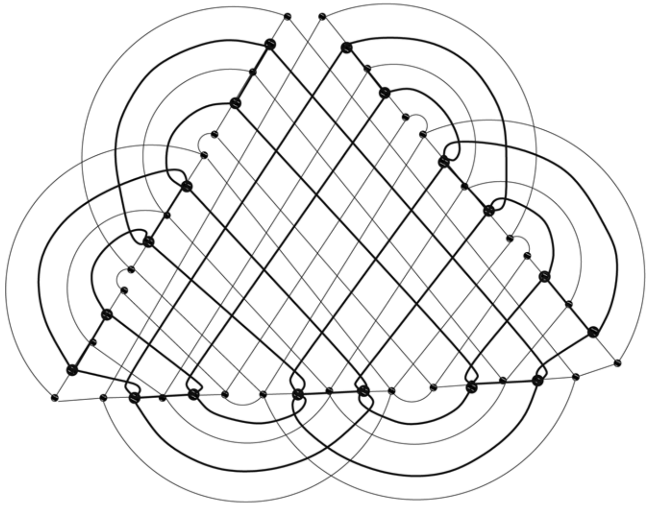

Fig. 3. Drawing of a cube $Q^L(3,3,5)$ in a plane with $Q(3,3,2)$ in bold.

$v_1, a_1, v_2, a_2, \ldots, a_n, (v_{n+1} = v_1)$ is called *non-periodic closed walk* if the following conditions are satisfied: $a_i = (v_i, v_{i+1}), a_i \in \{a_e, a_e^{-1} : e \in E\}, a_i \neq a_{i+1}^{-1}$ and $(a_1, \ldots, a_n) \neq Z^m$ for some sequence Z and $m > 1$. We let $X(p) = \prod_{i=1}^n x_{a_i}$. We further let $\text{sign}(p) = (-1)^{1+n(p)}$, where $n(p)$ is a *rotation number* of p, i.e. the number of integral revolutions of the tangent vector. Finally let $W(p) = \text{sign}(p)X(p)$.

Now assume that p is a closed aperiodic walk in 3D cubic lattice Q. Can we define its rotation number? The following construction provides a solution.

Assume that some vertices of a planar graph G embedded in the plane of degree 4 are marked. Then we call a walk p *correct* if it satisfies the 'crossover condition' at each marked vertex, i.e. it never enters and exits a marked vertex in a pair of neighbouring edges along the vertex.

We naturally get a planar graph from Q if we consider Q drawn as bold $Q(3,3,2)$ in Fig. 3 and put a new vertex to each edge-crossing. These new vertices will form the set of the marked vertices. Clearly there is a natural bijection which associates to each walk p of Q the corresponding correct walk p' in the new graph and we define rotation of p to be equal to rotation of p'. This defines $W(p)$ for each closed aperiodic walk p in Q.

There is a natural equivalence on non-periodic closed walks: p is equivalent to reversed p. Each equivalence class has two elements and will be denoted by $[p]$. We let $W([p]) = W(p)$ and note that this definition is correct since the equivalent walks have the same sign.

We denote by $\prod(1 + W([p]))$ the formal infinite product of $(1 + W([p]))$ over all equivalence classes of non-periodic closed walks of G.

Let D be an orientation of Q. We further let

$$\prod_D (1 + W([p])) = \prod (1 + W([p]))|_{x_e = \text{sign}(D,e)\alpha^{w_e}}.$$

Theorem 3. *The generating function of even subgraphs is*

$$\mathcal{H}(Q, \alpha) = 2^{-C} \sum_D (-1)^{|\{(e,e') \in R(Q); \text{sign}(D,e) = \text{sign}(D,e') = -1\}|} \prod_D (1 + W([p])),$$

where the sum is over the orientations of Q with all the vertical edges positive.

Remark. Theorem 3 expresses the 3D Ising partition function as a linear combination of infinite products over aperiodic closed walks. For a planar lattice this reduces to just one infinite product (conjectured by Feynman, proved in [6]) which provides an equivalence of the planar Ising model with the quantum field theory of free fermions. There have been several attempts to generalize this equivalence for 3D. Earlier papers (e.g. [7]) attempt to replace closed aperiodic walks by 2D surfaces. Recent promising developments mentioned in the introduction deal with discrete Riemann surfaces and the theta function.

