Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 244 (2008) 1395–1409

Journal of Differential Equations

www.elsevier.com/locate/jde

Existence results for quasilinear parabolic hemivariational inequalities ☆

Liu Zhenhai

Department of Mathematics, Central South University, Changsha, Hunan 410075, PR China

Received 14 May 2007; revised 12 August 2007

Available online 24 October 2007

Abstract

This paper is devoted to the periodic problem for quasilinear parabolic hemivariational inequalities at resonance as well as at nonresonance. By use of the theory of multi-valued pseudomonotone operators, the notion of generalized gradient of Clarke and the property of the first eigenfunction, we build a Landesman–Lazer theory in the nonsmooth framework of quasilinear parabolic hemivariational inequalities.

MSC: 35A15; 35K85; 49J40

Keywords: Parabolic hemivariational inequality; Generalized Clarke subdifferential; Pseudomonotone operator; Existence of solutions

1. Introduction

Let Ω be an open bounded subset of R^N, $Q = \Omega \times (0, T)$, $0 < T < +\infty$, $X = L^2(0, T; H_0^1(\Omega))$.

The aim of this paper is to study the existence of periodic solutions of parabolic hemivariational inequality:

Find $u \in X$, $\frac{\partial u}{\partial t} \in X^*$ (the dual of X) such that $u(x, 0) = u(x, T)$ and

$$\left\langle \frac{\partial u}{\partial t} + Au, v \right\rangle_X + \int_Q j^0(x, t; u; v)\,dx\,dt \geqslant \int_Q g(x, u)v\,dx\,dt, \quad \forall v \in X. \tag{1}$$

☆ Project partially supported by NNSF of China Grant No. 10671211.

E-mail address: zhhliu@mail.csu.edu.cn.

0022-0396/$ – see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.jde.2007.09.001

The operator $A : X \to X^*$ is assumed to be a second order quasilinear differential operator in divergence form of Leray–Lions type

$$Au(x,t) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\big(x, \nabla u(x,t)\big), \tag{2}$$

where $\nabla = (D_1, D_2, \ldots, D_N)$, $D_i = \frac{\partial}{\partial x_i}$; $j(x,t,\cdot)$ is a locally Lipschitz function. The notation $j^0(x,t; u(x,t); v(x,t))$ stands for the generalized Clarke derivative of $j(x,t,\cdot)$ at $u(x,t)$ in the direction $v(x,t)$ (see [4]).

Extensive attention has been paid to the existence results for evolution hemivariational inequalities by many authors in recent years, see, for example, Aizicovici, Papageorgiou and Staicu [2], Carl and Motreanu [3], Denkowski and Migorski [6], Liu [12,13], Migorski and Ochal [17].

A method of super-subsolutions has been established recently in [3] for quasilinear parabolic differential inclusion problems in the form

$$\frac{\partial u}{\partial t} + Au + \partial j(u) \ni f \quad \text{in } Q, \qquad u = 0 \quad \text{on } \Gamma, \qquad u(\cdot, 0) = 0 \quad \text{in } \Omega. \tag{3}$$

One can show that any solution of (3) is a solution of the hemivariational inequality (1) with zero initial value. The reverse is true only if the function j is regular in the sense of Clarke which means that the one-sided directional derivative and the generalized directional derivative coincide, cf. [5, Chapter 2.3].

However, little information is known for this kind of resonance parabolic problems with a nonsmooth potential (hemivariational inequality) like (1). Using the notion of the generalized gradient of Clarke and the property of the first eigenfunction, we shall study solvability of the parabolic hemivariational inequalities like (1) involving resonance.

2. Notation and hypotheses

Let $H_0^1(\Omega)$ denote the usual Sobolev space and $(H_0^1(\Omega))^*$ its dual space. Then $H_0^1(\Omega) \subset L^2(\Omega) \subset (H_0^1(\Omega))^*$ forms an evolution triple with all the embeddings being continuous, dense and compact. It is well known that [19] the L^2-norm of the gradient, defined by $\|\nabla u\|_{L^2(\Omega)} := (\int_\Omega |\nabla u|^2\, dx)^{1/2}$ is equivalent to the norm of the Sobolev space $H_0^1(\Omega)$.

We set $X = L^2(0,T; H_0^1(\Omega))$, whose dual space is $X^* = L^2(0,T; (H_0^1(\Omega))^*)$, and define a function space

$$W = \big\{u \in X\colon\ u_t \in X^*\big\},$$

where the derivative $u' := u_t = \partial u/\partial t$ is understood in the sense of vector-valued distributions, cf. [19], which is characterized by

$$\int_0^T u'(t)\phi(t)\, dt = -\int_0^T u(t)\phi'(t)\, dt, \quad \forall \phi \in C_0^\infty(0,T).$$

The space W endowed with the graph norm

$$\|u\|_W = \|u\|_X + \|u_t\|_{X^*}$$

is a Banach space which is separable and reflexive due to the separability and reflexivity of X and X^*, respectively. Furthermore it is well known that the embedding $W \subset C([0,T], L^2(\Omega))$ is continuous, cf. [19]. Finally, because $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, we have by Aubin's lemma a compact embedding of $W \subset L^2(Q)$, cf. [19]. Let $\|\cdot\|_X$ be the usual norm defined on X (and similarly on X^*):

$$\|u\|_X = \left(\int_0^T \|u(t)\|^2_{H_0^1(\Omega)}\, dt \right)^{1/2}.$$

The norm convergence in any Banach space B and its dual B^* is denoted by $\to$, and the weak convergence by $\rightharpoonup$. We also use the notation $\langle \cdot,\cdot \rangle_B$ for any of the dual pairings between B and B^*. For example, with $f \in X^*$, $u \in X$,

$$\langle f, u\rangle_X = \int_0^T \langle f(t), u(t)\rangle_{H_0^1(\Omega)}\, dt.$$

Let $L := \partial/\partial t$ and its domain of definition $D(L)$ given by

$$D(L) = \{u \in X\colon\ u_t \in X^* \text{ and } u(0) = u(T)\}.$$

The linear operator L, $D(L) \subset X \to X^*$ can be shown to be closed, densely defined and maximal monotone, e.g., cf. [19, Chapter 32].

