A Note on the Duality of Time-Varying Decomposable Systems*

G. Naudé

National Research Institute for Mathematical Sciences of the CSIR, Pretoria, South Africa

Previous results on the duality between reachability and observability for time-invariant decomposable systems in a category are extended to the time-varying case. In particular the results can be applied to time-varying infinite-dimensional linear systems in Banach spaces, and to time-varying linear systems over rings.

1. Introduction

Time-invariant decomposable systems were introduced by Arbib and Manes (1974) while the extension to time-varying systems appeared in Arbib and Manes (1975). The authors were, however, unable to apply their general results on the duality between reachability and observability to infinite-dimensional linear systems. Hegner (1978) used a slightly different approach to obtain a duality theory for time-invariant infinite-dimensional linear systems in linearly topologized spaces, while in Naudé (to appear) the Arbib-Manes theory is modified so that it can be applied to time-invariant infinite-dimensional linear systems in reflexive Banach spaces.

It is the purpose of this note to show that by using the Arbib-Manes approach to time-varying decomposable systems the latter results can easily be extended to the time-varying case. These results also yield a time-varying version of Hegner's theory as well as a duality for time-varying systems over rings.

2. A Duality Theory for Time-Varying Decomposable Systems

We briefly review some basic notions about decomposable systems. More details can be found in Arbib and Manes (1975).

A discrete-time constant finite-dimensional linear system is defined by equations of the form

\[ q(t + 1) = Fq(t) + Gi(t) \]
\[ y(t) = Hq(t), \]

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where $F: Q \to Q$, $G: I \to Q$ and $H: Q \to Y$ are linear transformations. We can represent the system by a 6-tuple $M = (Q F I G Y H)$.

A time-invariant decomposable system in a category $\mathbf{A}$ is simply such a 6-tuple $M$ with the vector spaces $Q$, $I$ and $Y$ replaced by $\mathbf{A}$-objects and the linear maps $F$, $G$ and $H$ replaced by $\mathbf{A}$-morphisms.

A time-varying decomposable system in a category $\mathbf{A}$ is a 6-tuple $M = (Q F I G Y H)$, where $Q$ is a sequence $(Q_k | k \in \mathbb{Z})$ (where $\mathbb{Z}$ denotes the set of all integers) of $\mathbf{A}$-objects, $I$ and $Y$ are $\mathbf{A}$-objects and $F$, $G$ and $H$ are sequences of $\mathbf{A}$-morphisms $(F_k: Q_k \to Q_{k+1} | k \in \mathbb{Z})$, $(G_k: I \to Q_k | k \in \mathbb{Z})$ and $(H_k: Q_k \to Y | k \in \mathbb{Z})$.

We note that we are considering systems with fixed input object $I$ and output object $Y$ but with variable state object, being $Q_k$ at time $k$. Of course, if $Q_k = Q$, $F_k = F$, $G_k = G$ and $H_k = H$ for all $k$ the system is again time-invariant.

We assume that $\mathbf{A}$ is such that $I$ has a countable copower $(\text{in}_j: I \to I^j | j \in \mathbb{N})$ and $Y$ has a countable power $(\pi_j: Y^j \to Y | j \in \mathbb{N})$. This enables us to define the reachability map $r$ and observability map $\sigma$ of a decomposable system. For example, in the time-invariant case $r$ is uniquely defined by $r_{in} = F^jG$ while $\sigma$ is characterized by $\pi_{\sigma r} = H^jF^j$.

For all $k \in \mathbb{Z}$, set $\Phi_{k,k} = 1_{Q_k}$, while for $k > l$ set $\Phi_{k,l} = F_{k-1} \cdots F_{l+1}F_l$. The reachability morphism $r_{k}: I^k \to Q_k$ of a time-varying $M$ at time $k$ is uniquely defined by $r_{k_{in}} = \Phi_{k,k-j}G_{k-j}$, while the observability morphism $\sigma_{k}: Q_k \to Y^k$ of $M$ at time $k$ is similarly defined by $\pi_{\sigma r_k} = H_{k+j}F_{k+j}$. A linear time-invariant system is said to be reachable if its reachability map is onto, and observable if its observability map is one to one. In the category of vector spaces onto and one-to-one maps are epimorphisms and monomorphisms, respectively. In more general categories various kinds of epi's and mono's arise so that it is profitable to axiomatize a class of possibilities. Thus an image factorization system $(\mathbf{E}, \mathbf{M})$ consists of a class $\mathbf{E}$ of epimorphisms and a class $\mathbf{M}$ of monomorphisms satisfying some axioms. A time-invariant decomposable system is called reachable if its reachability map is in $\mathbf{E}$ and observable if its observability map is in $\mathbf{M}$.