It is known that the planar Ising partition function in the thermodynamical limit behaves like a 'common term' times a Riemann theta function. For a torus, [2] asserts that in the thermodynamic limit and near to criticality, the Ising partition function behaves like a 'common term' times a linear combination of four Riemann theta functions corresponding to the torus. In [5,3,4], there is evidence that for the critical Ising model, the dependence of the determinants of adjacency matrices on the Kasteleyn orientations is exactly the same as the dependence of the determinants of the Dirac operator, of the corresponding conformal field theory, on the spin structures of the 2D Riemann surface; this is given in terms of theta functions of half-integer characteristics.

We believe that Theorem 3 may provide a new insight into these efforts.

4. Drawing cubic lattices

In this section we describe how we draw cubic lattices on 2D surfaces.

4.1. Theory of generalized g-graphs

Definition 4.1. A *surface polygonal representation* S_g of an orientable 2D surface of genus g consists of a *base* B_0 and $2g$ *bridges* B^i_j, $i = 1, \dots, g$ and $j = 1, 2$, where

(i) B_0 is a convex $4g$-gon with vertices $a_1, \dots, a_{4g}$ numbered clockwise;
(ii) B^i_1, $i = 1, \dots, g$, is a 4-gon with vertices $x^i_1, x^i_2, x^i_3, x^i_4$ numbered clockwise. It is glued with B_0 so that the edge $[x^i_1, x^i_2]$ of B^i_1 is identified with the edge $[a_{4(i-1)+1}, a_{4(i-1)+2}]$ of B_0 and the edge $[x^i_3, x^i_4]$ of B^i_1 is identified with the edge $[a_{4(i-1)+3}, a_{4(i-1)+4}]$ of B_0;
(iii) B^i_2, $i = 1, \dots, g$, is a 4-gon with vertices $y^i_1, y^i_2, y^i_3, y^i_4$ numbered clockwise. It is glued with B_0 so that the edge $[y^i_1, y^i_2]$ of B^i_2 is identified with the edge $[a_{4(i-1)+2}, a_{4(i-1)+3}]$ of B_0 and the edge $[y^i_3, y^i_4]$ of B^i_2 is identified with the edge $[a_{4(i-1)+4}, a_{4(i-1)+5}]$ of B_0. Indexing is mod $4g$.

Definition 4.2. A graph G is called a g-*graph* if it is embedded on S_g so that all the vertices belong to the base B_0, and the embedding of each edge uses at most one bridge. We denote the set of the edges embedded entirely on the base by E_0 and the set of the edges embedded on each bridge B^i_j by E^i_j, $i = 1, \dots, g$, $j = 1, 2$.
Moreover the following conditions need to be satisfied.

1. The outer face of $G_0 = (V, E_0)$ is a cycle, and it is embedded on the boundary of B_0,
2. If $e \in E^i_1$ then e is embedded entirely on B^i_1 with one end-vertex belonging to $[x^i_1, x^i_2]$ and the other one to $[x^i_3, x^i_4]$. Analogously for $e \in E^i_2$.

We need a generalization of the notion of a g-graph.

Definition 4.3. Any graph G obtained by the following construction will be called *generalized g-graph*.

1. Let $g = g_1 + \cdots + g_n$ be a partition of g into positive integers.
2. Let S_{g_i} be a polygonal representation of a surface of genus g_i, $i = 1, \ldots, n$. Let us denote the basis and the bridges of S_{g_i} by B_0^i and $B_{j,k}^i$, $i = 1, \ldots, n$, $j = 1, \ldots, g_i$ and $k = 1, 2$.
3. For $i = 1, \ldots, n$ let H_i be a g_i-graph with the property that the subgraph of H_i embedded on B_0^i is a cycle, embedded on the boundary of B_0^i. Let us denote it by C^i.
4. Let G_0 be a 2-connected graph properly embedded on the plane and let $F_1, \ldots, F_n$ be a subset of the faces of G_0. Let K^i be the cycle bounding F_i, $i = 1, \ldots, n$. Let each K^i be isomorphic to C^i.
5. Then G is obtained by gluing the H_i's into G_0 so that each K^i is identified with C_i.