For a locally Lipschitzian functional $h : B \to R$, we denote by $h^0(u,v)$ the Clarke generalized directional derivative of h at u in the direction v, that is

$$h^0(u,v) := \limsup_{\lambda \to 0+,\, w \to u} \frac{h(w+\lambda v) - h(w)}{\lambda}.$$

Recall also at this point that

$$\partial h(u) := \{u^* \in B^* \mid h^0(u,v) \geqslant \langle u^*, v\rangle_B,\ \forall v \in B\} \tag{4}$$

denotes the generalized Clarke subdifferential and the following assertion holds:

$$h^0(u,v) = \max\{\langle u^*, v\rangle_B\colon\ u^* \in \partial h(u)\}, \quad \forall v \in B. \tag{5}$$

In the following we assume that the coefficients a_i $(i = 1,\dots,N)$ in (2) are functions of $x \in \Omega$ and of $\xi \in R^N$ where $\xi = (\xi_1,\dots,\xi_N) \in R^N$. We assume that each $a_i(x,\xi)$ is a Carathéodory function, i.e., it is measurable in x for fixed $\xi \in R^N$ and continuous in ξ for almost all $x \in \Omega$. We suppose that $a_i(x,\xi)$ $(i = 1,\dots,N)$ satisfy:

(A_1) *There exist* $c_1 > 0$ *and* $b_1 \in L^2(\Omega)$ *such that*

$$\left|a_i(x,\xi)\right| \leqslant c_1|\xi| + b_1(x)$$

for a.e. $x \in \Omega$, *for all* $\xi \in R^N$.

(A_2) $\sum_{i=1}^{N}[a_i(x,\xi) - a_i(x,\xi')](\xi_i - \xi_i') > 0$ *for a.e.* $x \in \Omega$, *for all* $\xi, \xi' \in R^N$ *with* $\xi \neq \xi'$.

(A_3) *There exists a positive* c_2 *and a nonnegative function* $b_2 \in L^1(\Omega)$ *such that*

$$\sum_{i=1}^{n} a_i(x,\xi)\xi_i \geqslant c_2 \sum_{i=1}^{N} |\xi_i|^2 - b_2(x)$$

for a.e. $x \in \Omega$, *for all* $\xi, \xi' \in R^N$.

Concerning problem (1) we deal with the functional $J : X(\subseteq L^2(Q)) \to R$ of type

$$J(u) = \int_Q j\big(x,t;u(x,t)\big)\,dx\,dt, \quad u \in X. \tag{6}$$

We assume that $j : Q \times R \to R$ satisfies the following (H_1):

(a) $j(\cdot,\cdot,s) : Q \to R$ is measurable, $\forall s \in R$;
(b) $j(x,t,\cdot) : R \to R$ is locally Lipschitz, for almost all $(x,t) \in Q$;
(c) $j(\cdot,\cdot,0) \in L^1(Q)$;
(d) $|z| \leqslant b_3(x,t) + c_3|s|^{\sigma-1}$, $\forall s \in R$, a.e. $(x,t) \in Q$, $\forall z \in \partial_s j(x,t,s)$, with constants $c_3 > 0$ and $1 \leqslant \sigma < 2$ and $b_3 \in L^2(Q)$.

The assumptions (a)–(d) on j ensure that J is locally Lipschitz on X and

$$\int_Q j^0\big(x,t;u(x,t);v(x,t)\big)\,dx\,dt \geqslant J^0(u,v), \quad \forall u,v \in X. \tag{7}$$

In the following we also assume that (H_2):

(i) $g : \Omega \times R \to R$ is a Carathéodory function (i.e. $g(\cdot,s) : \Omega \to R$ is measurable, $\forall s \in R$ and $g(x,\cdot) : R \to R$ is continuous, for almost all $x \in \Omega$);
(ii) $\exists c_4 > 0$, and $b_4 \in L^2(\Omega)$ with $b_4 \geqslant 0$, a.e. in Ω such that $|g(x,s)| \leqslant c_4|s| + b_4(x)$ for a.e. $x \in \Omega$, $\forall s \in R$;
(iii) $\exists \gamma \in R$, and $b_5 \in L^2(\Omega)$ with $b_5 \geqslant 0$ a.e. in Ω such that $g(x,s)s \leqslant \gamma|s|^2 + b_5(x)|s|$ for a.e. $x \in \Omega$, $\forall s \in R$.

