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The fundamentals of non-singular dislocations in the theory of gradient elasticity: Dislocation loops and straight dislocations

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The fundamental problem of non-singular dislocations in the framework of the theory of gradient elasticity is presented in this work. Gradient elasticity of Helmholtz type and bi-Helmholtz type are used. A general theory of non-singular dislocations is developed for linearly elastic, infinitely extended, homogeneous, and isotropic media. Dislocation loops and straight dislocations are investigated. Using the theory of gradient elasticity, the non-singular fields which are produced by arbitrary dislocation loops are given. 'Modified' Mura, Peach–Koehler, and Burgers formulae are presented in the framework of gradient elasticity theory. These formulae are given in terms of an elementary function, which regularizes the classical expressions, obtained from the Green tensor of the Helmholtz–Navier equation and bi-Helmholtz–Navier equation. Using the mathematical method of Green's functions and the Fourier transform, exact, analytical, and non-singular solutions were found. The obtained dislocation fields are non-singular due to the regularization of the classical singular fields.

1. Introduction

Dislocations play an important role in the understanding of many phenomena in solid state physics, materials science, and engineering. They are the primary carriers of crystal plasticity. Dislocations are line defects which can be straight or curved lines. The internal geometry of generally curved dislocations, in deformed crystals is very complex. In the classical theory of dislocation loops in isotropic materials (DeWit, 1960; Lardner, 1974; Hirth and Lothe, 1982) two key equations are the Burgers formula (Burgers, 1939) for the displacement, and the Peach–Koehler formula (Peach and Koehler, 1950) for the stress. These equations are very important for the interaction between complex arrays of dislocations. The classical description of the elastic fields produced by dislocations is based on the theory of linear elasticity. There is the problem of mathematical singularities at the dislocation core, and an arbitrary core-cutoff radius which must be introduced to avoid divergence. In the classical continuum theory of dislocations (Kröner, 1958; DeWit, 1960; Nabarro, 1967; Lardner, 1974; Mura, 1987; Teodosiu, 1982; Li and Wang, 2008) the concept of Volterra dislocations is used, and the dislocation core is described by a Dirac delta function. This unsatisfactory situation can only be remedied, when the fact that physical dislocations have a finite core region and no singularities exist are taken into account.

As already pointed out by Kröner (1958) and Lothe (1992) the divergence can be avoided when dislocation distributions other than the delta function are used. Lothe (1992) considered a standard core model with constant density of dislocations in a planar strip with width d. However, the value of d remains undetermined and the expressions for the elastic fields are more complicated than their singular counterparts and difficult to use for generally curved dislocations. Moreover, for non-planar configurations the theory becomes much more complex.

In order to remove the singularities of dislocations and to model the dislocation core more realistically, continuum theories of generalized elasticity may be used. A very promising candidate of such a theory is the so-called gradient elasticity theory. The theory of gradient elasticity was originally proposed by Mindlin (1964); Mindlin (1965); and Mindlin and Eshel (1968) (see also Eshel and Rosenfeld, 1970). The correspondence between the strain gradient theory and the atomic structure of materials with the nearest and next nearest interatomic interactions was exhibited by Toupin and Grazis (1964). The original Mindlin theory possesses too many new material parameters. For isotropic materials, Mindlin's theory of first strain gradient elasticity (Mindlin, 1964; Mindlin and Eshel, 1968) involves two characteristic lengths, and Mindlin's theory of second strain gradient elasticity (Mindlin, 1965) possesses four characteristic lengths. The discrete nature of materials is inherently incorporated in the formulations through the characteristic

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lengths. The capability of strain gradient theories in capturing size effects is a direct manifestation of the involvement of characteristic lengths. Simplified versions, which are particular cases of Mindlin's theories, were proposed and used for dislocation modelling. Such simplified gradient elasticity theories are known as gradient elasticity of Helmholtz type (Lazar and Maugin, 2005), with only one material length scale parameter and gradient elasticity of bi-Helmholtz type (Lazar and Maugin, 2006a; Lazar et al., 2006a) which involves two material length scale parameters as new material coefficients. Gradient elasticity is a continuum model of dislocations with core spreading. Non-singular fields of straight dislocations were obtained in the framework of gradient elasticity of Helmholtz type by Gutkin and Aifantis (1999); Lazar and Maugin (2005); Lazar and Maugin (2006a); Lazar et al. (2005) and Gutkin (2000); Gutkin (2006) (see also, Gutkin and Ovid'ko, 2004). Surprisingly enough up until now, not a single work has been done in the direction of non-singular dislocation loops using strain gradient elasticity theory. The reason may be in the expected mathematical complexity of the problem. Such non-singular solutions of arbitrary dislocation loops could be very useful for the so-called discrete dislocation dynamics (e.g. Li and Wang, 2008; Ghoniem et al., 1999).

The aim of this paper is to present non-singular solutions of arbitrary dislocation loops, by using simplified gradient elasticity theories. We present the key-formulae of dislocations loops valid in the framework of gradient elasticity, and also reemphasize straight dislocations in gradient elasticity. The technique of Green functions for the key-formulae is used, and analytical closed-form solutions for the dislocation fields are derived.

The paper is organized as follows. In Section 2, the fundamentals of gradient elasticity of Helmholtz type are given. Dislocation loops and straight dislocations are investigated. In Section 3, the theory of gradient elasticity of bi-Helmholtz type is considered. Dislocation loops and straight dislocations will be examined in this framework. In Section 4, the conclusions are given. All the mathematical and technical details are given in the Appendices.

2. Gradient elasticity of Helmholtz type

A straightforward framework to obtain non-singular fields of dislocations is the so-called theory of gradient elasticity. A simplified theory of strain gradient elasticity is called gradient elasticity of Helmholtz type (Lazar and Maugin, 2005; Lazar and Maugin, 2006a). This gradient elasticity of Helmholtz type is a particular gradient elasticity theory evolving from Mindlin's general gradient elasticity theory (Mindlin, 1964; Mindlin and Eshel, 1968). This theory is also known as dipolar gradient elasticity theory (Georgiadis, 2003), simplified strain gradient elasticity theory (Gao and Ma, 2010a; Gao and Ma, 2010b) and special gradient elasticity theory (Altan and Aifantis, 1997). The theory of gradient elasticity of Helmholtz type is the gradient version of Eringen's theory of nonlocal elasticity of Helmholtz type (Eringen, 1983; Eringen, 2002) which is well-established.

The strain energy density of such a simplified gradient elasticity theory for an isotropic, linearly elastic material has the form (Lazar and Maugin, 2005; Gao and Ma, 2010a)

$$W = \frac{1}{2} C_{ijkl}\beta_{ij}\beta_{kl} + \frac{1}{2}\ell^2 C_{ijkl}\partial_m\beta_{ij}\partial_m\beta_{kl}, \tag{1}$$

where C_{ijkl} is the tensor of elastic moduli with the symmetry properties

$$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk} \tag{2}$$

and it reads for an isotropic material

$$C_{ijkl} = \mu\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big) + \lambda\delta_{ij}\delta_{kl}, \tag{3}$$

where μ and λ are the Lamé moduli. β_{ij} denotes the elastic distortion tensor. If the elastic distortion tensor is incompatible, it can be decomposed as follows

$$\beta_{ij} = \partial_j u_i - \beta^{\mathrm{P}}_{ij}, \tag{4}$$

where u_i and β^{P}_{ij} denote the displacement vector and the plastic distortion tensor, respectively. In addition, ℓ is the material length scale parameter of gradient elasticity of Helmholtz type. For dislocations, ℓ is related to the dislocation core radius and is proportional to a lattice parameter. Due to the symmetry of C_{ijkl}, Eq. (1) is equivalent to

$$W = \frac{1}{2} C_{ijkl}e_{ij}e_{kl} + \frac{1}{2}\ell^2 C_{ijkl}\partial_m e_{ij}\partial_m e_{kl}, \tag{5}$$

where $e_{ij} = 1/2(\beta_{ij} + \beta_{ji})$ is the elastic strain tensor. The condition for non-negative strain energy density, $W \geqslant 0$, gives

$$(2\mu + 3\lambda) \geqslant 0, \quad \mu \geqslant 0, \quad \ell^2 \geqslant 0. \tag{6}$$

The reason that the elastic and plastic distortion tensors are incompatible can be the presence of dislocations. Dislocations cause self-stresses that means stresses caused without the presence of body forces. The dislocation density tensor is defined in terms of the elastic and plastic distortion tensors as follows (e.g. Kröner, 1958)

$$\alpha_{ij} = \epsilon_{jkl}\partial_k\beta_{il} \tag{7}$$

$$\alpha_{ij} = -\epsilon_{jkl}\partial_k\beta^{\mathrm{P}}_{il} \tag{8}$$

and it fulfills the Bianchi identity of dislocations

$$\partial_j\alpha_{ij} = 0, \tag{9}$$

which means that dislocations do not end inside the body. Eq. (9) is a 'conservation' law and shows that dislocations are source-free fields.

From Eq. (1) it follows that the constitutive equations are

$$\sigma_{ij} = \frac{\partial W}{\partial\beta_{ij}} = \frac{\partial W}{\partial e_{ij}} = C_{ijkl}\beta_{kl} = C_{ijkl}e_{kl}, \tag{10}$$

$$\tau_{ijk} = \frac{\partial W}{\partial\partial_k\beta_{ij}} = \frac{\partial W}{\partial\partial_k e_{ij}} = \ell^2 C_{ijmn}\partial_k\beta_{mn} = \ell^2\partial_k\sigma_{ij}, \tag{11}$$

were σ_{ij} are the components of the Cauchy stress tensor, τ_{ijk} are the components of the so-called double stress tensor. It can be seen that ℓ is the characteristic length scale for double stresses. Using Eqs. (10) and (11), Eq. (5) can also be written as (Lazar and Maugin, 2005)

$$W = \frac{1}{2}\sigma_{ij}e_{ij} + \frac{1}{2}\ell^2\partial_k\sigma_{ij}\partial_k e_{ij}. \tag{12}$$

The strain energy density (12) exhibits the symmetry both in σ_{ij} and e_{ij} and in $\partial_k\sigma_{ij}$ and $\partial_k e_{ij}$.

The total stress tensor is given as a combination of the Cauchy stress tensor and the divergence of the double stress tensor

$$\sigma^0_{ij} = \sigma_{ij} - \partial_k\tau_{ijk} = (1 - \ell^2\Delta)\sigma_{ij} \tag{13}$$

and it fulfills the equilibrium condition for vanishing body forces

$$\partial_j\sigma^0_{ij} = \partial_j(\sigma_{ij} - \partial_k\tau_{ijk}) = 0. \tag{14}$$

The stress tensor σ^0_{ij} is called in the notation of Jaunzemis (1967) the polarization of the Cauchy stress σ_{ij}. Due to gradient elasticity of Helmholtz type, Eq. (13) reduces to an inhomogeneous Helmholtz equation where the total stress tensor is the inhomogeneous piece. As pointed out by Lazar and Maugin (2005); Lazar and Maugin (2006a), the total stress tensor may be identified with the singular classical stress tensor. This identifies that the inhomogeneous Helmholtz equation (13) is in full agreement with the equation for the stress proposed by Eringen (1983); Eringen (2002) in his theory of nonlocal elasticity of Helmholtz type.