Orientations. Let G be a g-graph and let $G_j^i = (V, E_0 \cup E_j^i)$. An orientation D_0 of G_0 such that each inner face of each 2-connected component of G_0 is clockwise odd in D_0 is called a *basic orientation* of G_0. Note that a basic orientation always exists for a planar graph. Further we define the orientation D_j^i of each G_j^i as follows: We consider G_j^i embedded on the plane by the planar projection of E_j^i outside B_0, and complete the basic orientation D_0 of G_0 to an orientation of G_j^i so that each inner face of each 2-connected component of G_j^i is clockwise odd. The orientation $-D_j^i$ is defined by complete reversing of the orientation D_j^i of G_j^i.

Observe that after fixing a basic orientation D_0, the orientation D_j^i is uniquely determined for each i, j.

Definition 4.4. Let G be a g-graph, $g \geq 1$. An orientation D of G which equals the *basic orientation* D_0 on G_0 and which equals D_j^i or $-D_j^i$ on E_j^i is called *relevant*. We define its *type* $r(D) \in \{+1, -1\}^{2g}$ as follows: For $i = 0, \ldots, g-1$ and $j = 1, 2$, $r(D)_{2i+j}$ equals $+1$ or -1 according to the sign of D_j^{i+1} in D.

Moreover we let $c(r(D))$ equal the product of c_i, $i = 0, \ldots, g-1$, where $c_i = c(r_{2i+1}, r_{2i+2})$ and $c(1, 1) = c(1, -1) = c(-1, 1) = 1/2$ and $c(-1, -1) = -1/2$.

Observe that $c(r(D)) = (-1)^n 2^{-g}$, where $n = |\{i; r_{2i+1} = r_{2i+2} = -1\}|$.

For each generalized g-graph G we define 4^g relevant orientations $D_1, \ldots, D_{4^g}$ with respect to a fixed basic orientation of G_0, and coefficients $c(r(D_i))$, $i = 1, \ldots, n$ in the same way as for a g-graph. Now we write theorem proved in [10] which will be essential in proof of Theorem 1.

Theorem 4. *Let G be a generalized g-graph with a perfect matching M_0 of G_0. Let D_0 be a basic orientation of G_0. If we order the vertices of G so that $s(D_0, M_0)$ is positive then*

$$\mathcal{P}(G, \alpha) = \sum_{i=1}^{4^g} c(r(D_i)) Pf_G(D_i, \alpha),$$

where D_i are all the relevant orientations of G.

The definition of $s(D_0, M_0)$ follows from the definition of the *Pfaffian*.

Let $G = (V, E)$ be a graph with $2n$ vertices and D an orientation of G. Denote by $A(D)$ the skew-symmetric matrix with the rows and the columns indexed by V, where $a_{uv} = \alpha^{w(u,v)}$ when (u, v) is an arc of D, $a_{u,v} = -\alpha^{w(u,v)}$ when (v, u) is an arc of D, and $a_{u,v} = 0$ otherwise.

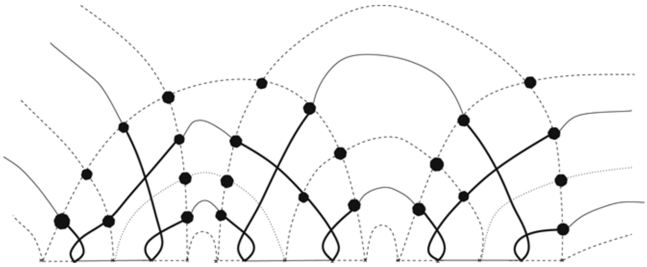

Fig. 4. Construction of Q' for part of the plane W_x (x odd), $V_{x(y-1)}$, V_{xy}, $V_{x(y+1)}$ (y even).

Definition 4.5. The *Pfaffian* is defined as

$$Pf_G(D,\alpha) = \sum_P s^*(P) a_{i_1 j_1} \cdots a_{i_n j_n},$$

where $P = \{\{i_1 j_1\}, \ldots, \{i_n j_n\}\}$ is a partition of the set $\{1, \ldots, 2n\}$ into pairs, $i_k < j_k$ for $k = 1, \ldots, n$, and $s^*(P)$ equals the sign of the permutation $i_1 j_1 \ldots i_n j_n$ of $12 \ldots (2n)$.

