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Topological properties defined by nets ☆

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A R T I C L E   I N F O

Dedicated to L. Michael Brown on the occasion of his 70th birthday

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A B S T R A C T

Nets are used to generalize a result of Murtinová and to define and study a class of properties related to sequential compactness.

1. Introduction

In recent years filters have been preferred to nets for studying convergence and cluster points in set-theoretic topology. Equally, compactness type properties are usually defined in terms of coverings rather than using nets and sequences. However, in a very recent paper [7], Hodel has shown the utility of a certain class of nets which he has called κ-nets and a similar approach was employed earlier in [8]. In this paper we use nets to generalize a result of Murtinová and to define and study properties related to sequential compactness.

Throughout, κ will denote an infinite cardinal. We say that a *net has cardinality* κ if its domain has cardinality κ and that a net is a κ*-net* in a set X if its domain is the set $\kappa^{<\omega}$ directed by inclusion. Clearly a κ-net has cardinality κ. In [7, Lemma 3.7] it was shown that if a net f of cardinality κ converges to $p \in X$ (respectively, has p as a cluster point), then there is a κ-net in the range of f which converges to p (respectively, has p as a cluster point).

A space X is κ-Fréchet (respectively, κ-net) if for each non-closed subset $A \subseteq X$ and each $x \in \mathrm{cl}(A) \setminus A$ (for some $x \in \mathrm{cl}(A) \setminus A$) there is a net of cardinality at most κ in A – or equivalently (see [7, Theorem 3.10]), for some $\lambda \leqslant \kappa$, a λ-net in A – which converges to x. The set A together with the limits of all such nets whose range is in A is called the κ*-net closure of* A, denoted by $\mathrm{cl}^\kappa(A)$. Clearly a space is κ-Fréchet if and only if $\mathrm{cl}^\kappa(A) = \mathrm{cl}(A)$ for each $A \subseteq X$.

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It is clear that if X is κ-Fréchet then it is also κ-net and is λ-Fréchet for each $\lambda \geqslant \kappa$. Every space X is a κ-net space and a λ-Fréchet space for some values of κ and λ. The minimum cardinal κ such that X is κ-net, denoted by $\sigma(X)$, is called the *net character of* X and the minimum cardinal λ such that X is λ-Fréchet is called the *Fréchet net character* of X and will be denoted by $\sigma_F(X)$. Clearly $\sigma(X) \leqslant \sigma_F(X) \leqslant \chi(X)$. An ω-net space is usually called a *sequential space* and an ω-Fréchet space is simply a *Fréchet space*. As usual, $\chi(p, X)$ denotes the minimal cardinality of a local base at p, $\chi(X) = \sup\{\chi(p, X)\colon\ p \in X\}$ and when $p \in \mathrm{cl}(A)$, $t(p, A) = \min\{|B|\colon\ B \subseteq A,\ \text{and } p \in \mathrm{cl}(B)\}$. All other notation is standard and undefined terms can be found in [6]. All topologies are assumed to be (at least) Hausdorff.

2. Compactness-type properties defined by κ-nets

If $\kappa \geqslant \lambda$, a κ-net g is a κ-subnet (a *finer κ-net* in the notation of [6, 1.6]) of a λ-net f if there is $\phi : \kappa^{<\omega} \to \lambda^{<\omega}$ such that $g = f \circ \phi$ and for each $F_0 \in \lambda^{<\omega}$ there is $G_0 \in \kappa^{<\omega}$ such that if $G \supseteq G_0$, then $\phi(G) \supseteq F_0$; if $\lambda = \kappa$, a map ϕ with these properties will be called a *λ-net map*. A map $\phi : \lambda^{<\omega} \to \lambda^{<\omega}$ will be said to be *expansive* if for all $F \in \lambda^{<\omega}$, $\phi(F) \supseteq F$ and (as usual) *monotone* if $F \subseteq G$ implies that $\phi(F) \subseteq \phi(G)$. It is easy to see that a monotone, expansive map is a λ-net map and a composition of monotone and expansive maps is monotone and expansive.