We also define operators $A, G : X \to X^*$ by

$$\langle Au, v\rangle_X := \sum_{i=1}^{N} \int_Q a_i\big(x,\nabla u(x,t)\big) D_i v(x,t)\,dx\,dt, \quad \forall u,v \in X, \tag{8}$$

$$\langle Gu, v\rangle_X := \int_Q g(x,u)v\,dx\,dt, \quad \forall u,v \in X. \tag{9}$$

Then the hemivariational inequality (1) is equivalent to the following:

Find $u \in D(L)$ such that

$$\langle Lu, v\rangle_X + \langle Au, v\rangle_X + \int\limits_Q j^0(x,t,u,v)\,dx\,dt \geqslant \langle Gu, v\rangle_X, \quad \forall v \in X. \tag{10}$$

We define the 1st eigenvalue of the operator A as

$$\lambda_1 = \liminf_{\|u\|_{L^2}\to\infty} \frac{\langle Au, u\rangle_X}{\|u\|^2_{L^2(Q)}}, \quad u \in X. \tag{11}$$

We say that $M : X \to 2^{X^*}$ is "L-pseudomonotone," if the following conditions hold:

(1) for every $v \in X$, $M(v)$ is a nonempty, weakly compact and convex subset of X^*;
(2) $M(\cdot)$ is usc from each finite-dimensional subspace of X into X^* furnished with the weak topology;
(3) if $\{v_n\}_{n\geqslant 1} \subseteq D(L)$, $v_n \rightharpoonup v$ in X, $Lv_n \rightharpoonup Lv$ in X^*, $v_n^* \in M(v_n)$, $n \geqslant 1$, $v_n^* \rightharpoonup v^* \in X^*$ and $\limsup_{n\to\infty}\langle v_n^*, v_n - v\rangle \leqslant 0$, then $v^* \in M(v)$ and $\langle v_n^*, v_n\rangle_X \to \langle v^*, v\rangle_X$.

The following lemma will be useful (cf. [16], [7, p. 71]).

Lemma 1. *If X is a reflexive Banach space which is strictly convex, $L : D(L) \subseteq X \to X^*$ is a linear, closed, densely defined and maximal monotone operator and $M : X \to 2^{X^*}$ is bounded, coercive, L-pseudomonotone operator, then $L + M$ is surjective, i.e., $R(L + M) = X^*$.*

In order to establish the existence results of the problem (1), we also need the following (see, for instance, [14,15], [7, p. 75]):

Lemma 2. *Suppose that the assumptions* (A_1)–(A_3) *and* (H_1)–(H_2) *hold. Then the sum operator $A - G + \partial J : X \to 2^{X^*}$ is bounded and L-pseudomonotone.*

3. Main results

Theorem 1. *Let assumptions* (A_1)–(A_3) *and* (H_1)–(H_2) *hold. Suppose furthermore $\gamma < \lambda_1$, where γ and λ_1 are defined in* (H_2) *and* (11)*, respectively. Then problem* (1) *has at least one solution.*

Proof. We first prove that the sum operator $A - G + \partial J : X \to 2^{X^*}$ is coercive. To this end, $\forall u_n \in X$ such that $\|u_n\|_X \to \infty$ as $n \to \infty$, $\forall u_n^* \in \partial J(u_n)$, we have

$$\begin{aligned}&\langle Au_n - Gu_n + u_n^*, u_n\rangle_X \\ &\quad= \int\limits_Q \sum_{i=1}^N a_i(x,\nabla u_n) D_i u_n\,dx\,dt - \int\limits_Q g(x,u_n)u_n\,dx\,dt + \langle u_n^*, u_n\rangle_X.\end{aligned} \tag{12}$$

In the case of $\|u_n\|_{L^2(Q)} \to \infty$: By $\gamma < \lambda_1$, we may choose $\varepsilon > 0$ such that $\gamma < \lambda_1 - \varepsilon\lambda_1$. In virtue of (H_2), (A_3), the definition of the least eigenvalue λ_1 and Hölder inequality, there exists $C_1 > 0$ such that

$$\begin{aligned}
&\int_Q \sum_{i=1}^N a_i(x, \nabla u_n) D_i u_n \, dx\, dt - \int_Q g(x, u_n) u_n \, dx\, dt \\
&\quad \geqslant \varepsilon \int_Q \sum_{i=1}^N a_i(x, \nabla u_n) D_i u_n \, dx\, dt + (\lambda_1 - \varepsilon\lambda_1 - \gamma)\|u_n\|_{L^2}^2 - C_1\|u_n\|_{L^2} \\
&\quad \geqslant \varepsilon c_2 \int_Q |\nabla u_n|^2 \, dx\, dt - \varepsilon \int_Q b_2(x)\, dx\, dt + (\lambda_1 - \varepsilon\lambda_1 - \gamma)\|u_n\|_{L^2}^2 - C_1\|u_n\|_{L^2},
\end{aligned} \tag{13}$$

as n is large enough.