Each nonzero term of the expansion of the Pfaffian equals $\alpha^{w(P)}$ or $-\alpha^{w(P)}$ where P is a perfect matching of G. If $s(D, P)$ denote the sign of the term $\alpha^{w(P)}$ in the expansion, we may write

$$Pf_G(D,\alpha) = \sum_P s(D,P) \alpha^{w(P)}.$$

4.2. *Cubic lattices as generalized g-graphs*

In this subsection we will describe how to realize $Q(m_1, m_2, k)$ as a subgraph of a generalized g-graph. Let us denote by Q^L a larger lattice $Q^L(m_1, m_2, 2k+1)$ (Q^L is a cubic lattice with all three indexes odd, we denote $n = 2k+1$).

How to draw Q^L on the plane. First draw the paths V_{xy} along a cycle in the linear order as in Definition 1.1. Next, draw the horizontal edges inside this cycle, and the width edges outside this cycle as depicted in Fig. 3 where $Q^L(3,3,5)$ is properly drawn. The figure also shows in bold how $Q(3,3,2)$ is embedded in $Q^L(3,3,5)$.

We keep the following rule: the interiors of the curves representing h_{xyz} and $h_{(x+1)yz}$ (w_{xyz} and $w_{x(y+1)z}$ respectively) intersect if and only if z is even. We denote by $C(e)$ the curve representing edge e.

Now we modify Q^L into a generalized g-graph Q'. We introduce new vertices to some edge-crossings of Q^L and delete and subdivide some edges. In this way we obtain a generalized g-graph Q' with an even subdivision of $Q(m_1, m_2, k)$ as its subgraph. The construction is analogous and independent for each plane, and so we describe it only for a plane W_x. Let us define $W_{xy} = \{w_{xyz}; z = 1, \ldots, n\}$.

The construction is described by Fig. 4 for the edges between $V_{x(y-1)}$, V_{xy} and between V_{xy}, $V_{x(y+1)}$, for x odd and $y < m_2 - 1$ even.

(1) For each y even let $\mathrm{Aux}_1 = \{w_{xyz}; z \text{ odd}\}$. For each edge e of Aux_1 introduce a new vertex to each intersection of $C(e)$ with the curves representing the edges of $W_{x(y-1)} \cup W$, where $W = W_{x(y+1)}$ in the case $y < m_2 - 1$ and $W = \emptyset$ otherwise. By this operation, each $e \in Aux_1$ is replaced by a path. Call each edge of this path *auxiliary*.

(2) For each y even let $\mathrm{Aux}_2 = \{w_{x(y-1)1}, w_{x(y-1)n}\} \cup A$, where $A = \{w_{x,(y+1)1}, w_{x(y+1)n}\}$ in the case $y < m_2 - 1$ and $A = \emptyset$ otherwise. For each edge e of Aux_2 introduce a new vertex to each intersection of $C(e)$ with the curves representing the edges of W_{xy}. Hence each $e \in Aux_2$ is replaced by a path. Call each edge of this path *auxiliary*. For each y even the edges $v_{xy1}, v_{xy(n-1)}$ and also $v_{x(y+1)1}, v_{x(y+1)(n-1)}$ will also be called *auxiliary*. In Fig. 4, the auxiliary edges are represented by dashed lines.

(3) The edges w_{xyz}, y even and z even will be called *relevant* for Q. If $y < m_2 - 1$ then the relevant edges are subdivided by two vertices (added in 2.) into three edges of Q'. The middle one will be called *special* and the other two *long*. If $y = m_2 - 1$ then the relevant edge w_{xyz} is subdivided by one vertex into two edges of Q'. The one incident to V_{xm_2} will be called *special* and the other one *long*. If e is a relevant edge of Q, then we choose a initial long edge f and we let $w(e) = w(f)$. We let the weight of the special edge and of the remaining long edge be equal to 0.

(4) The edges of $W_{x(y-1)} \cup W$ also got subdivided by new vertices introduced in step 1 and step 2.

(5) We delete all edges of the paths obtained from $w_{x(y-1)z}$ and $w_{x(y+1)z}$, $1 < z < n$ odd, as well as we delete all vertices in intersections of those edges with auxiliary edges. In Fig. 4, the deleted edges are represented by dotted lines.

(6) Each edge $e \in \{w_{x(y-1)z}, w_{x(y+1)z}; z \text{ even}\}$, is subdivided by new vertices introduced in step 1 into a path. We let the weights assigned to the edges of the path equal 0 except of one initial edge whose weight is set equal to $w(e)$. The edge e of this path such that the interior of $C(e)$ does not intersect interior of any curve representing a long edge will also be *special*. The others are called *short*.