Lemma 2.1. *Suppose that $f : \lambda^{<\omega} \to X$ is a λ-net in X and ϕ is a λ-net map such that the λ-subnet $f \circ \phi$ converges to $p \in X$. Then there is a monotone, expansive map ψ such that $f \circ \psi$ converges to p.*

Proof. The definition of $\psi(F)$ is by recursion. Let $\psi(\emptyset) = \emptyset$. Now suppose that for some $n \in \omega$ and all sets G of cardinality at most $n - 1$ we have defined $\psi(G)$ and suppose that $F \in \lambda^{<\omega}$ has cardinality n. Then, since ϕ is a λ-net map, we have:

(1) There is $H_F \in \lambda^{<\omega}$ such that for all $H \supseteq H_F$, $\phi(H) \supseteq F$.
(2) For each $G \subsetneq F$ there is $K_G \in \lambda^{<\omega}$ such that if $H \supseteq K_G$, then $\phi(H) \supseteq \psi(G)$.

Now define $L_F \in \lambda^{<\omega}$ as

$$L_F = F \cup H_F \cup \bigcup\{K_G\colon\ G \subsetneq F\}$$

and

$$\psi(F) = \phi(L_F).$$

This defines $\psi(F)$ for all $F \in \lambda^{<\omega}$ by recursion. Now, since $L_F \supseteq H_F$, we have from (1) that $\phi(L_F) \supseteq F \Rightarrow \psi(F) \supseteq F$ and hence ψ is expansive. Furthermore, whenever $G \subsetneq F$, we have from (2) that $L_F \supseteq K_G$ and so $\phi(L_F) \supseteq \psi(G) \Rightarrow \psi(F) \supseteq \psi(G)$, showing that ψ is monotone.

Finally, to show that $f \circ \psi$ converges to p, suppose that V is a neighborhood of p. Then, since $f \circ \phi$ converges to p, there is some $F_0 \in \lambda^{<\omega}$ such that if $F \supseteq F_0$, then $f(\phi(F)) \in V$. Thus if $F \supseteq F_0$, we have that $L_F \supseteq F \supseteq F_0$ and so $f(\psi(F)) = f(\phi(L_F)) \in V$ showing that $f \circ \psi$ converges to p. □

As mentioned in the introduction, it was shown in [7] (as part of an even stronger result) that if f is a net of cardinality κ which converges to p, then there is in the range of f a κ-net which converges to p. What was not explicitly stated in that article is the following slightly stronger result, which will be important later in this section.

Lemma 2.2. *If $f : (D, \preccurlyeq) \to X$ is a net of cardinality κ converging to p, then there is a κ-subnet of f which converges to p.*

Proof. For simplicity of notation, we identify D with κ and define $\psi : \kappa^{<\omega} \to D$ by $\psi(F) = \alpha$ where α is chosen so that $\beta \preccurlyeq \alpha$ for all $\beta \in F$. That $h = f \circ \psi$ is a subnet of f (and hence converges to p) follows from the fact that if $\gamma \in D$ and $F \supseteq \{\gamma\}$, then $\psi(F) \succcurlyeq \gamma$. □

It is easy to prove that a space is initially κ-compact (see [10]) if and only if every κ-net (or equivalently, every net of cardinality κ) has a cluster point if and only if each κ-net (or each net of cardinality κ) has a convergent λ-subnet for some cardinal λ. It may happen, as in the case of $\beta\omega$, λ is necessarily strictly larger than κ. This fact motivates the following definition: We say that a space is *strongly κ-compact* if for each $\lambda \leqslant \kappa$, each net of cardinality λ has a convergent subnet of cardinality at most λ (and hence by an argument very similar to that of [7, Theorem 3.8] it has a convergent subnet of cardinality λ). The following result is an immediate consequence of Lemma 2.2 and the preceding remarks.

Theorem 2.3. *A space is strongly κ-compact if and only if for each $\lambda \leqslant \kappa$, every λ-net has a convergent λ-subnet.* □

Since every countable net has a subnet which is a sequence, a space is strongly ω-compact if and only if it is sequentially compact. The space $\beta\omega$ is not strongly κ-compact for any κ and the ordinal ω_2 with the order topology is strongly ω_1-compact but not strongly ω_2-compact. The following lemma has a standard proof.