As shown by Lazar and Maugin (2005); Lazar and Maugin (2006a) the following governing equations for the displacement vector, the elastic distortion tensor, the dislocation density tensor, and the plastic distortion tensor can be derived in the framework of gradient elasticity of Helmholtz type

$$Lu_i = u_i^0, \tag{15}$$

$$L\beta_{ij} = \beta_{ij}^0, \tag{16}$$

$$L\alpha_{ij} = \alpha_{ij}^0, \tag{17}$$

$$L\beta_{ij}^{\mathrm{P}} = \beta_{ij}^{\mathrm{P},0}, \tag{18}$$

where

$$L = 1 - \ell^2\Delta \tag{19}$$

is the Helmholtz operator. The singular fields $u_i^0, \beta_{ij}^0, \alpha_{ij}^0$ and $\beta_{ij}^{\mathrm{P},0}$ are the sources of the inhomogeneous Helmholtz Equation (15)–(18). The Helmholtz Equation (15) and (16) can be further reduced to Helmholtz–Navier equations

$$LL_{ik}u_k = C_{ijkl}\partial_j\beta_{kl}^{\mathrm{P},0}, \tag{20}$$

$$LL_{ik}\beta_{km} = -C_{ijkl}\epsilon_{mlr}\partial_j\alpha_{kr}^0, \tag{21}$$

where $L_{ik} = C_{ijkl}\partial_j\partial_l$ is the differential operator of the Navier equation. For an isotropic material, it reads

$$L_{ik} = \mu\delta_{ik}\Delta + (\mu+\lambda)\partial_i\partial_k. \tag{22}$$

In Eqs. (20) and (21) the sources are now the plastic distortion $\beta_{kl}^{\mathrm{P},0}$ and the dislocation density α_{kr}^0 known from classical elasticity.

The corresponding three-dimensional Green tensor of the Helmholtz–Navier equation is defined by

$$LL_{ik}G_{kj} = -\delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{23}$$

and is calculated as (see Eq. (B.14))

$$G_{ij}(R) = \frac{1}{16\pi\mu(1-\nu)}\left[2(1-\nu)\delta_{ij}\Delta - \partial_i\partial_j\right]A(R), \tag{24}$$

with

$$A(R) = R + \frac{2\ell^2}{R}\left(1 - \mathrm{e}^{-R/\ell}\right) \tag{25}$$

and $R = |\boldsymbol{x}-\boldsymbol{x}'|$. In the limit $\ell \to 0$, the three-dimensional Green tensor of classical elasticity (Mura, 1987; Li and Wang, 2008) is recovered from Eqs. (24) and (25). It is important to note that $A(R)$ can be written as the convolution of R and $G(R)$:

$$A(R) = R * G(R), \tag{26}$$

where $*$ denotes the spatial convolution and G is the three-dimensional Green function of the Helmholtz equation

$$LG = \delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{27}$$

It reads (Wladimirow, 1971)

$$G(R) = \frac{1}{4\pi\ell^2 R}\mathrm{e}^{-R/\ell}. \tag{28}$$

In addition, it holds

$$\Delta\Delta R = -8\pi\delta(\boldsymbol{x}-\boldsymbol{x}'). \tag{29}$$

The function (25) fulfills the relations

$$L\Delta\Delta A(R) = -8\pi\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{30}$$

$$\Delta\Delta A(R) = -8\pi G(R), \tag{31}$$

$$LA(R) = R. \tag{32}$$

Thus, $A(R)$ is the Green function of Eq. (30) which is a Helmholtz-bi-Laplace equation.

Using Eqs. (A.3) and (A.4) for the differentiation of Eq. (24), the explicit form of the three-dimensional Green tensor of the Helmholtz–Navier equation is obtained

$$\begin{aligned} G_{ij}(R) = \frac{1}{16\pi\mu(1-\nu)}\Bigg[&\frac{\delta_{ij}}{R}\bigg((3-4\nu)\left(1-\mathrm{e}^{-R/\ell}\right) \\ &+\frac{1}{R^2}\left(2\ell^2 - \left(R^2+2\ell R+2\ell^2\right)\mathrm{e}^{-R/\ell}\right)\bigg) \\ &+\frac{R_iR_j}{R^3}\left(1-\frac{6\ell^2}{R^2}+\left(2+\frac{6\ell}{R}+\frac{6\ell^2}{R^2}\right)\mathrm{e}^{-R/\ell}\right)\Bigg], \end{aligned} \tag{33}$$

which is non-singular. It is worth noting as a check, that Eq. (33) is in agreement with the corresponding expressions derived by Polyzos et al. (2003) and Gao and Ma (2009) using slightly different approaches.

Note, that the Green tensor (33) gives the non-singular displacement field, $u_i = G_{ij}f_j$ (f_j is the constant value of the magnitude of the point force acting at the arbitrary position $\boldsymbol{x}'$ in an infinite body), of the Kelvin point force problem (e.g. Gurtin, 1972; Mura, 1987; Hetnarski and Ignaczyk, 2004) in the framework of gradient elasticity of Helmholtz type. The original solution of a concentrated force in an infinite body in the context of the classical continuum theory of elasticity was given by Kelvin (1882).

2.1. Dislocation loops

In this subsection, the characteristic fields of dislocation loops in the framework of gradient elasticity theory of Helmholtz type are calculated.

For a general (non-planar or planar) dislocation loop L, the classical dislocation density and the plastic distortion tensors are (e.g. DeWit, 1973a; Kossecka, 1974)

$$\alpha_{ij}^0 = b_i\delta_j(L) = b_i\oint_L \delta(\boldsymbol{x}-\boldsymbol{x}')\mathrm{d}L_j', \tag{34}$$

$$\beta_{ij}^{\mathrm{P},0} = -b_i\delta_j(S) = -b_i\int_S \delta(\boldsymbol{x}-\boldsymbol{x}')\mathrm{d}S_j', \tag{35}$$

where b_i is the Burgers vector of the dislocation line element $\mathrm{d}L_j'$ at $\boldsymbol{x}'$ and $\mathrm{d}S_j'$ is the dislocation loop area. The surface S is the dislocation surface, which is a cap of the dislocation line L. $\delta_j(L)$ is the Dirac delta function for a closed curve L and $\delta_j(S)$ is the Dirac delta function for a surface S with boundary L.

The solution of Eq. (17) can be written as the following convolution integral

$$\alpha_{ij} = G * \alpha_{ij}^0 = b_i\oint_L G(R)\mathrm{d}L_j', \tag{36}$$

where $G(R)$ denotes the three-dimensional Green function of the Helmholtz equation given by Eq. (28). The explicit solution of the dislocation density tensor for a dislocation loop in gradient elasticity is calculated as

$$\alpha_{ij}(\boldsymbol{x}) = \frac{b_i}{4\pi\ell^2}\oint_L \frac{\mathrm{e}^{-R/\ell}}{R}\mathrm{d}L_j', \tag{37}$$

describing a spreading dislocation core distribution. The plastic distortion tensor of a dislocation loop, which is the solution of Eq. (18), is given by the convolution integral

$$\beta_{ij}^{\mathrm{P}} = G * \beta_{ij}^{\mathrm{P},0} = -b_i\int_S G(R)\mathrm{d}S_j'. \tag{38}$$

It reads as

$$\beta^{\mathrm{P}}_{ij}(\boldsymbol{x}) = -\frac{b_i}{4\pi\ell^2}\int_S \frac{\mathrm{e}^{-R/\ell}}{R}\,\mathrm{d}S'_j. \tag{39}$$

Substituting Eq. (39) in Eq. (8) and using the Stokes theorem, we obtain formula (37).

Using the Green tensor (24), and after a straightforward calculation all the generalizations of the Mura, Peach–Koehler, and Burgers formulae towards gradient elasticity can be obtained. Starting with the elastic distortion tensor of a dislocation loop, the solution of Eq. (21) gives the representation as the following convolution integral

$$\beta_{im}(\boldsymbol{x}) = \int_{-\infty}^{\infty} \epsilon_{mnr} C_{jkln} G_{ij,k}(R)\alpha^0_{lr}(\boldsymbol{x}')\mathrm{d}V', \tag{40}$$

where $G_{ij,k} = \partial_k G_{ij}$. Substituting the classical dislocation density tensor of a dislocation loop (34) and carrying out the integration of the delta function, we find the modified Mura formula valid in gradient elasticity

$$\beta_{im}(\boldsymbol{x}) = \oint_L \epsilon_{mnr} b_l C_{jkln} G_{ij,k}(R)\mathrm{d}L'_r. \tag{41}$$

Substitute Eqs. (3) and (24) into Eq. (41) and obtain after rearranging terms

$$\begin{aligned}\beta_{ij}(\boldsymbol{x}) = \frac{1}{8\pi}\oint_L \epsilon_{jnr}\Big[&(b_i\partial_n - b_n\partial_i + b_l\delta_{in}\partial_l)\Delta \\ &+ \frac{1}{1-\nu}(b_n\partial_i\Delta - b_l\partial_i\partial_n\partial_l)\Big]A(R)\mathrm{d}L'_r.\end{aligned} \tag{42}$$

Using the identity

$$\begin{aligned}\epsilon_{rjn}(b_i\partial_n - b_n\partial_i) &= \epsilon_{rjn}\epsilon_{kin}\epsilon_{kst}b_s\partial_t = (\delta_{rk}\delta_{ji} - \delta_{ri}\delta_{jk})\epsilon_{kst}b_s\partial_t \\ &= (\epsilon_{rst}\delta_{ij} - \epsilon_{jst}\delta_{ir})b_s\partial_t = (\epsilon_{rkl}\delta_{ij} - \epsilon_{jkl}\delta_{ir})b_k\partial_l\end{aligned} \tag{43}$$

and the relation

$$\begin{aligned}&\oint_L \epsilon_{rjn}(b_n\partial_l - b_l\partial_n)\partial_l\partial_i A(R)\mathrm{d}L'_r \\ &= \oint_L b_s(\epsilon_{jst}\partial_r - \epsilon_{rst}\partial_j)\partial_t\partial_i A(R)\mathrm{d}L'_r \\ &= \oint_L \mathrm{d}\big(b_s\epsilon_{jst}\partial_t\partial_i A(R)\big) - \oint_L b_s\epsilon_{rst}\partial_t\partial_j\partial_i A(R)\mathrm{d}L'_r \\ &= -\oint_L b_s\epsilon_{rst}\partial_t\partial_j\partial_i A(R)\mathrm{d}L'_r = -\oint_L b_k\epsilon_{rkl}\partial_l\partial_j\partial_i A(R)\mathrm{d}L'_r,\end{aligned} \tag{44}$$

the non-singular elastic distortion (42) of a dislocation loop becomes

$$\begin{aligned}\beta_{ij}(\boldsymbol{x}) = -\frac{b_k}{8\pi}\oint_L\Big[&(\epsilon_{jkl}\delta_{ir} - \epsilon_{rkl}\delta_{ij} + \epsilon_{rij}\delta_{kl})\partial_l\Delta \\ &+ \frac{1}{1-\nu}\epsilon_{rkl}\partial_l\partial_i\partial_j\Big]A(R)\mathrm{d}L'_r.\end{aligned} \tag{45}$$

This is the 'Mura formula' for a dislocation loop in gradient elasticity. It is important to note that if Eq. (45) is substituted into (7) and the relation (31) is used, the dislocation density of a dislocation loop (37) is recovered.