We say that our time-varying system $\mathbf{M}$ is reachable at time $k$ if $r_k \in \mathbf{E}$; that it is completely reachable if $r_k \in \mathbf{E}$ for all $k \in \mathbb{Z}$; that it is observable at time $k$ if $\sigma_k \in \mathbf{M}$; and that it is completely observable if $\sigma_k \in \mathbf{M}$ for all $k \in \mathbb{Z}$.

We are now ready to develop the general duality theory. Let $\mathbf{A}$ and $\mathbf{B}$ be categories, each with countable powers and copowers, and suppose $(\mathbf{E}_1, \mathbf{M}_1)$ and $(\mathbf{E}_2, \mathbf{M}_2)$ are image factorization systems for $\mathbf{A}$ and $\mathbf{B}$, respectively.

Let $\mathbf{K}$ and $\mathbf{H}$ be subcategories of $\mathbf{A}$ and $\mathbf{B}$, respectively, such that $\mathbf{K}$ and $\mathbf{H}$ are dual equivalent under the functors $F: \mathbf{K}^\text{op} \to \mathbf{H}$ and $G: \mathbf{H}^\text{op} \to \mathbf{K}$.

Thus, for example, we can take $\mathbf{A} = \mathbf{B} = \text{vector spaces and linear maps}$, and $\mathbf{K} = \mathbf{H} = \text{finite-dimensional vector spaces}$. It is well known that the latter category is dual equivalent to itself via the functor which assigns to each space $V$ the space $V'$ of all linear functionals on $V$.
Further examples of dualities are reflexive Banach spaces as a subcategory of the category of all Banach spaces and contractions, and Morita dualities in the theory of modules.

The duality can be extended as follows. Let $A$ and $B$ be $K$-objects and $f: A^g \to B$. Define $Ff: FB \to (FA)^g$ by the following diagram:

\[
\begin{array}{c}
FB \\ \downarrow F(f \circ \eta) \\
FA
\end{array}
\]

Similarly for $h: A \to B^g$ we have $F'h: (FB)^g \to FA$, while for $g: X \to Y^g$, $k: X^g \to Y$ in $B$ we have $G'g: (GY)^g \to GX$ and $G'k: GY \to (GX)^g$.

Consider the following conditions:

A1 $e: A \to B \in E_1$ and $A \in K$ implies $B \in K$.
A2 $m: A \to B \in M_1$ and $B \in K$ implies $A \in K$.
A3 $e: X \to Y \in E_2$ and $X \in H$ implies $Y \in H$.
A4 $m: X \to Y \in M_2$ and $Y \in H$ implies $X \in H$.

In the case of vector spaces A1 simply states that the image of a finite-dimensional space under a linear map is again finite dimensional while A2 essentially states that a subspace of a finite-dimensional space is finite dimensional.

We note that if $e: V \to W$ is an onto linear map between finite-dimensional spaces, then its dual $e': W' \to V'$ is one to one.

In the following result, which is proved in Naude (to appear), this situation is investigated in the more general case.

**Lemma.** Suppose that for any $e$ in $K$ we have $e \in E_1$ if and only if $Fe \in M_2$ and let $f: A^g \to B$, $h: A \to B^g$ in $A$ and $g: X \to Y^g$, $k: X^g \to Y$ in $B$.

If A2, A3 hold then

(i) $f \in E_1$ if and only if $F'f \in M_2$;

(ii) $g \in M_2$ if and only if $G'g \in E_1$.

If A1, A4 hold, then

(iii) $h \in M_1$ if and only if $F'h \in E_2$;

(iv) $k \in E_2$ if and only if $G'k \in M_1$.

We call a time-varying system $M = (Q, F, I, G, Y, H)$ finite in $A$ if $I$, $Y$ and $Q_k$, for all $k \in Z$, are in $K$. 

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The dual of a finite system $M$ in $A$ is the system $FM = (FQ, FF, FY, FH, FI, FG)$, where

$$(FQ)_k = FQ_{-k},$$
$$(FF)_k = F(F_{-k-1}): FQ_{-k} \rightarrow FQ_{-k-1},$$
$$(FH)_k = F(H_{-k}): FY \rightarrow F(Q_{-k}),$$
$$(FG)_k = F(G_{-k}): F(Q_{-k}) \rightarrow FI.$$

Thus $FM$ is finite in $B$. We note that the direction of time is reversed in the dual.

We are now ready to prove our main result.

**Theorem (Duality for finite time-varying decomposable systems.)** Let $A, B, K, H, (E_1, M_1), (E_2, M_2)$ be as above and suppose that $e \in E_1$ if and only if $F_e \in M_2$. Let $M$ be a finite system in $A$ with reachability morphisms $(r_{-k}: P \rightarrow Q_{-k})$ and observability morphisms $(\sigma_{-k}: Q_{-k} \rightarrow Y_{-k})$.