Lemma 2.4. *If X is an initially κ-compact space and $\chi(X) \leqslant \kappa$, then each net in X of cardinality at most κ has a convergent subnet of cardinality at most κ.*

Proof. Suppose that $f : (D, \preccurlyeq) \to X$ is a net of cardinality at most κ in X. Since X is initially κ-compact, f has a cluster point p say. Let $\mathcal{B}$ be a local base at p of cardinality at most κ. Let $E = \{(U, d)\colon\ U \in \mathcal{B},\ d \in D \text{ and } f(d) \in U\}$ and define a direction $\leqslant$ on E by $(U_1, d_1) \leqslant (U_2, d_2)$ if and only if $U_1 \supseteq U_2$ and $d_1 \preccurlyeq d_2$. As in [6, 1.6.1], the net $g = f \circ \phi$ where $\phi : E \to D$ is defined by $\phi(U, d) = d$, is a subnet of f of cardinality at most κ which converges to p. $\square$

The well-known result that each first countable, countably compact space is sequentially compact is an immediate corollary of the previous lemma. However, if κ is uncountable it is not possible to conclude in the previous lemma that X is strongly κ-compact – $\beta\omega$ is a relevant counterexample. The following is true however.

Corollary 2.5. *If X is an initially κ^+-compact (in particular, if X is compact), strongly κ-compact space and $\chi(X) \leqslant \kappa^+$, then X is strongly κ^+-compact.* $\square$

The statements of the following lemma have simple standard proofs which we omit.

Lemma 2.6. *Both a closed subspace and a continuous image of a strongly κ-compact space are strongly κ-compact.* $\square$

Theorem 2.7. *If X is a space in which for every non-isolated $p \in X$, $t(p, X \setminus \{p\}) = \chi(p, X)$, then X is strongly κ-compact if and only if it is initially κ-compact.*

Proof. The necessity is obvious, so suppose that X is initially κ-compact and $f : D \to X$ is a net of cardinality $\lambda \leqslant \kappa$ in X. If p is a cluster point of f and f is frequently in $\{p\}$, then the result is obvious. Otherwise, we must have that $t(p, X \setminus \{p\}) \leqslant \lambda$ and so there is a local base $\mathcal{B}$ at p of cardinality at most λ. Let $\mathcal{P} = \{(d, U)\colon\ d \in D,\ U \in \mathcal{B} \text{ and } f(d) \in U\}$ with direction $(d_1, U_1) \leqslant (d_2, U_2)$ if and only if $d_1 \leqslant d_2$ and $U_1 \supseteq U_2$; the net $g : \mathcal{P} \to X$ defined by $g(d, U) = f(d)$ is a subnet of f of cardinality λ which converges to p. $\square$

Corollary 2.8. *A first countable space is strongly κ-compact if and only if it is initially κ-compact.* $\square$

Countable compactness and sequential compactness are equivalent in the class of generalized orderable or *GO-spaces* and while a *GO*-space $(X, <, \tau)$ does not necessarily satisfy the hypothesis of Theorem 2.7, what is true is that for each $p \in X$ which is not isolated in $[p, \rightarrow)$, $t(p, (p, \rightarrow)) = \chi(p, [p, \rightarrow))$ and similarly in $(\leftarrow, p]$. This leads to the following result.

Theorem 2.9. *A GO-space is strongly κ-compact if and only if it is initially κ-compact.*

Proof. Again the necessity is clear. For the sufficiency, suppose that $(X, <, \tau)$ is a *GO*-space and $p \in X$. Suppose that $f : D \to X$ is a net of cardinality $\lambda \leqslant \kappa$ in X and p is a cluster point of f which is not in the range of f. The net f is frequently in $(p, \rightarrow)$ or frequently in $(\leftarrow, p)$, so assume the former. We define a net $g : D \to X$ as follows: Let $q \in (p, \rightarrow)$ and

$$g(d) = \begin{cases} f(d) & \text{if } f(d) > p; \\ q & \text{if } f(d) < p. \end{cases}$$

It is easy to see that p is a cluster point of g and since $t(p, (p, \rightarrow)) = \chi(p, [p, \rightarrow))$ we can apply the previous theorem to obtain the required result. $\square$

Recall that a T_1-space is *scattered* if every non-empty subspace has an isolated point. For later use, note that if $f : (D, \leqslant) \to X$ is a net of cardinality at most κ which is frequently in $A \subseteq X$ (that is to say, for each $d_0 \in D$, there is $d \geqslant d_0$ such that $f(d) \in A$), then we can define $D_A = \{d \in D\colon\ f(d) \in A\}$ with the relation $\leqslant$ inherited from D. It is straightforward to show that $(D_A, \leqslant)$ is directed and that if $i : D_A \to D$ denotes the identity map, then $f \circ i : D_A \to X$ is a subnet of f (of cardinality at most κ).