In virtue of (5), (7), (H_1)(d) and Hölder inequality, there exists a positive constant C_2 such that

$$\begin{aligned}
\langle u_n^*, u_n \rangle_X &\geqslant -J^0(u_n, -u_n) \\
&\geqslant -\int_Q j^0(x, t, u_n, -u_n)\, dx\, dt \\
&\geqslant -\int_Q \big| j^0(x, t, u_n, -u_n)\big|\, dx\, dt \\
&\geqslant -\int_Q \max\big\{ \big|z(x,t) u_n(x,t)\big| \colon\, z(x,t) \in \partial j(x, t, u_n)\big\}\, dx\, dt \\
&\geqslant -\int_Q \big(b_3(x,t) + c_3|u_n|^{\sigma-1}\big)|u_n|\, dx\, dt \\
&\geqslant -C_2\big(\|u_n\|_{L^2} + \|u_n\|_{L^2}^{\sigma}\big).
\end{aligned} \tag{14}$$

It follows from (12)–(14) and $1 \leqslant \sigma < 2$ and Poincaré's inequality $\|u\|_{L^2(Q)} \leqslant \mathrm{Const} \cdot \|u\|_X$ that

$$\inf_{u_n^* \in \partial J(u_n)} \frac{\langle Au_n - Gu_n + u_n^*, u\rangle_X}{\|u_n\|_X} \to \infty, \quad \text{as } \|u_n\|_X \to \infty. \tag{15}$$

In the case of $\{\|u_n\|_{L^2(Q)}\}_{n=1}^\infty$ being bounded: By (A_3), (H_2), (14) and the Hölder inequality, we get

$$\begin{aligned}
&\langle Au_n - Gu_n + u_n^*, u\rangle_X \\
&\quad \geqslant c_2 \int_Q |\nabla u_n|^2 \, dx\, dt - \int_Q b_2(x)\, dx\, dt - \gamma\|u_n\|_{L^2}^2 - C_2\|u_n\|_{L^2} + \langle u_n^*, u_n\rangle_X \\
&\quad \geqslant c_2 \int_Q |\nabla u_n|^2 \, dx\, dt - \int_Q b_2(x)\, dx\, dt - \gamma\|u_n\|_{L^2}^2 - C_1\|u_n\|_{L^2} - C_2\big(\|u_n\|_{L^2} + \|u_n\|_{L^2}^{\sigma}\big),
\end{aligned}$$

which implies that (15) holds for the case of $\{\|u_n\|_{L^2(Q)}\}_{n=1}^\infty$ being bounded, too.

Therefore, from the discussion of the two cases above, we have shown that the sum operator $A-G+\partial J : X \to 2^{X^*}$ is coercive. In virtue of Lemmas 1 and 2, we get that there exists $u \in D(L)$ such that

$$0 \in Lu + Au - Gu + \partial J(u), \tag{16}$$

i.e., there exist $u \in X$ and $u^* \in \partial J(u)$ such that

$$Lu + Au - Gu + u^* = 0.$$

So we have

$$\langle Lu + Au - Gu + u^*, v \rangle_X = 0, \quad \forall v \in X.$$

By (5) and (7) we have

$$\langle u^*, v \rangle_X \leqslant J^0(u, v) \leqslant \int_Q j^0(x, t, u, v)\, dx\, dt, \quad \forall v \in X,$$

which implies that $u \in D(L)$ and

$$\langle Lu, v \rangle_X + \langle Au, v \rangle_X + \int_Q j^0(x, t, u, v)\, dx\, dt \geqslant \int_Q g(x, u) v\, dx\, dt, \quad \forall v \in X. \tag{17}$$

This ends the proof of the theorem. $\square$

Now we turn to the solvability of the problem (HVI) involving resonance. It is an easy matter in this case to give examples that show that Theorem 1 is false if $\gamma = \lambda_1$, since this is already well known if A given in (1) is linear. Consequently, a further condition is necessary to ensure that the conclusion of Theorem 1 holds for the situation $\gamma = \lambda_1$. Results of this nature are referred to in the literature as resonance results (see [1,8–11]). We shall present one such result here that will hold for the Hilbert space V $(= H_0^1(\Omega))$. In order to do this, we first recall some facts concerning linear elliptic theory.

Let $a : V \times V \to R$ be a continuous, symmetric, bilinear form which is coercive

$$a(u, u) \geqslant \alpha \|u\|^2_{H_0^1(\Omega)}, \quad \forall u \in V = H_0^1(\Omega),$$

with a constant $\alpha > 0$. Thus

$$\|\cdot\|_V := a(\cdot, \cdot)^{1/2}$$

is an equivalent norm on $V = H_0^1(\Omega)$, i.e., there exist two positive constants c_5 and c_6 such that

$$c_5 \|u\|^2_{H_0^1(\Omega)} \leqslant a(u, u) \leqslant c_6 \|u\|^2_{H_0^1(\Omega)}. \tag{18}$$

Similarly, we can define an equivalent norm on X by $\|u\|_X^2 = \int_0^T \|u\|_V^2 \, dt$. Denote by

$$\mu_1 < \mu_2 \leqslant \cdots \leqslant \mu_n \cdots \to +\infty \tag{19}$$

the sequence of eigenvalues of the linear problem

$$a(u, v) = \mu \langle u, v \rangle_{L^2}, \quad \forall v \in V. \tag{20}$$

We also consider a basis $\{\varphi_n\}_{n=1}^{\infty}$ for V consisting of eigenfunctions, where φ_n corresponds to μ_n, i.e., $u = \varphi_n$ and $\mu = \mu_n$ in (20), which is normalized in the following sense

$$a(\varphi_i, \varphi_j) = \delta_{ij}, \tag{21}$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$.

In this statement we use essentially the compactness of the embedding $V \subset L^2(\Omega)$. The fact that μ_1 is simple and the corresponding eigenfunction not changing sign (say $\varphi_1 > 0$) in Ω follows from Krein–Rutman Theorem (see [18], for example).