(7) All vertical edges which are not auxiliary (see Fig. 4) will be called *special*. In Fig. 4, the special edges are represented by normal lines.

(8) Let Aux denote the set of all auxiliary edges. Then deletion of Aux results in a subdivision of $Q(m_1 m_2 k)$. We subdivide some special edges (vertical and border width ones) so that the graph $\widetilde{Q} = Q' - Aux$ is an even subdivision of $Q(m_1 m_2 k)$. All these new edges will be special, and we set their weights equal 0.

This finishes the construction. In Fig. 4, the long and short edges are represented by fat lines.

Q' is a generalized g-graph. Its planar part Q^p is the set of all the auxiliary and special edges. Other edges (i.e. the short and long) are drawn on a face of Q^p and they may be drawn onto a pair of bridges above this face. One bridge contains one long edge, and the other bridge contains all the short edges.

Now let us set appropriate weights to all the auxiliary edges. In order to do that first subdivide and add some auxiliary edges and vertices to Q^p in such a way that there exists a matching M_{Aux} of $V(Q') - V(\widetilde{Q})$ consisting of auxiliary edges, and a perfect matching M_0 of Q^p. We set weights of edges from M_{Aux} as $w(e) = 0$. Weights of other auxiliary edges are $w(e) = -\infty$. With such properties of Q' we have an important result: $\mathcal{P}(Q', \alpha) = \mathcal{P}(Q(m_1, m_2, k), \alpha)$.

Now we introduce a *basic orientation* D^p of Q^p in order to prepare our stage to use Theorem 4 for the generalized g-graph Q'. Orientation D^p has the following properties:

1. D^p on special edges is in agreement with the natural ordering (Definition 1.1).
2. All the signs $s(D^p, P)$ in Pfaffian $Pf_{Q^p}(D^p, \alpha)$ (Definition 4.5) are positive.
3. The orientation of edges on a bridge has positive sign if and only if it is in agreement with the natural ordering (Definition 1.1).

The construction of such a D^p is possible according to Kasteleyn [11].

5. The proofs

5.1. *Proof of* Theorem 1

We use Theorem 4 for the graph Q'. Then we may write

$$\mathcal{P}(Q(m_1m_2k),\alpha) = \mathcal{P}(Q',\alpha) = \sum_{i=1}^{4^g} c(r(D_i))Pf_{Q'}(D_i,\alpha)$$
$$= \sum_{i=1}^{4^g} c(r(D_i))Pf_{\widetilde{Q}\cup M_{Aux}}(D_i,\alpha) = \sum_{i=1}^{4^g} c(r(D_i))Pf_{Q}(D_i^*,\alpha),$$

where the first equality is the result of previous section. The second equality holds because of Theorem 4. The third equality holds because of the setting of the weights of edges in Q' in the previous section. Also finally, orientation D_i^* being induced on Q by D_i on $\widetilde{Q}$, the fourth equality follows from the three facts: setting of the weights of edges, $\widetilde{Q}$ is even subdivision of Q, it is possible to permute vertices covered by M_{Aux} so that the sign is correct.

In Section 2 we have defined for a bipartite graph G a matrix $Z(D)$ depending on an orientation D. It is easy to check that for $Q\,Pf_Q(D_i^*,\alpha) = \det(Z(D_i^*))$ holds.

To proceed in proof of Theorem 1 we have to establish the dependence of $c(r(D_i))$ on the orientation D_i^* of Q and specify the orientations we sum over in the language of Q not Q'.

Each relevant orientation D_i of Q' is determined by the fixed basic orientation D^p of Q^p, and by a pair of signs for each pair of bridges. Each pair of bridges is associated with one long edge and set of short edges of Q'. Hence these signs may be given by specifying $(d^1_{D_i}(e), d^2_{D_i}(e)) \in \{+-\}^2$, for each long edge e, where $d^1_{D_i}(e)$ denotes the sign of the bridge containing e, and $d^2_{D_i}(e)$ denotes the sign of the other bridge containing the set of short edges.