The symmetric part of the elastic distortion tensor (45) gives the elastic strain tensor of a dislocation loop

$$\begin{aligned}e_{ij}(\boldsymbol{x}) = -\frac{b_k}{8\pi}\oint_L\Big[&\Big(\frac{1}{2}\epsilon_{jkl}\delta_{ir} + \frac{1}{2}\epsilon_{ikl}\delta_{jr} - \epsilon_{rkl}\delta_{ij}\Big)\partial_l\Delta \\ &+ \frac{1}{1-\nu}\epsilon_{rkl}\partial_l\partial_i\partial_j\Big]A(R)\mathrm{d}L'_r.\end{aligned} \tag{46}$$

The elastic dilatation of a dislocation loop is nothing but the trace of the elastic strain (46)

$$e_{ii}(\boldsymbol{x}) = \frac{(1-2\nu)b_k}{8\pi(1-\nu)}\oint_L \epsilon_{rkl}\partial_l\Delta A(R)\mathrm{d}L'_r. \tag{47}$$

The elastic rotation vector is defined as the skewsymmetric part of the elastic distortion tensor $\omega_l = \frac{1}{2}\epsilon_{ijl}\beta_{ij}$ and reads

$$\omega_l(\boldsymbol{x}) = -\frac{b_k}{8\pi}\oint_L\Big(\delta_{lr}\partial_k - \frac{1}{2}\delta_{kr}\partial_l\Big)\Delta A(R)\mathrm{d}L'_r. \tag{48}$$

Using the constitutive relation (10) with Eq. (3), the non-singular stress field produced by a dislocation loop is found

$$\begin{aligned}\sigma_{ij}(\boldsymbol{x}) = -\frac{\mu b_k}{8\pi}\oint_L\Big[&(\epsilon_{jkl}\delta_{ir} + \epsilon_{ikl}\delta_{jr})\partial_l\Delta \\ &+ \frac{2}{1-\nu}\epsilon_{rkl}(\partial_i\partial_j - \delta_{ij}\Delta)\partial_l\Big]A(R)\mathrm{d}L'_r,\end{aligned} \tag{49}$$

which can be interpreted as the Peach–Koehler formula within the framework of gradient elasticity. One may verify that the stress is divergence-less, $\partial_j\sigma_{ij} = 0$. The double stress tensor of a dislocation loop is easily obtained if Eq. (49) is substituted into Eq. (11).

The solution of Eq. (20) is the following convolution integral

$$u_i(\boldsymbol{x}) = -\int_{-\infty}^{\infty} C_{jkln}G_{ij,k}(R)\beta^{\mathrm{P},0}_{ln}(\boldsymbol{x}')\mathrm{d}V'. \tag{50}$$

Substituting the classical plastic distortion of a dislocation loop (35) into Eq. (50) gives the modified Volterra formula valid in gradient elasticity

$$u_i(\boldsymbol{x}) = \int_S b_l C_{jkln}G_{ij,k}(R)\mathrm{d}S'_n. \tag{51}$$

Substituting Eqs. (3) and (24) into Eq. (51) and rearranging terms yield

$$u_i(\boldsymbol{x}) = \frac{b_l}{8\pi}\int_S\Big[(\delta_{il}\partial_n + \delta_{in}\partial_l - \delta_{ln}\partial_i)\Delta + \frac{1}{1-\nu}(\delta_{ln}\Delta - \partial_l\partial_n)\partial_i\Big]A(R)\mathrm{d}S'_n. \tag{52}$$

Except the first term of Eq. (52), we apply the Stokes theorem in order to obtain line integrals with

$$\int_S(\delta_{in}\partial_l - \delta_{ln}\partial_i)\Delta A(R)\,\mathrm{d}S'_n = -\oint_L \epsilon_{ril}\Delta A(R)\mathrm{d}L'_r \tag{53}$$

and

$$\int_S(\delta_{nl}\Delta - \partial_l\partial_n)\partial_i A(R)\,\mathrm{d}S'_n = -\oint_L \epsilon_{rlj}\partial_j\partial_i A(R)\mathrm{d}L'_r. \tag{54}$$

In this way, the key-formula for the non-singular displacement vector in gradient elasticity is found

$$u_i(\boldsymbol{x}) = \frac{b_i}{8\pi}\int_S \Delta\partial_j A(R)\mathrm{d}S'_j + \frac{b_l\epsilon_{rlj}}{8\pi}\oint_L\Big\{\delta_{ij}\Delta - \frac{1}{1-\nu}\partial_i\partial_j\Big\}A(R)\mathrm{d}L'_r, \tag{55}$$

which is the Burgers formula in the framework of gradient elasticity of Helmholtz type. Eq. (55) determines the displacement field of a single dislocation loop. The Eqs. (45)–(55) are straightforward, simple, and closely resemble the singular solutions of classical elasticity theory. In the limit $\ell \to 0$, the classical expressions are recovered in Eqs. (45)–(55). The expressions (45), (49), and (55) retain most of the analytic structure of the classical Mura, Peach–Koehler, and Burgers formulae. The expressions (45)–(55) are given in terms of the elementary function $A(R)$ given in Eq. (25), instead of the classical expression R. The explicit expressions can be obtained by simple substitution of the formulae for the derivatives of A given in Eqs. (A.2)–(A.6). It is important to note that Eqs. (45)–(55) are non-singular due to the regularization of the classical singular expressions (see Appendix A). As an example, we substitute Eqs. (A.5) and (A.6) into Eq. (49) and obtain the explicit expression for the stress tensor

$$\sigma_{ij}(\boldsymbol{x}) = -\frac{\mu b_l}{8\pi}\oint_L \Bigg[\left(\epsilon_{jkl}\delta_{ir}+\epsilon_{ikl}\delta_{jr}-\frac{2}{1-\nu}\epsilon_{rkl}\delta_{ij}\right)\frac{2R_k}{R^3}\left[1-\left(1+\frac{R}{\ell}\right)e^{-R/\ell}\right] + \frac{2}{1-\nu}\epsilon_{rkl}\Bigg(\frac{\delta_{ij}R_k+\delta_{ik}R_j+\delta_{jk}R_i}{R^3}\left[1-\frac{6\ell^2}{R^2}(1-e^{-R/\ell})+\left(2+\frac{6\ell}{R}\right)e^{-R/\ell}\right] - \frac{3R_iR_jR_k}{R^5}\left[1-\frac{10\ell^2}{R^2}(1-e^{-R/\ell})+\left(4+\frac{10\ell}{R}+\frac{2R}{3\ell}\right)e^{-R/\ell}\right]\Bigg)\Bigg]\mathrm{d}L'_r. \quad (56)$$

To give the expression (55) more explicitly. Using Eq. (A.6) we introduce a generalized solid angle valid in gradient elasticity of Helmholtz type

$$\Omega(\boldsymbol{x},\ell) = -\frac{1}{2}\int_S \Delta\partial_j A(R)\,\mathrm{d}S'_j = \int_S \frac{R_j}{R^3}\left(1-\left(1+\frac{R}{\ell}\right)e^{-R/\ell}\right)\mathrm{d}S'_j. \quad (57)$$

Eq. (57) is non-singular and depends on the length scale ℓ. In the limit $\ell \to 0$, the usual solid angle (e.g. Li and Wang, 2008) is recovered. Thus, using Eq. (57) and carrying out some differentiations with Eqs. (A.2) and (A.4), we obtain from Eq. (55) the explicit gradient elasticity version of the Burgers formula

$$u_i(\boldsymbol{x}) = -\frac{b_i}{4\pi}\Omega(\boldsymbol{x},\ell) - \frac{b_l}{4\pi}\oint_L \epsilon_{ilr}\frac{1}{R}(1-e^{-R/\ell})\,\mathrm{d}L'_r - \frac{b_l}{8\pi(1-\nu)} \times \oint_L \epsilon_{ljr}\partial_i\frac{R_j}{R}\left(1-\frac{2\ell^2}{R^2}(1-e^{-R/\ell})+\frac{2\ell}{R}e^{-R/\ell}\right)\mathrm{d}L'_r. \quad (58)$$

The simplicity of our results is based on the use of gradient elasticity theory of Helmholtz type. Our results can be used in computer simulations of dislocation cores at nano-scale and in numerics as fast numerical sums of the relevant elastic fields as it is used for the classical equations (e.g. Ghoniem et al., 1999).

2.2. Straight dislocations

In this subsection, using the modified Mura equation of gradient elasticity of Helmholtz type (40), the non-singular elastic distortion fields of straight dislocations as a check of our general approach are calculated.

2.2.1. Screw dislocation

A screw dislocation corresponds to the anti-plane strain problem. The Green function of the anti-plane strain problem in gradient elasticity of Helmholtz type is nothing but the Green function of the two-dimensional Helmholtz–Laplace equation and it reads (see Eq. (B.21))

$$G_{zz}(R) = -\frac{1}{2\pi\mu}\{\gamma_E + \ln R + K_0(R/\ell)\}, \quad (59)$$

where $R = \sqrt{(x-x')^2+(y-y')^2}$, γ_E is the Euler constant and K_n is the modified Bessel function of order n. The Green function (59) is non-singular. The gradient of the Green function (59) is obtained as

$$G_{zz,k}(R) = -\frac{1}{2\pi\mu}\frac{R_k}{R^2}\left\{1-\frac{R}{\ell}K_1(R/\ell)\right\}. \quad (60)$$

Next, substituting Eq. (60) and the dislocation density of a screw dislocation $\alpha^0_{zz} = b_z\delta(x)\delta(y)$ into Eq. (40), the elastic distortion produced by a screw dislocation is obtained. For an infinite screw dislocation along the z-axis with Burgers vector b_z, the non-singular components for the elastic distortion are calculated as

$$\beta_{zx} = -\frac{b_z}{2\pi}\frac{y}{r^2}\left\{1-\frac{r}{\ell}K_1(r/\ell)\right\}, \quad (61)$$

$$\beta_{zy} = \frac{b_z}{2\pi}\frac{x}{r^2}\left\{1-\frac{r}{\ell}K_1(r/\ell)\right\}, \quad (62)$$

where $r = \sqrt{x^2+y^2}$. The expressions obtained earlier by Lazar (2003) and Lazar and Maugin (2006a) are recovered. In the limit $\ell \to 0$, the classical expressions given by DeWit (1973b) are recovered in Eqs. (61) and (62).

The Green function (59) gives the non-singular displacement field $u_z = -G_{zz}f_z$ of a line force with the magnitude f_z calculated by Lazar and Maugin (2006b) in the framework of gradient elasticity.