Now it is well known that

$$\mu_1 = \inf_{u \neq 0} \frac{a(u, u)}{\|u\|_{L^2(\Omega)}^2}, \quad u \in V = \big(H_0^1(\Omega)\big). \tag{22}$$

We extend the bilinear form $a(\cdot,\cdot)$ defined above from $H_0^1(\Omega)$ to X by

$$\tilde{a}(u, v) = \int_0^T a(u, v) \, dt, \quad \forall u, v \in X,$$

and observe from (22) that

$$\mu_1 = \inf_{u \neq 0} \frac{\tilde{a}(u, u)}{\|u\|_{L^2(Q)}^2}, \quad u \in W. \tag{23}$$

In the following theorem, we need the assumptions:

(A_4) Suppose that there exists a smooth function $T : \Omega \times R^N \to R$, such that $(T_{\xi_1}(x, \xi), \ldots, T_{\xi_N}(x, \xi)) = (a_1(x, \xi), \ldots, a_N(x, \xi))$ for $\xi \in R^N$, $x \in \Omega$, here $T_{\xi_i} = \frac{\partial T}{\partial \xi_i}$ $(1 \leqslant i \leqslant N)$.

(A_5) $\lambda_1 = \mu_1$ where λ_1 is given by (11), and

$$\liminf_{\|u\|_{L^2} \to \infty} \frac{\langle Au, u \rangle_X - \tilde{a}(u, u)}{\|u\|_{L^2}} \geqslant 0, \quad u \in X.$$

Also in Theorem 2, we shall set the following assumption:

(H_3) The functions $j_-^{+\infty}(x,t), j_+^{-\infty}(x,t) \in L^2(Q)$ satisfy the following inequalities:

$$\min\left\{\frac{1}{T}\int_Q j_-^{+\infty}(x,t)\varphi_1(x)\,dx\,dt, -\frac{1}{T}\int_Q j_+^{-\infty}(x,t)\varphi_1(x)\,dx\,dt\right\}$$
$$> \int_\Omega b_5(x)\varphi_1(x)\,dx, \tag{24}$$

where $j_-^{+\infty}(x,t) := \inf_{(z_n)}\{\liminf_{n\to\infty} z_n,\ \forall z_n \in \partial_s j(x,t,s_n) \in R \text{ with } s_n \to +\infty\}$; $j_+^{-\infty}(x,t) := \sup_{(z_n)}\{\limsup_{n\to\infty} z_n,\ \forall z_n \in \partial_s j(x,t,s_n) \in R \text{ with } s_n \to -\infty\}$; $b_5(x)$ appears in (H_2).

Remark. (H_3) is a condition of Landsman–Lazer type considered by many authors in connection with solvability of equations involving resonance, see, for example, [1,8–11] and references therein.

Theorem 2. *Let assumptions* (A_1)–(A_5) *and* (H_1)–(H_3) *hold and* $\gamma = \lambda_1$, *where* γ *and* λ_1 *are defined in* (H_2) *and* (11), *respectively. Then problem* (1) *has at least one solution.*

Proof. Set $g_n(x,s) = g(x,s) - n^{-1}s$. It then follows $g_n(x,s)$ meets condition (H_2)(iii) with $\gamma = \lambda_1 - n^{-1}$. Hence $\gamma < \lambda_1$ and the conditions of Theorem 1 are met. Therefore, there exists $u_n \in D(L)$ such that $\forall v \in X$

$$\int_Q u_{nt} v\,dx\,dt + \langle Au_n, v\rangle_X + \int_Q j^0(x,t,u_n,v)\,dx\,dt \geqslant \int_Q \left[g(x,u_n) - n^{-1}u_n\right]v\,dx\,dt. \tag{25}$$

Claim 1. $\exists C_3 > 0$ such that

$$\|u_n\|_{L^2} \leqslant C_3, \quad \forall n = 1,2,\ldots. \tag{26}$$

Suppose to the contrary that (26) is false. Then there exists a subsequence (which for ease of notation we take to be the full sequence) such that

$$\lim_{n\to\infty} \|u_n\|_{L^2} = \infty. \tag{27}$$

We shall show (27) leads to a contradiction. Taking $v = -u_n$ in (25) and using the fact

$$\int_Q uu_t\,dx\,dt = \frac{1}{2}\int_0^T \frac{\partial}{\partial t}\int_\Omega |u|^2\,dx\,dt = 0, \quad \forall u \in D(L),$$

we obtain

$$\langle Au_n, u_n\rangle + \|u_n\|_{L^2}^2/n \leqslant \int_Q j^0(x,t,u_n,-u_n)\,dx\,dt + \int_Q g(x,u_n)u_n\,dx\,dt. \tag{28}$$

Let $\epsilon > 0$ be given. Then it follows from (A_5) that $\exists n_0$ such that

$$\langle Au_n, u_n\rangle - \tilde{a}(u_n, u_n) \geqslant -\epsilon \|u_n\|_{L^2}, \quad \forall n \geqslant n_0.$$

Using the last inequality, we see from (28) that $\forall n \geqslant n_0$

$$\tilde{a}(u_n, u_n) + \|u_n\|_{L^2}^2/n \leqslant \int_Q j^0(x,t,u_n,-u_n)\,dx\,dt + \int_Q g(x,u_n)u_n\,dx\,dt + \epsilon\|u_n\|_{L^2}, \tag{29}$$

which implies that

$$\frac{\tilde{a}(u_n, u_n)}{\|u_n\|_{L^2}^2} + n^{-1} \leqslant \int_Q \frac{j^0(x,t,u_n,-u_n)}{\|u_n\|_{L^2}^2}\,dx\,dt + \int_Q \frac{g(x,u_n)u_n}{\|u_n\|_{L^2}^2}\,dx\,dt + \frac{\epsilon}{\|u_n\|_{L^2}}. \tag{30}$$