The relevant edges $w_{x(m_2-1)z}(Q(m_1m_2k))$ and $h_{(m_1-1)yz}(Q(m_1m_2k))$ are associated with only one long edge of Q'. If e is such a relevant edge of Q, we will call it a *border edge*, and we denote by e_1 the corresponding long edge. We let $d_{D_i}(e) = (d^1_{D_i}(e_1), d^2_{D_i}(e_1), +, +)$. Each relevant non-border edge e of Q has two long edges e_1, e_2 associated with it. We let $d_{D_i}(e) = (d^1_{D_i}(e_1), d^2_{D_i}(e_1), d^1_{D_i}(e_2), d^2_{D_i}(e_2))$. A *relevant vector* is any element r of $[\{+,-\}^4]^{\mathcal{R}}$ such that $r(e)_3 = r(e)_4 = +$ for each relevant border edge e of Q ($\mathcal{R}$ is the set of all relevant edges).

From the definition of $c(r(D_i))$ and from properties of the generalized g-graph Q' of genus g it follows that $c(r(D_i)) = 2^{-g}(-1)^n$, where n is number of faces of Q^p above them edges on both bridges are oriented in negative sense, i.e. against fixed ordering. There is a natural bijection between relevant orientations of Q' and relevant vectors. If r is a relevant vector, then let $D_i(r)$ denote the corresponding relevant orientation of Q' and let $\mathrm{sgn}(r)$ of a relevant vector r be calculated as follows: $\mathrm{sgn}(r) = 2^g c(r(D_i)) = (-1)^{|\{(e,i);i=0,1;r(e)_{2i+1}=r(e)_{2i+2}=-1\}|}$.

There are $C_r = km_1(m_2-1)/2 + km_2(m_1-1)/2$ relevant edges and $C_b = k(m_1+m_2)$ border edges. The genus of Q' is therefore $g = 2C_r - C_b$. There are $4^{2C_r-C_b}$ relevant vectors.

Let us consider a set $\mathcal{D}$ of relevant orientations D_i such that there is at least one relevant non-border edge e with $r(e)_2 \neq r(e)_4$, i.e. orientation of bridges with short edges incident to e are different.

Proposition 5.1. *The sum of the contributions of all orientations from $\mathcal{D}$ to $\mathcal{P}(Q,\alpha)$ is zero.*

Proof. For a given orientation $D \in \mathcal{D}$ define a set $A_D = \{e \quad \text{relevant}, r(e)_2 \neq r(e)_4\}$. Now fix a set of relevant edges A_f and consider all orientations D_f such that $A_{D_f} = A_f$. Fix also one relevant edge $e_f \in A_f$.

Remark. $c(r(D_1)) + c(r(D_2)) = 0$ if $r(D_1)$ and $r(D_2)$ differ only in one relevant edge e in such a way that $r_{D_1}(e)_2 = r_{D_2}(e)_2 \neq r_{D_1}(e)_4 = r_{D_2}(e)_4$ and $r_{D_1}(e)_1 = -r_{D_2}(e)_1$, $r_{D_1}(e)_3 = -r_{D_2}(e)_3$. In that case induced orientations are equal $D_1^* = D_2^*$.

Using this remark for $e = e_f$ we have for every A_f:

$$\sum_{D_i \in D_f} c(r(D_i)) Pf_Q(D_i^*, \alpha) = 0.$$

The claim in Proposition 5.1 follows directly. □

Let us call all $D_i \notin \mathcal{D}$ *useful orientations* and from now we sum only over such orientations. There are $2^{3C_r - 2C_b}$ useful orientations. Similarly we define useful relevant vectors.

Proposition 5.2. *Each orientation D of Q, which agrees with the fixed order on vertical edges, corresponds to $2^{C_r - C_b}$ useful orientations D_i of Q' and all those D_i have the same sign $c(r(D_i))$.*

Proof. Let r be a useful vector. Then $D(r)$ determines uniquely $r(e)_2$ and $r(e)_4$ for each relevant edge e and also $r(f)_1$ for each relevant border edge f. Hence $D(r)$ determines uniquely $r(f)$ for each relevant border edge f. Moreover $D(r)$ determines uniquely the product $r(e)_1 \times r(e)_3$ for each relevant non-border edge e. Since there are $C_r - C_b$ relevant non-border edges, there are $2^{C_r - C_b}$ orientations D_i corresponding to it.