2.2.2. Edge dislocation

Now the plane strain problem of an edge dislocation is investigated. The Green tensor of the plane strain problem in gradient elasticity of Helmholtz type is derived as (see Eq. (B.17))

$$G_{ij}(R) = -\frac{1}{2\pi\mu}\delta_{ij}\{\gamma_E+\ln R+K_0(R/\ell)\} + \frac{1}{16\pi\mu(1-\nu)}\partial_i\partial_j\left\{R^2(\gamma_E+\ln R)+4\ell^2(\gamma_E+\ln R+K_0(R/\ell))\right\}. \quad (63)$$

It is obvious that the terms proportional to the Euler constant do not contribute to the elastic distortion fields. The two-dimensional Green tensor (63) is non-singular. In the limit $\ell \to 0$, the two-dimensional Green tensor of classical elasticity (Mura, 1987; Li and Wang, 2008) is recovered in Eq. (63). The gradient of the Green tensor (63) is given by

$$G_{ij,k}(R) = -\frac{1}{8\pi\mu(1-\nu)}\Bigg[(3-4\nu)\delta_{ij}\frac{R_k}{R^2}-\delta_{ik}\frac{R_j}{R^2}-\delta_{jk}\frac{R_i}{R^2}+2\frac{R_iR_jR_k}{R^4} + \frac{2}{R^2}\left(\delta_{ij}R_k+\delta_{ik}R_j+\delta_{jk}R_i-4\frac{R_iR_jR_k}{R^2}\right)\left(\frac{2\ell^2}{R^2}-K_2(R/\ell)\right) - \left(4(1-\nu)\delta_{ij}\frac{R_k}{\ell R}-2\frac{R_iR_jR_k}{\ell R^3}\right)K_1(R/\ell)\Bigg]. \quad (64)$$

Substituting Eq. (64) and the dislocation density of an edge dislocation along z axis with Burgers vector b_x, $\alpha^0_{xz} = b_x\delta(x)\delta(y)$, into Eq. (40), the elastic distortion of an edge dislocation is obtained. Eventually, the non-vanishing components of the elastic distortion of an edge dislocation are calculated as

$$\beta_{xx} = -\frac{b_x}{4\pi(1-\nu)}\frac{y}{r^2}\left\{(1-2\nu)+\frac{2x^2}{r^2}+\frac{4\ell^2}{r^4}(y^2-3x^2) - \frac{2(y^2-3x^2)}{r^2}K_2(r/\ell)-\frac{2(y^2-\nu r^2)}{\ell r}K_1(r/\ell)\right\}, \quad (65)$$

$$\beta_{xy} = \frac{b_x}{4\pi(1-\nu)}\frac{x}{r^2}\left\{(3-2\nu)-\frac{2y^2}{r^2}-\frac{4\ell^2}{r^4}(x^2-3y^2) + \frac{2(x^2-3y^2)}{r^2}K_2(r/\ell)-\frac{2(y^2+(1-\nu)r^2)}{\ell r}K_1(r/\ell)\right\}, \quad (66)$$

$$\beta_{yx} = -\frac{b_x}{4\pi(1-\nu)}\frac{x}{r^2}\left\{(1-2\nu)+\frac{2y^2}{r^2}+\frac{4\ell^2}{r^4}(x^2-3y^2) - \frac{2(x^2-3y^2)}{r^2}K_2(r/\ell)+\frac{2(y^2-(1-\nu)r^2)}{\ell r}K_1(r/\ell)\right\}, \quad (67)$$

$$\beta_{yy} = -\frac{b_x}{4\pi(1-\nu)}\frac{y}{r^2}\left\{(1-2\nu)-\frac{2x^2}{r^2}-\frac{4\ell^2}{r^4}(y^2-3x^2) + \frac{2(y^2-3x^2)}{r^2}K_2(r/\ell)-\frac{2(x^2-\nu r^2)}{\ell r}K_1(r/\ell)\right\}, \quad (68)$$

which are non-singular and agree with the formulae given by Lazar (2003) and Lazar and Maugin (2006a). In the limit $\ell \to 0$, we obtain in Eqs. (65)–(68) the classical expressions given by DeWit (1973b). As discussed by Lazar et al. (2006a), the dislocation core radius can be defined straightforwardly in the framework of gradient elasticity as $R_c \simeq 6\ell$. If $\ell \simeq 0.4a$, where a denotes the lattice parameter, is adopted as proposed by Eringen (1983), the dislocation core radius is $R_c \simeq 2.5a$. Using $\ell \simeq 0.4a$, the internal length reduces to $\ell \simeq 1.97$ Å for lead (Pb) with $a = 4.95$ Å.

Note that the two-dimensional Green function (63) gives the non-singular displacement field, $u_i = -G_{ij}f_j$, of a line force with magnitude f_j calculated by Lazar and Maugin (2006b) in the framework of gradient elasticity.

3. Gradient elasticity of bi-Helmholtz type

In this section, gradient elasticity theory of higher order is considered. Gradient elasticity theory of higher order was originally introduced by Mindlin (1965); Mindlin (1972) (see also, Jaunzemis, 1967; Wu, 1992; Agiasofitou and Lazar, 2009). Mindlin's theory of second strain gradient elasticity involves for isotropic materials, in addition to the two Lamé constants, sixteen additional material constants. These constants produce four characteristic length scales.

A simple and robust gradient elasticity of higher order which is called gradient elasticity theory of bi-Helmholtz type was introduced by Lazar et al. (2006a) and Lazar and Maugin (2006a) and successfully applied to the problems of straight dislocations (Lazar et al., 2006a; Lazar and Maugin, 2006a), straight disclinations (Deng et al., 2007) and point defects (Zhang et al., 2006). Lazar et al., 2006a and Lazar and Maugin, 2006a have shown that all state quantities are non-singular. By means of this second order gradient theory it is possible to eliminate not only the singularities of the strain and stress tensors, but also the singularities of the double and triple stress tensors and of the dislocation density tensors of straight dislocations at the dislocation line. In general, all fields calculated in the theory of gradient elasticity of bi-Helmholtz type are smoother than those calculated by gradient elasticity theory of Helmholtz type. In general, there a two main motivations for the use of gradient elasticity of bi-Helmholtz type: a consistent regularization of all state quantities, and a more realistic modelling of dispersion relations. A simple higher-order gradient theory in order to investigate dislocation loops should be used. The theory of gradient elasticity of bi-Helmholtz type is the gradient version of nonlocal elasticity of bi-Helmholtz type (Lazar et al., 2006b).

The strain energy density of gradient elasticity theory of bi-Helmholtz type for an isotropic, linearly elastic material has the form (Lazar et al., 2006a)

$$W = \frac{1}{2}C_{ijkl}\beta_{ij}\beta_{kl} + \frac{1}{2}\ell_1^2 C_{ijkl}\partial_m\beta_{ij}\partial_m\beta_{kl} + \frac{1}{2}\ell_2^4 C_{ijkl}\partial_n\partial_m\beta_{ij}\partial_n\partial_m\beta_{kl}, \quad (69)$$

where $\ell_1 = \ell, \ell_2$ is another characteristic length scale and C_{ijkl} is given in (3). Due to the symmetry of C_{ijkl}, Eq. (69) is equivalent to

$$W = \frac{1}{2}C_{ijkl}e_{ij}e_{kl} + \frac{1}{2}\ell_1^2 C_{ijkl}\partial_m e_{ij}\partial_m e_{kl} + \frac{1}{2}\ell_2^4 C_{ijkl}\partial_n\partial_m e_{ij}\partial_n\partial_m e_{kl}. \quad (70)$$

In addition to the constitutive Eqs. (10) and (11) another one is present in such a higher-order gradient theory,

$$\tau_{ijkl} = \frac{\partial W}{\partial\partial_l\partial_k\beta_{ij}} = \frac{\partial W}{\partial\partial_l\partial_k e_{ij}} = \ell_2^4 C_{ijmn}\partial_l\partial_k\beta_{mn} = \ell_2^4\partial_l\partial_k\sigma_{ij}, \quad (71)$$

where τ_{ijkl} is called the triple stress tensor. It can be seen that ℓ_2 is the characteristic length scale for triple stresses. On the other hand, ℓ_1 is the characteristic length scale for double stresses. Using Eqs. (10), (11), and (71), Eq. (70) can also be written as (Lazar et al., 2006a)

$$W = \frac{1}{2}\sigma_{ij}e_{ij} + \frac{1}{2}\ell_1^2\partial_k\sigma_{ij}\partial_k e_{ij} + \frac{1}{2}\ell_2^4\partial_l\partial_k\sigma_{ij}\partial_l\partial_k e_{ij}. \quad (72)$$

The strain energy density (72) exhibits the symmetry in σ_{ij} and e_{ij}, in $\partial_k\sigma_{ij}$ and $\partial_k e_{ij}$, and in $\partial_l\partial_k\sigma_{ij}$ and $\partial_l\partial_k e_{ij}$. The condition for non-negative strain energy density, $W \geqslant 0$, gives

$$\ell_1^2 \geqslant 0, \quad \ell_2^4 \geqslant 0, \quad (73)$$

in addition to $(3\mu + 2\lambda) \geqslant 0$ and $\mu \geqslant 0$.

The total stress tensor reads now

$$\sigma_{ij}^0 = \sigma_{ij} - \partial_k\tau_{ijk} + \partial_l\partial_k\tau_{ijkl}. \quad (74)$$

In absence of body forces, the equation of equilibrium has the following form

$$\partial_j\sigma_{ij}^0 = \partial_j(\sigma_{ij} - \partial_k\tau_{ijk} + \partial_l\partial_k\tau_{ijkl}) = 0. \quad (75)$$

Using Eqs. (11) and (71), the total stress tensor (74) can be written

$$\sigma_{ij}^0 = L\sigma_{ij}, \quad (76)$$

where the differential operator L is given by

$$L = (1 - \ell_1^2\Delta + \ell_2^4\Delta\Delta) = (1 - c_1^2\Delta)(1 - c_2^2\Delta) \quad (77)$$

with

$$c_1^2 = \frac{\ell_1^2}{2}\left(1 + \sqrt{1 - 4\frac{\ell_2^4}{\ell_1^4}}\right), \quad (78)$$

$$c_2^2 = \frac{\ell_1^2}{2}\left(1 - \sqrt{1 - 4\frac{\ell_2^4}{\ell_1^4}}\right) \quad (79)$$

and

$$\ell_1^2 = c_1^2 + c_2^2, \quad (80)$$

$$\ell_2^4 = c_1^2 c_2^2. \quad (81)$$

Due to its structure as a product of two Helmholtz operators, the differential operator (77) is called bi-Helmholtz operator.

An important point, is the question concerning the mathematical character of the length scales c_1 and c_2. Mindlin, 1965 (see also, Mindlin, 1972; Wu, 1992) pointed out that the conditions for non-negative W supply no indications of the character, real or complex, of the characteristic lengths. Mindlin (1965) and Wu (1992) have treated the characteristic lengths as if they were real and positive. They also pointed out that a complex character of the lengths is equally admissible. The character, real or complex, of the lengths dictates the behaviour of the field variables. In the theory of gradient elasticity of bi-Helmholtz type the condition for the character, real or complex, of the length scales c_1 and c_2 can be obtained from the condition if the argument of the square root in Eqs. (78) and (79) is positive or negative. Thus, c_1 and c_2 are real if

$$\ell_1^4 - 4\ell_2^4 \geqslant 0, \quad (82)$$

and c_1 and c_2 are complex if

$$\ell_1^4 - 4\ell_2^4 < 0. \quad (83)$$

If the lengths c_1 and c_2 are complex, then the behaviour of the solutions of the field quantities would be oscillatory. In this case, the far-field behaviour of the strain and stress fields of dislocations would not agree with the classical behaviour. The limit from gradient elasticity of bi-Helmholtz type to gradient elasticity of Helmholtz type is: $c_2 \to 0, \ell_2 \to 0$ and $c_1 \to \ell_1$. If c_1 is complex, then also ℓ_1 becomes complex what would be rather strange. Thus, a real character of the length scales c_1 and c_2 seems to be more realistic and more physical. In addition, Zhang et al., 2006 determined, in an atomistic calculation, the length scales c_1 and c_2 as positive and real for graphene. In what follows, the length scales c_1 and c_2 will be treated as if they are real and positive.

The Green tensor of the bi-Helmholtz–Navier equation is calculated as (see Eq. (B.27))

$$G_{ij}(R) = \frac{1}{16\pi\mu(1-\nu)}\left[2(1-\nu)\delta_{ij}\Delta - \partial_i\partial_j\right]A(R), \quad (84)$$

where the elementary function (25) is changed to

$$A(R) = R + \frac{2(c_1^2+c_2^2)}{R} - \frac{2}{c_1^2-c_2^2}\frac{1}{R}\left(c_1^4 e^{-R/c_1} - c_2^4 e^{-R/c_2}\right). \tag{85}$$

Eq. (85) is the Green function of the three-dimensional bi-Helmholtz-bi–Laplace equation. It is worth noting that the Green tensor (84) with (85) is in agreement with the corresponding expression derived by Zhang et al. (2006). On the other hand, the Green function of the bi-Helmholtz equation is given by (e.g. Lazar et al., 2006b)

$$G(R) = \frac{1}{4\pi(c_1^2-c_2^2)R}\left(e^{-R/c_1} - e^{-R/c_2}\right). \tag{86}$$

In the framework of gradient elasticity of bi-Helmholtz, the differential operator of bi-Helmholtz type (77) appears in Eqs. (15)–(18).