Similar to (14), there exists a positive constant C_2 such that

$$\int_Q \frac{|j^0(x,t,u_n,-u_n)|}{\|u_n\|_{L^2}^2}\,dx\,dt \leqslant \frac{C_2(\|u_n\|_{L^2} + \|u_n\|_{L^2}^\sigma)}{\|u_n\|_{L^2}^2},$$

which implies from $1 \leqslant \sigma < 2$ that

$$\int_Q \frac{|j^0(x,t,u_n,-u_n)|}{\|u_n\|_{L^2}^2}\,dx\,dt \to 0, \quad \text{as } \|u_n\|_{L^2} \to \infty. \tag{31}$$

By (H_2) and $\gamma = \lambda_1$ it is clear that

$$\int_Q \frac{g(x,u_n)u_n}{\|u_n\|_{L^2}^2}\,dx\,dt \leqslant \lambda_1 + \int_Q \frac{b_5(x)|u_n|}{\|u_n\|_{L^2}^2}\,dx\,dt. \tag{32}$$

From (30)–(32), we obtain

$$\mu_1 \leqslant \liminf_{n\to\infty} \frac{\tilde{a}(u_n, u_n)}{\|u_n\|_{L^2}^2} \leqslant \limsup_{n\to\infty} \frac{\tilde{a}(u_n, u_n)}{\|u_n\|_{L^2}^2} \leqslant \mu_1.$$

On relabeling if necessary, we can assume that $w_n := u_n/\|u_n\|_{L^2} \rightharpoonup w$ in X.

In virtue of the definition of μ_1, the weak lower semi-continuity of the norm, we have

$$\mu_1 \leqslant \|w\|_X^2 \leqslant \liminf_{n\to\infty} \|w_n\|_X^2 \leqslant \limsup_{n\to\infty} \|w_n\|_X^2 \leqslant \mu_1, \tag{33}$$

which implies that

$$\|w\|_X^2 = \mu_1, \qquad \|w\|_{L^2} = 1. \tag{34}$$

Since $u_n(x,0) = u_n(x,T)$, it is easy to get from (A_4) that

$$\begin{aligned}\left\langle Au_n, \frac{\partial u_n}{\partial t}\right\rangle_X &= \sum_{i=1}^{N}\int_Q a_i\big(x, \nabla u_n(x,t)\big)\frac{\partial D_i u_n(x,t)}{\partial t}\,dx\,dt\\ &= \int_0^T \frac{\partial}{\partial t}\int_\Omega T(x, \nabla u_n)\,dx\,dt\\ &= 0, \end{aligned}\tag{35}$$

$$\int_Q \big[g(x,u_n) - n^{-1}u_n\big]\frac{\partial u_n}{\partial t}\,dx\,dt = \int_0^T\int_\Omega \frac{\partial}{\partial t}\int_0^{u_n}\big[g(x,s) - n^{-1}s\big]\,ds\,dx\,dt = 0. \tag{36}$$

Taking $v = -u_{nt}$ in (25), we observe from (H_1), (35), (36) and Hölder inequality, $\exists C_4 > 0$

$$\begin{aligned}\int_Q \left|\frac{\partial u_n}{\partial t}\right|^2 dx\,dt &\leqslant -\left\langle Au_n, \frac{\partial u_n}{\partial t}\right\rangle_X + \int_Q j^0\left(x,t,u_n,-\frac{\partial u_n}{\partial t}\right)dx\,dt\\ &\quad + \int_Q \big[g(x,u_n) - n^{-1}u_n\big]\frac{\partial u_n}{\partial t}\,dx\,dt\\ &= \int_Q j^0\left(x,t,u_n,-\frac{\partial u_n}{\partial t}\right)dx\,dt\\ &\leqslant \int_Q \max\left\{\left|z(x,t)\frac{\partial u_n}{\partial t}\right|:\ z(x,t)\in\partial j(x,t,u_n)\right\}dx\,dt\\ &\leqslant \int_Q \big(b_3(x,t) + c_3|u_n|^{\sigma-1}\big)\left|\frac{\partial u_n}{\partial t}\right|dx\,dt\\ &\leqslant C_4\big(1 + \|u_n\|_{L^2}^{\sigma-1}\big)\left\|\frac{\partial u_n}{\partial t}\right\|_{L^2}. \end{aligned}\tag{37}$$

Dividing both sides of the above inequality by $\|u_n\|_{L^2}\|\frac{\partial u_n}{\partial t}\|_{L^2}$, we easily conclude from $1 \leqslant \sigma < 2$ that

$$\lim_{n\to\infty}\left\|\frac{\partial w_n}{\partial t}\right\|_{L^2} = 0. \tag{38}$$