It was shown in [2] that countable compactness and sequential compactness are also equivalent in the class of scattered T_3-spaces. The next theorem generalizes this result.

Theorem 2.10. *An initially κ-compact scattered T_3-space is strongly κ-compact.*

Proof. Let X be scattered; we first show that every point has a neighborhood which is strongly κ-compact. Suppose that

$$A = \{x \in X\colon\ x \text{ has no strongly } \kappa\text{-compact neighborhood}\};$$

we will show that $A = \emptyset$. If not, then since A is scattered, it has an isolated point $z \in A$ say and then z has a closed neighborhood V in X such that $V \cap A = \{z\}$. We show that V is strongly κ-compact which gives a contradiction. Suppose that $f : (D, \leqslant) \to V$ is a net of cardinality at most κ in V which we assume to be not convergent. Since V is initially κ-compact, f has a cluster point $y \in V$; if $y \neq z$, then since y has a strongly κ-compact neighborhood, we are done. So suppose that z is a cluster point of f; then there is some open neighborhood $W \subseteq V$ of z such that f is frequently in $A = V \setminus \mathrm{cl}(W)$. Applying the remarks prior to this theorem, since $V \setminus W$ is initially κ-compact, the subnet $f \circ i : D_A \to V$ has an accumulation point in $V \setminus W$ and we are reduced to the previous case.

Finally note that if $f : (D, \leqslant) \to X$ is a net of cardinality κ in X, then since X is initially κ-compact, f has an accumulation point, x say, and x has a closed strongly κ-compact neighborhood U. Again by the remarks preceding this theorem, $f \circ i : (D_U, \leqslant) \to X$ is a subnet of f in U of cardinality at most κ which in its turn has a convergent subnet of cardinality at most κ. □

Recall that a space X is *weakly Whyburn* (respectively, *Whyburn*) if whenever $A \subseteq X$ is not closed, there is $B \subseteq A$ such that $|\mathrm{cl}(B) \setminus A| = 1$ (respectively, for each $x \in \mathrm{cl}(A) \setminus A$ there is $B \subseteq A$ such that $\mathrm{cl}(B) \setminus A = \{x\}$). A Whyburn (respectively, weakly Whyburn) space was formerly called an AP-space (respectively, WAP-space). Every Fréchet T_2-space is Whyburn and each sequential T_2-space is weakly Whyburn. Scattered T_3-spaces are weakly Whyburn and a countably compact, weakly Whyburn Hausdorff space is sequentially compact (see [3]), thus we ask:

Question 2.11. Is it true that an initially κ-compact weakly Whyburn space is strongly κ-compact?

It is a simple exercise to show that the product of two strongly κ-compact spaces is strongly κ-compact, but to show that this property is preserved under countably infinite products requires a little more work. Here is where Lemma 2.1 and Theorem 2.3 come in useful.

Theorem 2.12. *A countable product of strongly κ-compact spaces is strongly κ-compact.*

Proof. Suppose that for each $n \in \omega$, X_n is strongly κ-compact and let $X = \Pi\{X_n\colon n \in \omega\}$. Let $f : \lambda^{<\omega} \to X$ be a λ-net in X for some $\lambda \leqslant \kappa$, then $\pi_1 \circ f$ is a λ-net in X_1 which has a λ-subnet, $\pi_1 \circ f \circ \phi_1 : \lambda^{<\omega} \to X_1$, convergent to p_1 say, where ϕ_1 is a λ-net map. The net $\pi_2 \circ f \circ \phi_1 : \lambda^{<\omega} \to X_2$ is a λ-net in X_2 which in its turn has a λ-subnet $\pi_2 \circ f \circ \phi_1 \circ \phi_2 : \lambda^{<\omega} \to X_2$, convergent to p_2, say. Continuing thus, we obtain for each $n \in \omega$, λ-nets, $\pi_1 \circ f \circ \Phi_1$, $\pi_2 \circ f \circ \Phi_2, \ldots, \pi_n \circ f \circ \Phi_n, \ldots$ where for each $n \in \omega$, $\Phi_n = \phi_1 \circ \phi_2 \circ \cdots \circ \phi_n$ and $\pi_n \circ f \circ \Phi_n$ converges to $p_n \in X_n$.