If we use Eqs. (A.9) and (A.10) for the differentiation of Eq. (84), we obtain the explicit form of the three-dimensional Green tensor of the bi-Helmholtz–Navier equation

$$\begin{aligned} G_{ij}(R) = \frac{1}{16\pi\mu(1-\nu)}\Bigg[&\frac{\delta_{ij}}{R}\Bigg((3-4\nu)\left(1-\frac{1}{c_1^2-c_2^2}\left(c_1^2 e^{-R/c_1}-c_2^2 e^{-R/c_2}\right)\right)\\ &+\frac{2(c_1^2+c_2^2)}{R^2}-\frac{2}{c_1^2-c_2^2}\frac{1}{R^2}\left(c_1^4 e^{-R/c_1}-c_2^4 e^{-R/c_2}\right)\\ &-\frac{2}{c_1^2-c_2^2}\frac{1}{R}\left(c_1^3 e^{-R/c_1}-c_2^3 e^{-R/c_2}\right)-\frac{1}{c_1^2-c_2^2}\left(c_1^2 e^{-R/c_1}-c_2^2 e^{-R/c_2}\right)\Bigg)\\ &+\frac{R_iR_j}{R^3}\Bigg(1-\frac{6(c_1^2+c_2^2)}{R^2}+\frac{6}{c_1^2-c_2^2}\frac{1}{R^2}\left(c_1^4 e^{-R/c_1}-c_2^4 e^{-R/c_2}\right)\\ &+\frac{6}{c_1^2-c_2^2}\frac{1}{R}\left(c_1^3 e^{-R/c_1}-c_2^3 e^{-R/c_2}\right)+\frac{2}{c_1^2-c_2^2}\left(c_1^2 e^{-R/c_1}-c_2^2 e^{-R/c_2}\right)\Bigg)\Bigg]. \end{aligned} \tag{87}$$

The Green tensor (87) gives the non-singular displacement field $u_i = G_{ij}f_j$ of the Kelvin point force problem, in the framework of gradient elasticity of bi-Helmholtz type.

3.1. Dislocation loops

The calculation of the characteristic fields of a dislocation loop in gradient elasticity of bi-Helmholtz type, is analogous to the technique used in gradient elasticity of Helmholtz type. The only difference in the results is that now the Green function (86) and the elementary function (85) of bi-Helmholtz type enter the characteristic fields of a dislocation loop. In gradient elasticity of bi-Helmholtz type, the dislocation density tensor (36) and the plastic distortion tensor (38) are given in terms of the Green function of bi-Helmholtz type (86). Thus, they are calculated as

$$\alpha_{ij}(\boldsymbol{x}) = \frac{b_i}{4\pi(c_1^2-c_2^2)}\oint_L \frac{e^{-R/c_1}-e^{-R/c_2}}{R}\,dL'_j, \tag{88}$$

$$\beta^{P}_{ij}(\boldsymbol{x}) = -\frac{b_i}{4\pi(c_1^2-c_2^2)}\int_S \frac{e^{-R/c_1}-e^{-R/c_2}}{R}\,dS'_j. \tag{89}$$

In the limit $R \to 0$, the integrands of Eqs. (88) and (89) are non-singular at the dislocation line in contrast to the corresponding ones, Eqs. (37) and (39), calculated in gradient elasticity of Helmholtz type. On the other hand, the elastic distortion tensor (45), the elastic strain tensor (46), the elastic dilatation (47), the elastic rotation vector (48), the stress tensor (49), and the displacement vector (55) are given in terms of the elementary function (85) and only (85) has to be substituted in these formulae. The explicit formulae are not reproduced. The only difference between the fields of a dislocation loop in gradient elasticity of bi-Helmholtz type, and of Helmholtz type is that the Green function of bi-Helmholtz type (86) and the elementary function (85) have to be substituted instead of the Green function of Helmholtz type (28) and the elementary function (17). For the derivatives of the function (85), Eqs. (A.7)–(A.12) can be substituted into the corresponding formulae. The characteristic fields of a dislocation loop in gradient elasticity of bi-Helmholtz type retain all the analytical tensor structure of the corresponding classical formulae.

The triple stress tensor of a dislocation loop is easily obtained if the stress tensor σ_{ij} is substituted into Eq. (71). In gradient elasticity of bi-Helmholtz type the fields produced by a dislocation loop are smoother that those predicted by gradient elasticity of Helmholtz type.

3.2. Straight dislocations

In this subsection, the modified Mura Equation (40) is used for gradient elasticity of bi-Helmholtz type. The technique of Green functions is used in order to determine the non-singular elastic distortion of straight dislocations.

3.2.1. Screw dislocation

The Green function of the anti-plane strain problem in gradient elasticity of bi-Helmholtz type is the Green function of the two-dimensional bi-Helmholtz–Laplace equation and is given by (see Eq. (B.34))

$$G_{zz}(R) = -\frac{1}{2\pi\mu}\left\{\gamma_E + \ln R + \frac{1}{c_1^2-c_2^2}\left[c_1^2K_0(R/c_1) - c_2^2K_0(R/c_2)\right]\right\}, \tag{90}$$

where $R = \sqrt{(x-x')^2+(y-y')^2}$. The gradient of the Green function (90) is calculated as

$$G_{zz,k}(R) = -\frac{1}{2\pi\mu}\frac{R_k}{R^2}\left\{1 - \frac{1}{c_1^2-c_2^2}\left[c_1RK_1(R/c_1) - c_2RK_1(R/c_2)\right]\right\}. \tag{91}$$

If Eq. (91) and $\alpha^0_{zz} = b_z\delta(x)\delta(y)$ are substituted into Eq. (40), the elastic distortion produced by a screw dislocation with Burgers vector b_z is obtained

$$\beta_{zx} = -\frac{b_z}{2\pi}\frac{y}{r^2}\left\{1 - \frac{1}{c_1^2-c_2^2}\left[c_1rK_1(r/c_1) - c_2rK_1(r/c_2)\right]\right\}, \tag{92}$$

$$\beta_{zy} = \frac{b_z}{2\pi}\frac{x}{r^2}\left\{1 - \frac{1}{c_1^2-c_2^2}\left[c_1rK_1(r/c_1) - c_2rK_1(r/c_2)\right]\right\}, \tag{93}$$

where $r = \sqrt{x^2+y^2}$. Eqs. (92) and (93) are in agreement with the expressions obtained by Lazar and Maugin (2006a).

The Green function (90) gives the non-singular displacement field $u_z = -G_{zz}f_z$ of a line force with the magnitude f_z in the framework of gradient elasticity of bi-Helmholtz type.

3.2.2. Edge dislocation

The plane strain problem of an edge dislocation is now investigated. The Green tensor of the plane strain problem in gradient elasticity of bi-Helmholtz type is found as (see Eq. (B.30))

$$\begin{aligned} G_{ij}(R) = &-\frac{1}{2\pi\mu}\delta_{ij}\left\{\gamma_E + \ln R + \frac{1}{c_1^2-c_2^2}\left[c_1^2K_0(R/c_1) - c_2^2K_0(R/c_2)\right]\right\}\\ &+\frac{1}{16\pi\mu(1-\nu)}\partial_i\partial_j\Big\{R^2(\gamma_E+\ln R) + 4(c_1^2+c_2^2)(\gamma_E+\ln R)\\ &+\frac{4}{c_1^2-c_2^2}\left[c_1^4K_0(R/c_1) - c_2^4K_0(R/c_2)\right]\Big\}. \end{aligned} \tag{94}$$

The gradient of the Green tensor Eq. (94) is calculated as

$$
\begin{aligned}
G_{ij,k}(R)=&-\frac{1}{8\pi\mu(1-\nu)}\bigg[(3-4\nu)\delta_{ij}\frac{R_k}{R^2}-\delta_{ik}\frac{R_j}{R^2}-\delta_{jk}\frac{R_i}{R^2}+2\frac{R_iR_jR_k}{R^4}\\
&+\frac{2}{R^2}\left(\delta_{ij}R_k+\delta_{ik}R_j+\delta_{jk}R_i-4\frac{R_iR_jR_k}{R^2}\right)\\
&\times\left(\frac{2(c_1^2+c_2^2)}{R^2}-\frac{1}{c_1^2-c_2^2}\left[c_1^2K_2(R/c_1)-c_2^2K_2(R/c_2)\right]\right)\\
&-\left(4(1-\nu)\delta_{ij}\frac{R_k}{R}-2\frac{R_iR_jR_k}{R^3}\right)\frac{1}{c_1^2-c_2^2}\left[c_1K_1(R/c_1)-c_2K_1(R/c_2)\right]\bigg].
\end{aligned}
\tag{95}
$$

If Eq. (95) and $\alpha^0_{xz}=b_x\delta(x)\delta(y)$ are substituted into Eq. (40), the non-vanishing components of the elastic distortion of an edge dislocation are found as

$$
\begin{aligned}
\beta_{xx}=&-\frac{b_x}{4\pi(1-\nu)}\frac{y}{r^2}\bigg\{(1-2\nu)+\frac{2x^2}{r^2}+\frac{4(c_1^2+c_2^2)}{r^4}(y^2-3x^2)\\
&-\frac{2(y^2-\nu r^2)}{r^2(c_1^2-c_2^2)}\left[c_1rK_1(r/c_1)-c_2rK_1(r/c_2)\right]\\
&-\frac{2(y^2-3x^2)}{(c_1^2-c_2^2)r^2}\left[c_1^2K_2(r/c_1)-c_2^2K_2(r/c_2)\right]\bigg\},
\end{aligned}
\tag{96}
$$

$$
\begin{aligned}
\beta_{xy}=&\frac{b_x}{4\pi(1-\nu)}\frac{x}{r^2}\bigg\{(3-2\nu)-\frac{2y^2}{r^2}-\frac{4(c_1^2+c_2^2)}{r^4}(x^2-3y^2)\\
&-\frac{2(y^2+(1-\nu)r^2)}{(c_1^2-c_2^2)r^2}\left[c_1rK_1(r/c_1)-c_2rK_1(r/c_2)\right]\\
&+\frac{2(x^2-3y^2)}{(c_1^2-c_2^2)r^2}\left[c_1^2K_2(r/c_1)-c_2^2K_2(r/c_2)\right]\bigg\},
\end{aligned}
\tag{97}
$$

$$
\begin{aligned}
\beta_{yx}=&-\frac{b_x}{4\pi(1-\nu)}\frac{x}{r^2}\bigg\{(1-2\nu)+\frac{2y^2}{r^2}+\frac{4(c_1^2+c_2^2)}{r^4}(x^2-3y^2)\\
&+\frac{2(y^2-(1-\nu)r^2)}{(c_1^2-c_2^2)r^2}\left[c_1rK_1(r/c_1)-c_2rK_1(r/c_2)\right]\\
&-\frac{2(x^2-3y^2)}{(c_1^2-c_2^2)r^2}\left[c_1^2K_2(r/c_1)-c_2^2K_2(r/c_2)\right]\bigg\},
\end{aligned}
\tag{98}
$$

$$
\begin{aligned}
\beta_{yy}=&-\frac{b_x}{4\pi(1-\nu)}\frac{y}{r^2}\bigg\{(1-2\nu)-\frac{2x^2}{r^2}-\frac{4(c_1^2+c_2^2)}{r^4}(y^2-3x^2)\\
&-\frac{2(x^2-\nu r^2)}{r^2(c_1^2-c_2^2)}\left[c_1rK_1(r/c_1)-c_2rK_1(r/c_2)\right]\\
&+\frac{2(y^2-3x^2)}{(c_1^2-c_2^2)r^2}\left[c_1^2K_2(r/c_1)-c_2^2K_2(r/c_2)\right]\bigg\},
\end{aligned}
\tag{99}
$$

which are in agreement with the formulae given by Lazar and Maugin (2006a).