Let r, s be useful and both lead to the same D of Q. Then $r(e)_2 = s(e)_2 = s(e)_4 = r(e)_4$ for each relevant non-border edge e and $r(e)_2 = s(e)_2$ and $s(e)_1 = r(e)_1$ for each relevant border edge. This implies that $\text{sgn}(r) = \text{sgn}(s)$. □

Concluding from what we did till now we can write the dimer partition function as

$$\mathcal{P}(Q, \alpha) = 2^{-C_r} \sum_D \text{sgn}(D) \det(Z(D)),$$

where the sum is over all orientations of Q with all vertical edges positive. At this point the only thing missing in the proof of Theorem 1 is to concretize $\text{sgn}(D)$.

Proposition 5.3. $\text{sgn}(D) = (-1)^{|\{(e,e') \in R(Q); \text{sign}(D,e) = \text{sign}(D,e') = -1\}|}$, *where the set of edges $R(Q)$ is defined in 1.2.*

Proof. Recall that $\text{sgn}(D) = (-1)^n$ where n is number of faces where both bridges above them are negative. Each non-border relevant edge e corresponds to two faces with bridges. The contribution of the two corresponding faces to sign can be negative only if the orientation of e is negative ($r(e)_1 \neq r(e)_3$). Moreover if the orientation of corresponding short edges is negative ($r(e)_2 = r(e)_4 = -1$). It follows from the construction that this happens only if product of orientation of all e' (as defined in 1.2) is negative. Considering all the relevant edges we obtain the claim of Proposition 5.3. □

5.2. *Proof of* Theorem 2

Proposition 5.4. *The average of* $\det(Z(D))$ *over all orientations of* Q *with all vertical edges fixed positive equals* $\alpha^{w(M)}$*, where* M *is perfect matching of* Q *consisting only from vertical edges.*

Proof. By the linearity of expectation the contribution of other than vertical edges cancel out when we calculate the average of $\det(Z(D))$. Since $Q(m_1m_2k)$ has exactly one perfect matching consisting of vertical edges only, Proposition 5.4 follows. □

There are 2^{2C} orientations D in Theorem 1 to be summed over. $2^{C-1}(2^C + 1)$ of them have positive sign. Proof: Looking at Fig. 2 we can see that as soon as there is at least one negative relevant (shaded) edge there are as many orientations with positive as with negative sign. Only if all the relevant edges are positive will the sign be also positive. This means there are 2^C more positive orientation than negative.

By Theorem 1 and Proposition 5.4 we have then

$$\mathcal{P}(Q, \alpha) = 2^{-C}[-2^{2C}\alpha^{w(M)} + 2\mathcal{E}(2^{C-1}(2^C + 1))] = -2^C\alpha^{w(M)} + \mathcal{E}(2^C + 1),$$

where $\mathcal{E}$ equals the average of $\det(Z(D))$ over all D of Q with positive sign. Also Theorem 2 is proved. □

5.3. *Proof of* Theorem 3

We use a straightforward reformulation of a theorem of [9]:

Theorem 5. *Let G be a generalized g-graph. Then*

$$\mathcal{H}(G, \alpha) = \sum_{i=1}^{4^g} c(r(D_i)) \prod_{D_i} (1 + W([p])),$$

where D_i *are all the relevant orientations of G, and rotation of* p *and* $\prod_{D_i}(1 + W([p]))$ *are defined in the same way as described for* Q *in Section* 3.

The proof of this theorem can be found in [9] and for planar graphs in [6].

Let $D_i, i = 1, \ldots, 4^g$ be the relevant orientations of generalized g-graph Q'. If we apply Theorem 5 to Q', from the choice of weights we get

$$\mathcal{H}(Q, \alpha) = \sum_{i=1}^{4^g} c(r(D_i)) \prod_{D_i^*} (1 + W([p])).$$

The same analysis as in the previous section finishes the proof of Theorem 3.

6. Conclusions

We expressed the 3D Dimer partition function on a finite lattice as a linear combination of determinants of oriented adjacency matrices in Theorem 1, and the 3D Ising partition function as a linear combination of products over aperiodic closed walks in Theorem 3.

The methodology we use is embedding cubic lattices on 2D surfaces of large genus. By this embedding the cubic lattice is transformed to a generalized g-graph. Then we use theorems from [10,9], which generalize the approach of [11].

We believe this approach may relate to recent developments in conformal field theory [5,3,4] and topological string theory [1], and ultimately teach us something about 3D statistical physics problems.

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