By Lemma 2.1 we may assume that each of the λ-net maps ϕ_n is monotone and expansive. Now if $F \in \lambda^{<\omega}$, then $|F| = n$ for some $n \in \omega$ and we define $\Phi : \lambda^{<\omega} \to \lambda^{<\omega}$ by $\Phi(F) = \Phi_n(F)$. We claim that Φ is a monotone, expansive λ-net map and that $f \circ \Phi$ is a convergent λ-subnet of f. To prove the claim, note first that each map Φ_n, being a composition of monotone and expansive maps, has the same property; the map Φ is monotone, since if $G \subseteq H$, say $|G| = n$ and $|H| = n + k$, then

$$\begin{aligned}\Phi(H) &= \Phi_{n+k}(H) = \phi_1 \circ \cdots \circ \phi_n \circ \cdots \circ \phi_{n+k}(H)\\ &= \Phi_n \circ \phi_{n+1} \circ \cdots \circ \phi_{n+k}(H) \supseteq \Phi_n(H) \supseteq \Phi_n(G) = \Phi(G).\end{aligned}$$

The map Φ is also expansive since if $G \in \lambda^{<\omega}$, say $|G| = m$, then $\Phi(G) = \Phi_m(G) \supseteq G$. Thus $f \circ \Phi$ is a λ-subnet of f and it remains only to show that $f \circ \Phi$ is convergent in X and for this, it suffices to show that for each $n \in \omega$, $\pi_n \circ f \circ \Phi$ converges to p_n. So suppose that U is a neighborhood of p_n; since $\pi_n \circ f \circ \Phi_n$ converges to p_n, there is some $G \in \lambda^{<\omega}$ such that whenever $H \supseteq G$, $\pi_n \circ f \circ \Phi_n(H) \in U$. Then if $H \supseteq G$ and $|H| = m \geqslant n$; we have that

$$\pi_n \circ f \circ \Phi(H) = \pi_n \circ f \circ \Phi_m(H) = \pi_n \circ f \circ \Phi_n \circ \phi_{n+1} \circ \cdots \circ \phi_m(H)$$

and the result follows since $\phi_{n+1} \circ \cdots \circ \phi_m(H) \supseteq H \supseteq G$. □

The following result has a simpler proof.

Theorem 2.13. *If X is initially κ-compact and Y is strongly κ-compact, then $X \times Y$ is initially κ-compact.*

Proof. Suppose that $f : D \to X \times Y$ defined by $f(d) = (x_d, y_d)$ is a net of cardinality $\lambda \leqslant \kappa$ in $X \times Y$. Since Y is strongly κ-compact, there is a subnet $\pi_Y \circ f \circ \phi : E \to Y$ of $\pi_Y \circ f$ with $|E| \leqslant \lambda$ which converges to $y \in Y$, say. Since X is initially κ-compact, the corresponding subnet $\pi_X \circ f \circ \phi : E \to X$ has an accumulation point $x \in X$ and it is straightforward to prove that (x, y) is an accumulation point of the original net $f : D \to X \times Y$. □

We say that a space is *locally strongly κ-compact* if each point has a closed neighborhood which is strongly κ-compact. In a T_3-space which is locally strongly κ-compact, each point has a local base of closed neighborhoods which are strongly κ-compact. Furthermore, in a κ-net space, an initially κ-compact subspace is necessarily closed.

The product of a Fréchet–Urysohn fan and a convergent sequence is not Fréchet and hence the product of two ω-Fréchet spaces, one of which is compact and sequentially compact need not be ω-Fréchet. The product of two arbitrary Fréchet spaces need not even have countable tightness (see [1]); however, we have the following result.