The two-dimensional Green function (94) gives the non-singular displacement field, $u_i=-G_{ij}f_j$, of a line force with magnitude f_j calculated in the framework of gradient elasticity of bi-Helmholtz type.

4. Conclusions

Non-singular dislocation fields are presented in the framework of gradient elasticity. The technique of Green functions is used. The Green tensors of all relevant partial differential equations of generalized Navier type were calculated. For the first time, the elastic distortion, plastic distortion, stress, displacement, and dislocation density of a closed dislocation loop, using the theories of gradient elasticity of Helmholtz type and of bi-Helmholtz type were calculated. Straight dislocations using Green tensors were revisited. Such generalized continuum theories allow dislocation core spreading in a straightforward manner. In classical dislocation theory the dislocation function is a Dirac delta function, $\delta(\boldsymbol{x})$, without core spreading. In the non-singular approaches by Cai et al. (2006) and Lazar, presented in the present paper, the dislocation spreading functions are w and G, respectively (see Table 1). In the theory of gradient elasticity all formulae are closed in contrast to the theory of Cai et al. (2006) where the spreading function w is determined in a sophisticated way in order to obtain $R_a=[R^2+a^2]^{1/2}$. Due to the use of simplified theories of gradient elasticity, the dislocation fields retain most of the analytical structure of the classical expressions for these quantities but remove the singularity at the dislocation core due to the mathematical regularization of the classical singular expressions. In gradient elasticity of Helmholtz type, the characteristic length ℓ takes into account the information from atomistic calculations as discussed in this paper. In a straightforward manner, the length ℓ determines the dislocation core radius. Therefore, in gradient elasticity it is not necessary to introduce an artificial core-cutoff radius. It should be mentioned that the characteristic lengths which arise in first strain gradient elasticity (e.g. Maranganti and Sharma, 2007; Shodja and Tehranchi, 2010) and in second strain gradient elasticity (e.g. Zhang et al., 2006; Shodja et al., 2012) have been recently computed using atomistic approaches.

The obtained results can be used in computer simulations and numerics of dislocation cores, discrete dislocation dynamics, and arbitrary 3D dislocation configurations. The results can be implemented in dislocation dynamics codes (finite element implementation, technique of fast numerical sums), and compared to atomistic models (e.g. Ghoniem et al., 1999; Li and Wang, 2008).

Table 1
Comparison of the basic quantities in different dislocation theories (classical dislocation theory, Cai et al. and gradient theory of Helmholtz type).

Classical theory (DeWit, 1960)	Cai et al., 2006	Lazar [this paper]
R	R_a	$A(R)$
$\Delta\Delta R=-8\pi\delta(\boldsymbol{x})$	$\Delta\Delta R_a=-8\pi w$	$\Delta\Delta A(R)=-8\pi G$
	$R_a=R*w$	$A(R)=R*G$
	w chosen to obtain:	$LG=\delta(\boldsymbol{x})$, $L=1-\ell^2\Delta$
	$R_a=[R^2+a^2]^{1/2}$	$A(R)=R+\frac{2\ell^2}{R}(1-e^{-R/\ell})$
	a – arbitrary constant	ℓ – characteristic length
	$w=15a^4/[8\pi(r^2+a^2)^{7/2}]$	$G=e^{-r/\ell}/[4\pi\ell^2r]$

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Appendix A. Appendix: *A* and its derivatives

In gradient elasticity theory, the stress tensor, the elastic distortion tensor, the elastic strain tensor, and the displacement vector of a dislocation loop are given in terms of derivatives of the elementary function A.

A.1. Helmholtz type

For gradient elasticity of Helmholtz type, the elementary function A is given by

$$
A=R+\frac{2\ell^2}{R}\left(1-e^{-R/\ell}\right).
\tag{A.1}
$$

Higher-order derivatives of A are given by the following set of equations

$$A_{,i} = \frac{R_i}{R}\left[1 - \frac{2\ell^2}{R^2}\left(1 - e^{-R/\ell}\right) + \frac{2\ell}{R} e^{-R/\ell}\right], \tag{A.2}$$

where $R_i = x_i - x'_i$,

$$A_{,ij} = \frac{\delta_{ij}}{R}\left[1 - \frac{2\ell^2}{R^2}\left(1 - e^{-R/\ell}\right) + \frac{2\ell}{R} e^{-R/\ell}\right] - \frac{R_iR_j}{R^3}\left[1 - \frac{6\ell^2}{R^2}\left(1 - e^{-R/\ell}\right) + \left(2 + \frac{6\ell}{R}\right) e^{-R/\ell}\right], \tag{A.3}$$

$$A_{,ii} = \frac{2}{R}\left(1 - e^{-R/\ell}\right), \tag{A.4}$$

$$A_{,ijk} = -\frac{\delta_{ij}R_k + \delta_{ik}R_j + \delta_{jk}R_i}{R^3}\left[1 - \frac{6\ell^2}{R^2}\left(1 - e^{-R/\ell}\right) + \left(2 + \frac{6\ell}{R}\right) e^{-R/\ell}\right] + \frac{3R_iR_jR_k}{R^5}\left[1 - \frac{10\ell^2}{R^2}\left(1 - e^{-R/\ell}\right) + \left(4 + \frac{10\ell}{R} + \frac{2R}{3\ell}\right) e^{-R/\ell}\right] \tag{A.5}$$

and

$$A_{,iik} = -\frac{2R_k}{R^3}\left(1 - \left(1 + \frac{R}{\ell}\right) e^{-R/\ell}\right). \tag{A.6}$$

The expressions (A.1)–(A.6) are non-singular. For $R \to 0$, they are either zero or finite.

A.2. Bi-Helmholtz type

In gradient elasticity of bi-Helmholtz type, the elementary function A reads

$$A = R + \frac{2(c_1^2 + c_2^2)}{R} - \frac{2}{c_1^2 - c_2^2}\frac{1}{R}\left(c_1^4 e^{-R/c_1} - c_2^4 e^{-R/c_2}\right). \tag{A.7}$$

The higher-order derivatives of A are given by

$$A_{,i} = \frac{R_i}{R}\left[1 - \frac{2(c_1^2 + c_2^2)}{R^2} + \frac{2}{c_1^2 - c_2^2}\frac{1}{R^2}\left(c_1^4 e^{-R/c_1} - c_2^4 e^{-R/c_2}\right) + \frac{2}{c_1^2 - c_2^2}\frac{1}{R}\left(c_1^3 e^{-R/c_1} - c_2^3 e^{-R/c_2}\right)\right], \tag{A.8}$$

$$\begin{aligned} A_{,ij} = {}& \frac{\delta_{ij}}{R}\left[1 - \frac{2(c_1^2 + c_2^2)}{R^2} + \frac{2}{c_1^2 - c_2^2}\frac{1}{R^2}\left(c_1^4 e^{-R/c_1} - c_2^4 e^{-R/c_2}\right) + \frac{2}{c_1^2 - c_2^2}\frac{1}{R}\left(c_1^3 e^{-R/c_1} - c_2^3 e^{-R/c_2}\right)\right] \\ & - \frac{R_iR_j}{R^3}\left[1 - \frac{6(c_1^2 + c_2^2)}{R^2} + \frac{6}{c_1^2 - c_2^2}\frac{1}{R^2}\left(c_1^4 e^{-R/c_1} - c_2^4 e^{-R/c_2}\right) + \frac{6}{c_1^2 - c_2^2}\frac{1}{R}\left(c_1^3 e^{-R/c_1} - c_2^3 e^{-R/c_2}\right) + \frac{2}{c_1^2 - c_2^2}\left(c_1^2 e^{-R/c_1} - c_2^2 e^{-R/c_2}\right)\right], \end{aligned} \tag{A.9}$$

$$A_{,ii} = \frac{2}{R}\left[1 - \frac{1}{c_1^2 - c_2^2}\left(c_1^2 e^{-R/c_1} - c_2^2 e^{-R/c_2}\right)\right], \tag{A.10}$$

$$\begin{aligned} A_{,ijk} = {}& -\frac{\delta_{ij}R_k + \delta_{ik}R_j + \delta_{jk}R_i}{R^3}\left[1 - \frac{6(c_1^2 + c_2^2)}{R^2} + \frac{6}{c_1^2 - c_2^2}\frac{1}{R^2}\left(c_1^4 e^{-R/c_1} - c_2^4 e^{-R/c_2}\right) + \frac{6}{c_1^2 - c_2^2}\frac{1}{R}\left(c_1^3 e^{-R/c_1} - c_2^3 e^{-R/c_2}\right) + \frac{2}{c_1^2 - c_2^2}\left(c_1^2 e^{-R/c_1} - c_2^2 e^{-R/c_2}\right)\right] \\ & + \frac{3R_iR_jR_k}{R^5}\left[1 - \frac{10(c_1^2 + c_2^2)}{R^2} + \frac{10}{c_1^2 - c_2^2}\frac{1}{R^2}\left(c_1^4 e^{-R/c_1} - c_2^4 e^{-R/c_2}\right) + \frac{10}{c_1^2 - c_2^2}\frac{1}{R}\left(c_1^3 e^{-R/c_1} - c_2^3 e^{-R/c_2}\right) + \frac{4}{c_1^2 - c_2^2}\left(c_1^2 e^{-R/c_1} - c_2^2 e^{-R/c_2}\right) + \frac{2R}{3(c_1^2 - c_2^2)}\left(c_1 e^{-R/c_1} - c_2 e^{-R/c_2}\right)\right] \end{aligned} \tag{A.11}$$

and

$$A_{,iik} = -\frac{2R_k}{R^3}\left[1 - \frac{1}{c_1^2 - c_2^2}\left(c_1^2 e^{-R/c_1} - c_2^2 e^{-R/c_2}\right) - \frac{R}{c_1^2 - c_2^2}\left(c_1 e^{-R/c_1} - c_2 e^{-R/c_2}\right)\right]. \tag{A.12}$$

The expressions (A.7)–(A.12) are non-singular. In the limit $c_2 \to 0$ and $c_1 = \ell$, Eqs. (A.7)–(A.12) reduce to Eqs. (A.1)–(A.6).