We thus conclude from (33), (38) that $\{w_n\}_{n=1}^{\infty}$ is a bounded sequence in W. Therefore, we may assume that $\frac{\partial w_n}{\partial t} \rightharpoonup \frac{\partial w}{\partial t}$ in X^* and

$$\frac{\partial w}{\partial t} = 0 \quad \text{a.e. in } Q, \tag{39}$$

which implies that w is independent of variable t, i.e., $w = w(x)$. Furthermore, we obtain from the compact embedding theorem that there is a subsequence (which for ease of notation we take to be the full sequence) such that

$$\lim_{n\to\infty} \|w_n - w\|_{L^2} = 0, \tag{40}$$

$$\lim_{n\to\infty} w_n(x,t) = w(x) \quad \text{a.e. in } Q, \tag{41}$$

which implies from the property of the eigenfunction and (33) that $w \equiv \pm\frac{1}{\sqrt{T}}\varphi_1$. By ($H_2$), ($A_5$) and $\gamma = \lambda_1$, $\forall\varepsilon > 0$, $\exists n_0 > 0$, such that $\forall n \geqslant n_0$, we get that

$$\langle Au_n, u_n\rangle_X - \int_Q \big[g(x,u_n) - n^{-1}u_n\big]u_n\,dx\,dt \geqslant -\int_Q b_5(x)|u_n|\,dx\,dt - \varepsilon\|u_n\|_{L^2}.$$

It follows from (28) that $\forall n \geqslant n_0$

$$\int_Q \frac{j^0(x,t,u_n,-u_n)}{\|u_n\|_{L^2}}\,dx\,dt + \int_Q \frac{b_5(x)|u_n|}{\|u_n\|_{L^2}}\,dx\,dt + \varepsilon \geqslant 0. \tag{42}$$

Then a well-known property of the generalized gradient (cf. [4]) implies for each $n \geqslant n_0$, there exists $z_n \in \partial j(x,t,u_n)$ such that

$$\int_Q \frac{j^0(x,t,u_n,-u_n)}{\|u_n\|_{L^2}}\,dx\,dt = \int_Q \frac{-z_n u_n}{\|u_n\|_{L^2}}\,dx\,dt = -\int_Q z_n w_n\,dx\,dt. \tag{43}$$

If $w = \frac{1}{\sqrt{T}}\varphi_1$, i.e., $w_n = u_n/\|u_n\|_{L^2} \to \frac{1}{\sqrt{T}}\varphi_1$ as $n \to \infty$, then $u_n(x,t) \to +\infty$ for a.e. $(x,t) \in Q$ as $n \to \infty$. Due to (H_3) we arrive at the conclusion that

$$j_-^{+\infty}(x,t) \leqslant \liminf_{n\to\infty} z_n(x,t). \tag{44}$$

Therefore we conclude by Fatou's lemma and (42)–(44) that

$$\begin{aligned}
\int_Q j_-^{+\infty}(x,t)w\,dx\,dt &\leqslant \liminf_{n\to\infty} \int_Q z_n(x,t)w_n(x,t)\,dx\,dt \\
&\leqslant \limsup_{n\to\infty} \int_Q z_n(x,t)w_n(x,t)\,dx\,dt \\
&= -\liminf_{n\to\infty} \int_Q \frac{j^0(x,t,u_n,-u_n)}{\|u_n\|_{L^2}}\,dx\,dt \\
&\leqslant \int_Q b_5(x)w\,dx\,dt + \varepsilon,
\end{aligned}$$

which implies that

$$\frac{1}{T}\int_{Q} j_{-}^{+\infty}(x,t)\varphi_1\,dx\,dt \leqslant \int_{\Omega} b_5(x)\varphi_1(x)\,dx.$$

This contradicts (H_3). Analogously, if $w = -\frac{1}{\sqrt{T}}\varphi_1$, then $u_n(x,t) \to -\infty$. The same argument above implies that

$$-\frac{1}{T}\int_{Q} j_{+}^{-\infty}(x,t)\varphi_1\,dx\,dt \leqslant \int_{\Omega} b_5(x)\varphi_1(x)\,dx,$$

which contradicts (H_3) too. Therefore we have shown that the inequality (26) holds true.

By (26), (28) and the assumptions (A_3), (H_1), (H_2), we easily obtain that the sequence $\{\|u_n\|_X\}_{n=1}^{\infty}$ is bounded. Furthermore, it follows from (26), (37) that $\{\|\frac{\partial u_n}{\partial t}\|_{L^2}\}_{n=1}^{\infty}$ is bounded, i.e., $\{\|Lu_n\|_X\}_{n=1}^{\infty}$ is bounded. Therefore, $\exists C_5$ such that

$$\big\{\|u_n\|_W\big\}_{n=1}^{\infty} \leqslant C_5, \quad \forall n = 1,2,\ldots.$$

From this last inequality we observe as before that there exists a subsequence (which for ease of notation we take to be the full sequence) and $u \in W$ such that $u_n \rightharpoonup u$ in W:

$$Lu_n \rightharpoonup Lu \quad \text{in } X^*, \qquad u_n \rightharpoonup u \quad \text{in } X. \tag{45}$$

Furthermore, from the compact embedding theorem for Sobolev spaces, the following facts prevail:

$$\lim_{n\to\infty} u_n(x,t) = u(x,t) \quad \text{for a.e. } (x,t) \in Q, \tag{46}$$

$$\lim_{n\to\infty} \|u_n - u\|_{L^2} = 0. \tag{47}$$

Claim 2. u solves problem (1).