Theorem 2.14. *The product of two κ-net spaces, one of which is T_3 and locally strongly κ-compact is a κ-net space.*

Proof. Suppose that X, Y are κ-net spaces where Y is T_3 and locally strongly κ-compact and $A \subseteq X \times Y$ is such that $\mathrm{cl}(A) \setminus A \neq \emptyset$, say $(x, y) \in \mathrm{cl}(A) \setminus A$. We will construct a net of cardinality κ in A which converges out of A. Let K be a closed strongly κ-compact neighborhood of y; we consider the space $X \times K$. If $(x, y) \in \mathrm{cl}(A \cap (\{x\} \times K))$, then since K is closed in Y, K is a κ-net space and so there is by [7, Lemma 3.7], some κ-net in $A \cap (\{x\} \times K)$ converging to a point $(x, p) \notin A$ and we are done. Thus we may assume that $(x, y) \notin \mathrm{cl}(A \cap (\{x\} \times K))$ and hence there is a closed (hence κ-net) strongly κ-compact neighborhood U of y such that $U \cap (A \cap (\{x\} \times K)) = \emptyset$. Thus we are reduced to the case in which Y is strongly κ-compact, $(x, y) \in \mathrm{cl}(A) \setminus A$ and $A \cap (\{x\} \times Y) = \emptyset$.

Since X is a κ-net space and $\pi_X(A)$ is not closed, there is some κ-net $f : \kappa^{<\omega} \to \pi_X(A)$ which converges to a point $p \notin \pi_X(A)$. For each $F \in \kappa^{<\omega}$, let $f(F) = x_F$ and choose $y_F \in Y$ so that $(x_F, y_F) \in A$. Then $g : \kappa^{<\omega} \to \pi_Y(A)$ defined by $g(F) = y_F$ is a κ-net in Y and since Y is strongly κ-compact, this net has a subnet of cardinality at most κ, $g \circ \phi : D \to \pi_Y(A)$, where $\phi : D \to \kappa^{<\omega}$ and $|D| \leqslant \kappa$, which converges to $q \in Y$, say. The net $h : D \to X \times Y$ defined by $h(d) = ((f \circ \phi)(d), (g \circ \phi)(d))$ is a net of cardinality κ in A which converges to $(p, q) \notin A$. □

As corollaries, we have two theorems of Boehme (see [4] or [6, 3.3J and 3.10J]); note that by definition, a locally sequentially compact space is T_3.

Corollary 2.15. *The product of two sequential spaces, one of which is locally sequentially compact is sequential.* □

Corollary 2.16. *The product of two sequential spaces, one of which is locally countably compact and T_3 is sequential.*

Proof. A countably compact sequential space is sequentially compact. □

Question 2.17. Can the condition locally strongly κ-compact be replaced by locally compact in Theorem 2.14?

3. The κ-Fréchet property in countable spaces

For $f, g \in \omega^\omega$, $f \leqslant^* g$ means that $f(m) \leqslant g(m)$ for cofinitely many $m \in \omega$. A set $\mathcal{F} \subseteq \omega^\omega$ is *dominating* if for each $g \in \omega^\omega$ there is $f \in \mathcal{F}$ such that $g \leqslant^* f$. The minimum cardinality of a dominating family is denoted by $\mathfrak{d}$. It is well known and easy to prove that $\omega_1 \leqslant \mathfrak{d} \leqslant \mathfrak{c}$; more details can be found in [5].

A countable space has character at most $\mathfrak{c}$, but such spaces are not necessarily Whyburn (as defined following Theorem 2.10) even when they are sequential – the Arens–Franklin space of Lemma 3.2 below is such a space. Below we use nets of cardinality κ to generalize [9, Proposition 2] which states that a countable space of character less than $\mathfrak{d}$ is Whyburn.

Theorem 3.1. *If a countable Hausdorff space X is κ-Fréchet for some $\kappa < \mathfrak{d}$, then it is Whyburn.*

Proof. Suppose that $A \subseteq X$ and $x \in \mathrm{cl}(A) \setminus A$ and let $f : D \to A$ be a net of cardinality at most κ which converges to x, where $|D| < \mathfrak{d}$. Let $T_c = \{f(d)\colon\ c \leqslant d \in D\}$. If there is a neighborhood U of x such that for every neighborhood V of x, $(A \cap U) \setminus V$ is finite, then $A \cap U$ is a sequence which converges to x and since X is Hausdorff, $\mathrm{cl}(A \cap U) = (A \cap U) \cup \{x\}$.