Appendix B. Green tensors of generalized Navier equations

The following notation is used for the n-dimensional Fourier transform (Guelfand and Chilov, 1962)

$$\tilde{f}(\boldsymbol{k}) \equiv \mathcal{F}_{(n)}[f(\boldsymbol{r})] = \int_{-\infty}^{+\infty} f(\boldsymbol{r}) e^{+i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{r}, \tag{B.1}$$

$$f(\boldsymbol{r}) \equiv \mathcal{F}^{-1}_{(n)}\left[\tilde{f}(\boldsymbol{k})\right] = \frac{1}{(2\pi)^n}\int_{-\infty}^{+\infty} \tilde{f}(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{k}. \tag{B.2}$$

We have (Wladimirow, 1971; Nowacki, 1986)

$$\mathcal{F}^{-1}_{(2)}\left[\frac{1}{k^2}\right] = -\frac{1}{2\pi}\left(\gamma_E + \ln\sqrt{x^2 + y^2}\right), \tag{B.3}$$

$$\mathcal{F}^{-1}_{(3)}\left[\frac{1}{k^2}\right] = \frac{1}{4\pi}\frac{1}{\sqrt{x^2 + y^2 + z^2}}, \tag{B.4}$$

$$\mathcal{F}^{-1}_{(2)}\left[\frac{1}{k^4}\right] = \frac{1}{8\pi}(x^2 + y^2)\left(\gamma_E + \ln\sqrt{x^2 + y^2}\right), \tag{B.5}$$

$$\mathcal{F}^{-1}_{(3)}\left[\frac{1}{k^4}\right] = -\frac{1}{8\pi}\sqrt{x^2 + y^2 + z^2}, \tag{B.6}$$

$$\mathcal{F}^{-1}_{(2)}\left[\frac{1}{k^2 + \frac{1}{c^2}}\right] = \frac{1}{2\pi}K_0\left(\sqrt{x^2 + y^2}/c\right), \tag{B.7}$$

$$\mathcal{F}^{-1}_{(3)}\left[\frac{1}{k^2 + \frac{1}{c^2}}\right] = \frac{1}{4\pi\sqrt{x^2 + y^2 + z^2}}\exp\left(-\sqrt{x^2 + y^2 + z^2}/c\right). \tag{B.8}$$

B.1. Green tensor of the Helmholtz–Navier equation

The Green tensor of the Helmholtz–Navier equation is defined by

$$(1 - \ell^2\Delta)(\mu\delta_{il}\Delta + (\lambda + \mu)\partial_i\partial_l)G_{lj}(r) = -\delta_{ij}\delta(\boldsymbol{x}). \tag{B.9}$$

The Fourier transform of Eq. (B.9) reads

$$(1 + \ell^2k^2)\left(\mu\delta_{il}k^2 + (\lambda + \mu)k_ik_l\right)\tilde{G}_{lj}(k) = \delta_{ij}, \tag{B.10}$$

where $\lambda = 2\mu\nu/(1 - 2\nu)$ and ν is Poisson's ratio. The Fourier transformed Green tensor is found as

$$\tilde{G}_{ij}(k) = \frac{1}{\mu}\left[\frac{\delta_{ij}}{k^2} - \frac{1}{2(1 - \nu)}\frac{k_ik_j}{k^4}\right]\frac{1}{1 + \ell^2k^2}. \tag{B.11}$$

Using partial fractions and the inverse Fourier transform, we find

$$\begin{aligned} \mathcal{F}^{-1}_{(3)}\left(\frac{k_ik_j}{k^4(1 + \ell^2k^2)}\right) &= -\partial_i\partial_j\mathcal{F}^{-1}_{(3)}\left(\frac{1}{k^4(1 + \ell^2k^2)}\right) \\ &= -\partial_i\partial_j\mathcal{F}^{-1}_{(3)}\left(\frac{1}{k^4} - \frac{\ell^2}{k^2} + \frac{\ell^2}{k^2 + \frac{1}{\ell^2}}\right) \\ &= \frac{\partial_i\partial_j}{8\pi}\left(r + \frac{2\ell^2}{r} - \frac{2\ell^2}{r}e^{-r/\ell}\right) \end{aligned} \tag{B.12}$$

and

$$\mathcal{F}^{-1}_{(3)}\left(\frac{1}{k^2(1+\ell^2k^2)}\right)=-\Delta\mathcal{F}^{-1}_{(3)}\left(\frac{1}{k^4(1+\ell^2k^2)}\right)=\frac{\Delta}{8\pi}\left(r+\frac{2\ell^2}{r}-\frac{2\ell^2}{r}\mathrm{e}^{-r/\ell}\right), \tag{B.13}$$

the three-dimensional Green tensor of the Helmholtz–Navier equation is calculated as

$$G_{ij}(r)=\frac{1}{16\pi\mu(1-\nu)}\left[2(1-\nu)\delta_{ij}\Delta-\partial_i\partial_j\right]\left[r+\frac{2\ell^2}{r}\left(1-\mathrm{e}^{-r/\ell}\right)\right], \tag{B.14}$$

where $r=\sqrt{x^2+y^2+z^2}$.

On the other hand, using

$$\begin{aligned}\mathcal{F}^{-1}_{(2)}\left(\frac{k_ik_j}{k^4(1+\ell^2k^2)}\right)&=-\partial_i\partial_j\mathcal{F}^{-1}_{(2)}\left(\frac{1}{k^4(1+\ell^2k^2)}\right)\\&=-\partial_i\partial_j\mathcal{F}^{-1}_{(2)}\left(\frac{1}{k^4}-\frac{\ell^2}{k^2}+\frac{\ell^2}{k^2+\frac{1}{\ell^2}}\right)\\&=-\frac{\partial_i\partial_j}{8\pi}\left(r^2(\gamma_E+\ln r)+4\ell^2(\gamma_E+\ln r+K_0(r/\ell))\right)\end{aligned} \tag{B.15}$$

and

$$\mathcal{F}^{-1}_{(2)}\left(\frac{1}{k^2(1+\ell^2k^2)}\right)=\mathcal{F}^{-1}_{(2)}\left(\frac{1}{k^2}-\frac{1}{k^2+\frac{1}{\ell^2}}\right)=-\frac{1}{2\pi}(\gamma_E+\ln r+K_0(r/\ell)), \tag{B.16}$$

the two-dimensional Green tensor of the Helmholtz–Navier equation is obtained as

$$\begin{aligned}G_{ij}(r)=&-\frac{1}{2\pi\mu}\delta_{ij}\{\gamma_E+\ln r+K_0(r/\ell)\}\\&+\frac{1}{16\pi\mu(1-\nu)}\partial_i\partial_j\{r^2(\gamma_E+\ln r)+4\ell^2(\gamma_E+\ln r+K_0(r/\ell))\},\end{aligned} \tag{B.17}$$

where $r=\sqrt{x^2+y^2}$.

B.2. Green function of the Helmholtz–Laplace equation

For the anti-plane strain problem, the Green tensor of the Navier–Helmholtz equation reduces to the Green function of the two-dimensional Helmholtz–Laplace equation which is defined by

$$(1-\ell^2\Delta)\Delta G_{zz}(r)=-\frac{1}{\mu}\delta(\boldsymbol{x}). \tag{B.18}$$

The Fourier transform of Eq. (B.18) reads

$$(1+\ell^2k^2)k^2\widetilde{G}_{zz}(k)=\frac{1}{\mu}. \tag{B.19}$$

The Fourier transformed Green function is

$$\widetilde{G}_{zz}(k)=\frac{1}{\mu}\frac{1}{k^2(1+\ell^2k^2)}. \tag{B.20}$$

Using Eq. (B.16), the two-dimensional Green function is calculated as

$$G_{zz}(r)=-\frac{1}{2\pi\mu}\{\gamma_E+\ln r+K_0(r/\ell)\}. \tag{B.21}$$

B.3. Green tensor of the bi-Helmholtz–Navier equation

The Green tensor of the bi-Helmholtz–Navier equation is defined by

$$(1-c_1^2\Delta)(1-c_2^2\Delta)(\mu\delta_{il}\Delta+(\lambda+\mu)\partial_i\partial_l)G_{lj}(r)=-\delta_{ij}\delta(\boldsymbol{x}). \tag{B.22}$$

The Fourier transform of Eq. (B.22) reads

$$(1+c_1^2k^2)(1+c_2^2k^2)\Big(\mu\delta_{il}k^2+(\lambda+\mu)k_ik_l\Big)\widetilde{G}_{lj}(k)=\delta_{ij}. \tag{B.23}$$

The Fourier space Green tensor is

$$\widetilde{G}_{ij}(k)=\frac{1}{\mu}\left[\frac{\delta_{ij}}{k^2}-\frac{1}{2(1-\nu)}\frac{k_ik_j}{k^4}\right]\frac{1}{(1+c_1^2k^2)(1+c_2^2k^2)}. \tag{B.24}$$

Using

$$\begin{aligned}&\mathcal{F}^{-1}_{(3)}\left(\frac{k_ik_j}{k^4(1+c_1^2k^2)(1+c_2^2k^2)}\right)\\&=-\partial_i\partial_j\mathcal{F}^{-1}_{(3)}\left(\frac{1}{k^4(1+c_1^2k^2)(1+c_2^2k^2)}\right)\\&=-\partial_i\partial_j\mathcal{F}^{-1}_{(3)}\left(\frac{1}{k^4}-\frac{c_1^2+c_2^2}{k^2}+\frac{1}{c_1^2-c_2^2}\left(\frac{c_1^4}{k^2+\frac{1}{c_1^2}}-\frac{c_2^4}{k^2+\frac{1}{c_2^2}}\right)\right)\\&=\frac{\partial_i\partial_j}{8\pi}\left(r+\frac{2(c_1^2+c_2^2)}{r}-\frac{2}{c_1^2-c_2^2}\frac{1}{r}\left(c_1^4\mathrm{e}^{-r/c_1}-c_2^4\mathrm{e}^{-r/c_2}\right)\right)\end{aligned} \tag{B.25}$$

and

$$\begin{aligned}&\mathcal{F}^{-1}_{(3)}\left(\frac{1}{k^2(1+c_1^2k^2)(1+c_2^2k^2)}\right)\\&=-\Delta\mathcal{F}^{-1}_{(3)}\left(\frac{1}{k^4(1+c_1^2k^2)(1+c_2^2k^2)}\right)\\&=\frac{\Delta}{8\pi}\left(r+\frac{2(c_1^2+c_2^2)}{r}-\frac{2}{c_1^2-c_2^2}\frac{1}{r}\left(c_1^4\mathrm{e}^{-r/c_1}-c_2^4\mathrm{e}^{-r/c_2}\right)\right),\end{aligned} \tag{B.26}$$

the three-dimensional Green tensor of the bi-Helmholtz–Navier equation is found as

$$\begin{aligned}G_{ij}(r)=&\frac{1}{16\pi\mu(1-\nu)}\left[2(1-\nu)\delta_{ij}\Delta-\partial_i\partial_j\right]\Bigg[r+\frac{2(c_1^2+c_2^2)}{r}\\&-\frac{2}{c_1^2-c_2^2}\frac{1}{r}\left(c_1^4\mathrm{e}^{-r/c_1}-c_2^4\mathrm{e}^{-r/c_2}\right)\Bigg],\end{aligned} \tag{B.27}$$

where $r=\sqrt{x^2+y^2+z^2}$.