Taking $v = u - u_n$ in (25), we obtain

$$\begin{aligned}\langle Au_n, u_n - u\rangle_X \leqslant{}& \int_{Q} u_{nt}(u-u_n)\,dx\,dt + \int_{Q} j^0(x,t,u_n,u-u_n)\,dx\,dt \\ &+ \int_{Q}\big[g(x,u_n) - n^{-1}u_n\big](u_n-u)\,dx\,dt.\end{aligned} \tag{48}$$

By (45) and (47) it is easy to get

$$\lim_{n\to\infty}\int_{Q} u_{nt}(u-u_n)\,dx\,dt = 0. \tag{49}$$

Applying the upper semicontinuity of the generalized directional derivative of the locally Lipschitz functions, it follows from (46) and Fatou's lemma that

$$\limsup_{n\to\infty}\int_Q j^0(x,t,u_n,u-u_n)\,dx\,dt \leqslant \int_Q \limsup_{n\to\infty} j^0(x,t,u_n,u-u_n)\,dx\,dt$$
$$\leqslant \int_Q j^0(x,t,u,0)\,dx\,dt = 0. \tag{50}$$

In virtue of (H_2) and the continuity of the Nemytskii operators, we observe from (47) that

$$\lim_{n\to\infty}\int_Q g(x,u_n)(u-u_n)\,dx\,dt = 0. \tag{51}$$

Therefore, using (49)–(51) in (48) we have

$$\limsup_{n\to\infty}\langle Au_n, u_n-u\rangle_X \leqslant 0. \tag{52}$$

By the pseudomonotonicity of A, cf. [14], it follows from (45) and (52)

$$Au_n \rightharpoonup Au \quad \text{in } X^*, \qquad \langle Au_n,u_n\rangle_X \to \langle Au,u\rangle_X. \tag{53}$$

Using the same arguments in (50) and (51), we easily obtain

$$\lim_{n\to\infty}\int_Q g(x,u_n)v\,dx\,dt = \int_Q g(x,u)v\,dx\,dt, \quad \forall v\in X, \tag{54}$$

$$\limsup_{n\to\infty}\int_Q j^0(x,t,u_n,v)\,dx\,dt \leqslant \int_Q \limsup_{n\to\infty} j^0(x,t,u_n,v)\,dx\,dt$$
$$\leqslant \int_Q j^0(x,t,u,v)\,dx\,dt, \quad \forall v\in X. \tag{55}$$

Passing to the limit as $n\to\infty$ on both sides of (25) and using (45)–(55), we obtain

$$\langle Lu,v\rangle_X + \langle Au,v\rangle_X + \int_Q j^0(x,t,u,v)\,dx\,dt \geqslant \int_Q g(x,u)v\,dx\,dt.$$

This completes the proof. $\square$

References

[1] S. Ahmad, A.C. Lazer, J.L. Paul, Elementary critical point theory and perturbations of elliptic boundary value problems at resonance, Indiana Univ. Math. J. 25 (1976) 933–944.

[2] S. Aizicovici, N.S. Papageorgiou, V. Staicu, Periodic solutions of nonlinear evolution inclusions in Banach spaces, J. Nonlinear Convex Anal. 7 (2) (2006) 163–177.

[3] S. Carl, D. Motreanu, Extremal solutions of quasilinear parabolic inclusions with generalized Clarke's gradient, J. Differential Equations 191 (2003) 206–233.

[4] K.C. Chang, Variational methods for nondifferentiable functionals and their applications to partial differential equations, J. Math. Anal. Appl. 80 (1981) 102–129.

[5] F.H. Clarke, Optimization and Nonsmooth Analysis, SIAM, Philadelphia, 1990.

[6] Z. Denkowski, S. Migorski, A system of evolution hemivariational inequalities modeling thermoviscoelastic frictional contact, Nonlinear Anal. 60 (8) (2005) 1415–1441.

[7] Z. Denkowski, S. Migorski, N.S. Papageorgiou, An Introduction to Nonlinear Analysis: Applications, Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York, 2003.

[8] D. Goeleven, D. Motreanu, P.D. Panagiotopoulos, Eigenvalue problems for variational–hemivariational inequalities at resonance, Nonlinear Anal. 33 (2) (1998) 161–180.

[9] P. Hess, On a theorem of Landesman and Lazer, Indiana Univ. Math. J. 23 (1974) 827–829.

[10] E.M. Landesman, A.C. Lazer, Nonlinear perturbations of linear elliptic boundary value problems, J. Math. Mech. 7 (1970) 609–623.

[11] Zhenhai Liu, On eigenvalue problems for elliptic hemivariational inequalities, Proc. Edinb. Math. Soc., in press.

[12] Zhenhai Liu, Browder–Tikhonov regularization of non-coercive evolution hemivariational inequalities, Inverse Problems 21 (1) (2005) 13–20.

[13] Zhenhai Liu, A class of evolution hemivariational inequalities, Nonlinear Anal. 36 (1999) 91–100.

[14] Zhenhai Liu, On doubly degenerate quasilinear parabolic equations of higher order, Acta Math. Sin. (Engl. Ser.) 21 (1) (2005) 197–208.

[15] Zhenhai Liu, On quasilinear elliptic hemivariational inequalities, Appl. Math. Mech. 20 (2) (1999) 225–230.

[16] Zhenhai Liu, Shisheng Zhang, On the degree theory for multivalued (S_+) type mappings, Appl. Math. Mech. 19 (1998) 1141–1149.

[17] S. Migorski, A. Ochal, Existence of solutions for second order evolution inclusions with application to mechanical contact problems, Optimization 55 (1–2) (2006) 101–120.

[18] Q.X. Ye, Z.Y. Li, Introduction to Reaction–Diffusion Equations, Kexue Press, Beijing, 1990.

[19] E. Zeidler, Nonlinear Functional Analysis and Its Applications, IIA and IIB, Springer, New York, 1990.