Let $\{U_n\colon\ n \in \omega\}$ be such that $\bigcap\{\mathrm{cl}(U_n)\colon\ n \in \omega\} = \{x\}$. Repeatedly applying the argument of the previous paragraph, we may suppose that $S_n = (U_n \setminus U_{n+1}) \cap A$ is infinite for each $n \in \omega$. We identify S_n with the set $\{n\} \times \omega$ and note that each set T_c must have non-empty intersection with an infinite number of the sets S_n, for otherwise, the net is contained in the complement of some open set U_n. Hence for each $c \in D$, we can define a partial function $f_c : \mathrm{dom}(f_c) \to \omega$ in such a way that $(k, f_c(k)) \in U_k \cap T_c$ for each $k \in \omega$ and note that the domain of each function f_c is infinite. Since $|D| = \kappa < \mathfrak{d}$, there is, by [5, Theorem 3.10], a function $g : \omega \to \omega$ such that for all $c \in D$ there exists $n \in \mathrm{dom}(f_c)$ such that $g(n) > f_c(n)$. The set $S = \{(n, m)\colon\ m \leqslant g(n)\}$ is contained in A and for every n, $S \setminus U_n$ is finite; thus x is the only possible cluster point of S. Furthermore, for each $c \in D$, $S \cap T_c \neq \emptyset$ and hence $x \in \mathrm{cl}(S)$. □

Recall that the *Arens–Franklin space* A is the set $(\omega \times (\omega + 1)) \cup \{p\}$ (where $p \notin \omega \times (\omega + 1)$) endowed with the following topology:

The sets ω and $\omega + 1$ have the order topology and $\omega \times (\omega + 1)$ has the product topology; the topology at p is the strongest in order that the sequence $\{(n, \omega)\}_{n \in \omega}$ converges to p.

It is well known that this is a countable sequential space which is not Fréchet and hence is not Whyburn.

Lemma 3.2. *There is no net of cardinality κ, for any $\kappa < \mathfrak{d}$, in $\omega \times \omega$ in the Arens–Franklin space A which converges to p, that is to say, $\sigma_F(A) \geqslant \mathfrak{d}$.*

Proof. Suppose that $|D| = \kappa < \mathfrak{d}$ and $f : (D, \leqslant) \to \omega \times \omega$ is a net in A. We will show that f does not converge to p. For each $c \in D$, let $x_c = f(c)$ and $A_c = \{x_d\colon d \geqslant c\}$. We will construct a neighborhood U of p such that for each $c \in D$, $A_c \nsubseteq U$.

If for some $k \in \omega$ there is $c \in D$ such that $A_c \subseteq \bigcup\{L_n\colon n \leqslant k\}$ (where $L_n = \{n\} \times \omega$), then $U = A \setminus \bigcup\{L_n\colon n \leqslant k\}$ is the required neighborhood of p. Thus we may assume that for each $c \in D$, A_c meets an infinite number of the sets L_n.

Now given $c = c_0 \in D$, $x_{c_0} \in L_{k_0}$ for some $k_0 \in \omega$; since for all $d \geqslant c$, we have $A_d \nsubseteq \bigcup\{L_n\colon n \leqslant k_0\}$, there is some $k_1 > k_0$ and $c_1 > c_0$ such that $x_{c_1} \in L_{k_1}$. Repeating this argument, we obtain a strictly increasing sequence $(k_n)_{n\in\omega}$ of integers and a sequence $(c_n)_{n\in\omega}$ of elements of A_c such that $x_{c_n} \in L_{k_n}$ for each $n \in \omega$. Define a partial function $f_c : \mathrm{dom}(f_c) \to \omega$ by $f_c(k_n) = \pi_2(x_{c_n})$ (where π_2 denotes the projection onto $\omega + 1$). As in Theorem 3.1, since $|D| = \kappa < \mathfrak{d}$, there is a function $g : \omega \to \omega$ such that for all $c \in D$ there exists $n \in \mathrm{dom}(f_c)$ such that $g(n) > f_c(n)$ and we let $U = \{(k, \ell) \in \omega \times \omega\colon \ell > g(k)\} \cup \{p\}$. It is clear that for each $c \in D$, $A_c \nsubseteq U$, and our result is proved. □

Combining the last two results we have:

Corollary 3.3. *The minimum Fréchet net character of a countable space which is not Whyburn is $\mathfrak{d}$; that is, $\mathfrak{d} = \min\{\lambda\colon$ there is a countable Hausdorff space X with $\sigma_F(X) = \lambda$ and which is not Whyburn$\}$.*

Proof. It is easy to see that in the previous lemma, $\chi(p, A) = \mathfrak{d}$ and so $\sigma_F(A) = \mathfrak{d}$. □

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