(i) $F'(r_{-k}): (FQ)_k \rightarrow (FI)_k$ is the observability morphism of $FM$ at time $k$.

(ii) $F'(\sigma_{-k}): (FY)_k \rightarrow (FQ)_k$ is the reachability morphism of $FM$ at time $k$.

(iii) If $M_1, E_2$ satisfy $A_2, A_3$ then $M$ is $E_1$-reachable at time $-k$ if and only if $FM$ is $M_2$-observable at time $k$. $M$ is completely $E_1$-reachable if and only if $FM$ is completely $M_2$-observable.

(iv) If $M_2, E_1$ satisfy $A_1, A_4$ then $M$ is $M_1$-observable at time $-k$ if and only if $FM$ is $E_2$-reachable at time $k$. $M$ is completely $M_1$-observable if and only if $FM$ is completely $E_2$-reachable.

Similar results hold for a finitary system $N$ in $B$ and its dual $GN$ in $A$.

**Proof.** We only have to verify (i), since (ii) is similar, and (iii) and (iv) follow from the Lemma.

To prove (i) we note that the observability morphism $\bar{\sigma}_k$ of $FM$ at time $k$ is characterized by $\pi_j \bar{\sigma}_k = (FG)_{k+j} \bar{\psi}_{k+j,k}$, where $\bar{\psi}_{k,k} = 1_{(FO)_k}$ and $\bar{\psi}_{k,i} = (FF)_{k-1} \cdots (FF)_1$ for $k > l$. We note that $\bar{\psi}_{k,l} = F(\Phi_{-l,-k})$. We have

$$\pi_j F'(r_{-k}) = F(r_{-k} \sigma_{-k}) = F(\Phi_{-k,-l} G_{-k,-l}) = F(G_{-k,-l}) F(\Phi_{-k,-l}) = (FG)_{k+l} (F\Phi)_{k+l}.$$

Thus $F'(r_{-k})$ is indeed the observability morphism of $FM$ at time $k$. \[\qed\]
DUALITY OF TIME-VARYING SYSTEMS

EXAMPLES

1. **Finite-Dimensional Linear Systems**

The duality for time-varying linear systems over a field $K$ (Arbib and Manes (1975)) is obtained by letting $A = B =$ the category of all vector spaces over $K$. The full subcategory of all finite-dimensional spaces is self-dual. We choose $E_1$ and $E_2$ both to be all epimorphisms while $M_1$ and $M_2$ are all monomorphisms. The condition \( e \in E_1 \) if and only if \( Fe \in M_2 \) as well as A1–A4 are satisfied.

2. **Infinite-Dimensional Linear Systems in Banach Space**

In the case of infinite-dimensional time-varying systems we let $A = B =$ the category of Banach spaces and contractive linear maps, and $K = H =$ the subcategory of reflexive Banach spaces.

As image factorization systems we take $E_1 =$ all dense maps, $M_1 =$ all isometric embeddings, $E_2 =$ all surjective maps for which the norm on the codomain is the quotient norm, and $M_2 =$ all injective maps. $M_1$ satisfies A2, while $E_2$ satisfies A3. Furthermore $e \in E_1$ if and only if $Fe \in M_2$ so that (i), (ii) and (iii) of the theorem can be applied.

3. **Infinite-Dimensional Systems in Linearly Topologized Spaces**

In Hegner (1978) a duality theory is developed for infinite-dimensional systems in various linearly topologized spaces. It is mentioned that apparently the theory presented can be extended to the time-varying case, although this has not been verified. The various categories which are used are mostly self-dual with countable powers and copowers. The time-varying version of the theory follows easily from our result by choosing $A = K = H = B$, etc.

4. **Linear Systems over Rings**

The theory in Naudé (to appear) can also be applied to time-invariant linear systems over rings (Naudé and Nolte, to appear). The results easily carry over to the time-varying case. In particular, let $R$ and $S$ be arbitrary rings and let $A$ and $B$ be the categories of left $R$- and right $S$-modules, respectively. Let $RU_S$ be a bimodule defining a Morita duality between the subcategories $R_U [U]$ and $R[U]_S$ of $U$-reflexive submodules. Let $K = R[U]$ and $H = R[U]_S$ while $E_1 =$ all epimorphisms $= E_2$, and $M_1 =$ all monomorphisms $= M_2$. All the conditions necessary for applying the theorem are satisfied.

In the particular case when $R = S$ and $RU_S = R_R$ we have a Morita duality between the subcategories of finitely generated left and right $R$-modules (so that we can apply the theory to finitely generated systems) if and only if $R$ is a quasi-Frobenius ring.

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REFERENCES


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