In two dimensions, we use the formulae

$$\begin{aligned}&\mathcal{F}^{-1}_{(2)}\left(\frac{1}{k^2(1+c_1^2k^2)(1+c_2^2k^2)}\right)\\&=\mathcal{F}^{-1}_{(2)}\left(\frac{1}{k^2}-\frac{1}{c_1^2-c_2^2}\left(\frac{c_1^2}{k^2+\frac{1}{c_1^2}}-\frac{c_2^2}{k^2+\frac{1}{c_2^2}}\right)\right)\\&=-\frac{1}{4\pi}\left(\gamma_E+\ln r+\frac{1}{c_1^2-c_2^2}\left[c_1^2K_0(r/c_1)-c_2^2K_0(r/c_2)\right]\right)\end{aligned} \tag{B.28}$$

and

$$\begin{aligned}&\mathcal{F}^{-1}_{(2)}\left(\frac{k_ik_j}{k^4(1+c_1^2k^2)(1+c_2^2k^2)}\right)\\&=-\partial_i\partial_j\mathcal{F}^{-1}_{(2)}\left(\frac{1}{k^4(1+c_1^2k^2)(1+c_2^2k^2)}\right)\\&=-\partial_i\partial_j\mathcal{F}^{-1}_{(2)}\left(\frac{1}{k^4}-\frac{c_1^2+c_2^2}{k^2}+\frac{1}{c_1^2-c_2^2}\left(\frac{c_1^4}{k^2+\frac{1}{c_1^2}}-\frac{c_2^4}{k^2+\frac{1}{c_2^2}}\right)\right)\\&=-\frac{\partial_i\partial_j}{8\pi}\Bigg(r^2(\gamma_E+\ln r)+4(c_1^2+c_2^2)(\gamma_E+\ln r)\\&\quad+\frac{4}{c_1^2-c_2^2}\left[c_1^4K_0(r/c_1)-c_2^4K_0(r/c_2)\right]\Bigg).\end{aligned} \tag{B.29}$$

Eventually, the two-dimensional Green tensor of the bi-Helmholtz–Navier equations is obtained as

$$G_{ij}(r) = -\frac{1}{2\pi\mu}\delta_{ij}\left\{\gamma_E + \ln r + \frac{1}{c_1^2 - c_2^2}\left[c_1^2 K_0(r/c_1) - c_2^2 K_0(r/c_2)\right]\right\} + \frac{1}{16\pi\mu(1-\nu)}\partial_i\partial_j\left\{R^2(\gamma_E + \ln r) + 4(c_1^2 + c_2^2)(\gamma_E + \ln r) + \frac{4}{c_1^2 - c_2^2}\left[c_1^4 K_0(r/c_1) - c_2^4 K_0(r/c_2)\right]\right\}. \quad \text{(B.30)}$$

B.4. Green function of the bi-Helmholtz-Laplace equation

For the anti-plane strain problem, the Green tensor of the Navier–Helmholtz equation reduces to the Green function of the two-dimensional bi-Helmholtz-Laplace equation which is defined by

$$(1 - c_1^2\Delta)(1 - c_2^2\Delta)\Delta G_{zz}(r) = -\frac{1}{\mu}\delta(\boldsymbol{x}). \quad \text{(B.31)}$$

The Fourier transform of Eq. (B.31) reads

$$(1 + c_1^2k^2)(1 + c_2^2k^2)k^2\widetilde{G}_{zz}(k) = \frac{1}{\mu}. \quad \text{(B.32)}$$

The Fourier transformed Green function is

$$\widetilde{G}_{zz}(k) = \frac{1}{\mu}\frac{1}{k^2(1 + c_1^2k^2)(1 + c_2^2k^2)}. \quad \text{(B.33)}$$

Using Eq. (B.28), the two-dimensional Green function is obtained as

$$G_{zz}(r) = -\frac{1}{2\pi\mu}\left\{\gamma_E + \ln r + \frac{1}{c_1^2 - c_2^2}\left[c_1^2 K_0(r/c_1) - c_2^2 K_0(r/c_2)\right]\right\}. \quad \text{(B.34)}$$

References

Agiasofitou, E.K., Lazar, M., 2009. Conservation and balance laws in linear elasticity of grade three. J. Elast. 94, 69–85.

Altan, B.C., Aifantis, E.C., 1997. On some aspects in the special theory of gradient elasticity. J. Mech. Behav. Mater. 8, 231–282.

Burgers, J.M., 1939. Some considerations on the fields of stress connected with dislocations in a regular lattice III. Proc. Konikl. Nederl. Acad. Wetenschap. 42 (293–325), 378–399.

Cai, W., Arsenlis, A., Weinberger, C.R., Bulatov, V.V., 2006. A non-singular continuum theory of dislocations. J. Mech. Phys. Solids 54, 561–587.

Deng, S., Liu, J., Liang, N., 2007. Wedge and twist disclinations in second strain gradient elasticity. Int. J. Solids Struct. 44, 3646–3665.

DeWit, R., 1960. The continuum theory of stationary dislocations. Solid State Phys. 10, 249–292.

DeWit, R., 1973a. Theory of disclinations II, J. Res. Nat. Bur. Stand. (U.S.) 77A, 49–100.

DeWit, R., 1973b. Theory of disclinations III, J. Res. Nat. Bur. Stand. (U.S.) 77A, 359–368.

Eringen, A.C., 1983. On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. J. Appl. Phys. 54, 4703–4710.

Eringen, A.C., 2002. Nonlocal Continuum Field Theories. Springer, New York.

Eshel, N.N., Rosenfeld, G., 1970. Effects of strain-gradient on the stress-concentration at a cylindrical hole in a field of unaxial tension. J. Eng. Math. 4, 97–111.

Gao, X.-L., Ma, H.M., 2009. Green's function and Eshelby's tensor based on a simplified strain gradient elasticity theory. Acta Mech. 207, 163–181.

Gao, X.-L., Ma, H.M., 2010a. Solution of Eshelby's inclusion problem with a bounded domain and Eshelby's tensor for a spherical inclusion in a finite matrix based on a simplified strain gradient elasticity theory. J. Mech. Phys. Solids 88, 779–797.

Gao, X.-L., Ma, H.M., 2010b. Strain gradient solution for Eshelby's ellipsoidal inclusion problem. Proc. R. Soc. Lond. A 466, 2425–2446.

Georgiadis, H.G., 2003. The mode III crack problem in microstructured solids governed by dipolar gradient elasticity. ASME J. Appl. Mech. 70, 517–530.

Ghoniem, N.M., Sun, L.Z., 1999. Fast-sum method for the elastic field of three-dimensional dislocation ensembles. Phys. Rev. B 60, 128–140.

Guelfand, I.M., Chilov, G.E., 1962. Les Distributions I. Dunod, Paris.

Gurtin, M.E., 1972. The linear theory of elasticity. In: Flügge, S. (Ed.), Handbuch der Physik VIa/2. Springer, Berlin, pp. 1–345.

Gutkin, M.Yu., Aifantis, E.C., 1999. Dislocations in the theory of gradient elasticity. Scripta Mater. 40, 559–566.

Gutkin, M.Yu., 2000. Nanoscopics of dislocations and disclinations in gradient elasticity. Rev. Adv. Mater. Sci. 1, 27–60.

Gutkin, M.Yu., Ovid'ko, I., 2004. Plastic Deformation in Nanocrystalline Materials. Springer, Berlin.

Gutkin, M.Yu., 2006. Elastic behaviour of defects in nanomaterials I. Rev. Adv. Mater. Sci. 13, 125–161.

Hetnarski, R.B., Ignaczyk, J., 2004. Mathematical Theory of Elasticity. Taylor and Francis, London.

Hirth, J.P., Lothe, J., 1982. Theory of Dislocations, 2nd edition. John Wiley, New York.

Jaunzemis, W., 1967. Continuum Mechanics. The Macmillan Company, New York.

Kelvin, Lord, 1882. Mathematical and Physical Papers. Cambridge University Press, Cambridge, vol. 1, p. 97.

Kossecka, E., 1974. Mathematical theory of defects. Part I. Statics. Arch. Mech. 26, 995–1010.

Kröner, E., 1958. Kontinuumstheorie der Versetzungen und Eigenspannungen. Springer, Berlin.

Lardner, R.W., 1974. Mathematical Theory of Dislocations and Fracture. University of Toronto Press, Toronto.

Lazar, M., 2003. Dislocations in the field theory of elastoplasticity. Comput. Mater. Sci. 28, 419–428.

Lazar, M., Maugin, G.A., 2005. Nonsingular stress and strain fields of dislocations and disclinations in first strain gradient elasticity. Int. J. Eng. Sci. 43, 1157–1184.

Lazar, M., Maugin, G.A., Aifantis, E.C., 2005. On dislocations in a special class of generalized elasticity. Phys. Stat. Sol. B 242, 2365–2390.

Lazar, M., Maugin, G.A., 2006a. Dislocations in gradient elasticity revisited. Proc. R. Soc. Lond. A 462, 3465–3480.

Lazar, M., Maugin, G.A., 2006b. A note on line forces in gradient elasticity. Mech. Res. Commun. 33, 674–680.

Lazar, M., Maugin, G.A., Aifantis, E.C., 2006a. Dislocations in second strain gradient elasticity. Int. J. Solids Struct. 43, 1787–1817; [Addendum, Int. J. Solids Struct. 47, 738–739 (2010)].

Lazar, M., Maugin, G.A., Aifantis, E.C., 2006b. On the theory of nonlocal elasticity of bi-Helmholtz type and some applications. Int. J. Solids Struct. 43, 1404–1421.

Li, S., Wang, G., 2008. Introduction to Micromechanics and Nanomechanics. World Scientific, Singapore.

Lothe, J., 1992. Dislocations in Continuous Elastic Media. In: V.L. Indenbom, J. Lothe, (Eds.) Elastic Strain Fields and Dislocation Mobility, Elsevier, pp. 175–235.

Maranganti, R., Sharma, P., 2007. A novel atomistic approach to determine strain-gradient elasticity constants: tabulation and comparison for various metals, semiconductors, silica, polymers and the (Ir) relevance for nanotechnologies. J. Mech. Phys. Solids 55, 1823–1852.

Mindlin, R.D., 1964. Micro-structure in linear elasticity. Arch. Ration. Mech. Anal. 16, 51–78.

Mindlin, R.D., 1965. Second gradient of strain and surface-tension in linear elasticity. Int. J. Solids Struct. 1, 417–438.

Mindlin, R.D., Eshel, N.N., 1968. On first strain-gradient theories in linear elasticity. Int. J. Solids Struct. 4, 109–124.

Mindlin, R.D., 1972. Elasticity, piezoelectricity and crystal lattice dynamics. J. Elast. 2, 217–282.

Mura, T., 1987. Micromechanics of Defects in Solids, 2nd ed. Martinus Nijhoff, Dordrecht.

Nabarro, F.R.N., 1967. Theory of Crystal Dislocations. Oxford University Press, Oxford.

Nowacki, W., 1986. Theory of Asymmetric Elasticity. PWN-Polish Scientific Publishers, Warszawa.

Peach, M.O., Koehler, J.S., 1950. Forces extended on dislocations and the stress field produced by them. Phys. Rev. 80, 436–439.

Polyzos, D., Tsepoura, K.G., Tsinopoulos, S.V., Beskos, D.E., 2003. A boundary element method for solving 2-D and 3-D static gradient elastic problems. Part I: Integral formulation. Comput. Methods Appl. Mech. Eng. 192, 2845–2873.

Shodja, H.M., Tehranchi, A., 2010. A formulation for the characteristic lengths of fcc materials in first strain gradient elasticity via Sutton–Chen potential. Phil. Mag. 90, 1893–1913.

Shodja, H.M., Ahmadpoor, F., Tehranchi, A., 2012. Calculation of the additional constants for fcc materials in second strain gradient elasticity: behavior of a nano-size Bernoulli–Euler beam with surface effects. Journal of Applied Mechanics 79, 021008 (8 pages).

Teodosiu, C., 1982. Elastic Models of Crystal Defects. Springer-Verlag, Berlin.

Toupin, R.A., Grazis, D.C., 1964. Surface effects and initial stress in continuum and lattice models of elastic crystals. In: Wallis, R.F. (Ed.), Proceedings of the International Conference on Lattice Dynamics. Pergamon Press, Copenhagen, pp. 597–602.

Wladimirow, W.S., 1971. Equations of Mathematical Physics. Deutscher Verlag der Wissenschaften, Berlin (in German).

Wu, C.H., 1992. Cohesive elasticity and surface phenomena. Quart. Appl. Math. L. 1, 73–103.

Zhang, X., Jiao, K., Sharma, P., Yakobson, B.I., 2006. An atomistic and non-classical continuum field theoretic perspective of elastic interactions between defects (force dipoles) of various symmetries and application to graphene. J. Mech. Phys. Solids 54, 2304–2329.